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Preservation of completeness under mappings in asymmetric topology

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Abstract. The preservation of various completeness properties in the quasi-metric (and quasi-uniform) setting under open, closed and uniformly open mappings is investigated. In particular, it is noted that between quasi-uniform spaces the property that each costable filter has a cluster point is preserved under uniformly open continuous surjections. Furthermore in the realm of quasi-uniform spaces conditions under which almost uniformly open mappings are uniformly open are given which generalize corresponding classical results for uniform spaces. As a by-product it is shown that a quasi-metrizable Moore space admits a left K-complete quasi-metric if and only if it is a complete Aronszajn space.

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1. Introduction

In Section 2 the preservation of topological completeness properties related to left K-completeness under open continuous mappings between quasi-metrizable spaces is studied. Section 3 contains similar investigations for uniformly open mappings between quasi-metric and quasi-uniform spaces. Furthermore Section 4 deals with the classical problem of determining conditions under which almost uniformly open mappings are uniformly open, again for the case of mappings between quasi-metric and quasi-uniform spaces. Those two sections show that in order to obtain satisfactory results, for our purposes it is appropriate to

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work under some conditions of supercompleteness. Section 5 finally records several results on the preservation of these completeness properties under closed continuous mappings between quasi-metrizable spaces.

2. Preservation of completeness properties under open mappings

Looking for an adequate version of completeness for the present investigations, we found that conditions from the area of left K-completeness were especially useful. Therefore in the following we shall concentrate on such properties and notions. To fix our notation and terminology let us recall the following basic concepts and conventions. By $\mathbf{N}$ we shall denote the positive integers. Let X be a set. As usual, a function $d : X \times X \to [0, \infty)$ that satisfies $d(x, y) = 0$ if and only if $x = y$, and $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$ is called a *quasi-metric* on X. The induced topology $\tau(d)$ is the topology generated by the base consisting of the balls $B_{2^{-n}}(x) = \{y \in X : d(x, y) < 2^{-n}\}$ where $x \in X$ and $n \in \mathbf{N}$. A sequence $(x_n)_{n \in \mathbf{N}}$ in a quasi-metric space (X, d) is called *left K-Cauchy* provided that for each $k \in \mathbf{N}$ there is $n_k \in \mathbf{N}$ such that $d(x_n, x_m) < 2^{-k}$ whenever $n, m \in \mathbf{N}$ and $n_k \leq n \leq m$. A quasi-metric space (X, d) is *left K-complete* provided that each left K-Cauchy sequence converges in $(X, \tau(d))$ (compare [32]). For further concepts from the theory of quasi-uniform spaces we refer the reader to [9]. (Note however that we shall use $\mathcal{U}^s$ instead of $\mathcal{U}^*$ to denote the coarsest uniformity finer than some given quasi-uniformity $\mathcal{U}$.)

It is well known that Hausdorff showed that each open continuous image of a completely metrizable space is completely metrizable provided that it is metrizable. A modern proof of this fact is now usually based on the result that a paracompact open continuous (Hausdorff) image of a Čech complete space is Čech complete (compare [4, pp. 114–116]). On the other hand we do not know whether each open continuous quasi-metrizable image of a left K-complete quasi-metric space admits a left K-complete quasi-metric. (Recall that Kofner [20] showed that quasi-metrizability need not be preserved under open compact continuous mappings.) For the further discussion of that problem it is useful to be aware of the characterization of R_0-spaces possessing a λ-base given by Wicke and Worrell in [38, Theorems 3.2 and 3.3]: A topological R_0-space X has a λ-base if and only if there exists a sequence $(\mathcal{G}_n)_{n \in \mathbf{N}}$ of bases for X such that every decreasing representative $(G_n)_{n \in \mathbf{N}}$ of $(\mathcal{G}_n)_{n \in \mathbf{N}}$ with nonempty terms converges to some $x \in X$ and also to every element of $\cap_{n \in \mathbf{N}} G_n$. (In the following we shall call such a sequence of bases a *λ-base sequence*.) It is straightforward to verify that a quasi-metric space (X, d) is left K-complete if and only if $(\mathcal{G}_n)_{n \in \mathbf{N}}$, where $\mathcal{G}_n = \{B_{2^{-k}}(x) : x \in X, k \geq n, k \in \mathbf{N}\}$ whenever $n \in \mathbf{N}$, is a λ-base sequence (compare [33, Theorem 1]). Since each open continuous R_0-image of a space possessing a λ-base has a λ-base [37, Theorem 1], the question arises whether each quasi-metrizable space with a λ-base admits a left K-complete quasi-metric. We observe that it was shown in [24, Propositions 10 and 11] that a (Tychonoff) Čech complete or scattered quasi-metrizable space admits a left K-complete quasi-metric. We also note that for regular spaces, because a regular T_0-space is a complete Aronszajn space if and only if it has a λ-base (see [37, p. 256]), our problem was already formulated by Romaguera (Question 3 of [33]) when he asked whether each complete Aronszajn quasi-metrizable space admits a left K-complete quasi-metric. While the latter question remains open, in this section we shall show that Romaguera's problem has a positive answer in the class of quasi-metrizable Moore spaces. Our method of proof may be of independent interest, since as a by-product of our slightly more general argument we obtain a new proof of Kofner's classical result [19] that each γ-space with an ortho-pair-base is quasi-metrizable which seems easier to comprehend than the original one. Our proof will make use of some ideas contained in Junnila's thesis [11] (see also [12]) and in [14]. In particular, the following concept will be used: A neighbornet U of a topological space X is called *unsymmetric* provided that $x, y \in X, x \in U(y)$ and $y \in U(x)$ imply that $U(x) = U(y)$.

The definition of an ortho-pair-base (for T_1-spaces) is due to Kofner [19]. A collection $\mathcal{B}$ of pairs (G, G') (with $G \subseteq G'$) of open sets in a topological space X is called a *pair-base* for X provided that whenever H is open and $x \in H$ then there is $(G, G') \in \mathcal{B}$ such that $x \in G \subseteq G' \subseteq H$. The concept of a *local pair-base* at some $x \in X$ will now be self-explanatory. A pair-base $\mathcal{P}$ of a topological space X is called an *ortho-pair-base* provided that for each subcollection $\mathcal{P}_0$ of $\mathcal{P}$ and each $x \in \cap\{G : (G, G') \in \mathcal{P}_0\}$ such that $x \notin \operatorname{int} \cap \{G' : (G, G') \in \mathcal{P}_0\}$, the collection $\mathcal{P}_0$ is a local pair-base at x.

Proposition 2.1. *Let X be a topological space that possesses an ortho-pair-base. Then for every unsymmetric neighbornet S of X there exists a neighbornet V of X such that $V^2 \subseteq S$.*

Proof. Suppose that $\mathcal{G}$ is an ortho-pair-base for X that is well-ordered by $\leq$. For each $x \in X$ choose the first element $(G, G') \in \mathcal{G}$ with respect to $\leq$ such that $x \in G \subseteq G' \subseteq S(x)$ and call it (G_x, G'_x); furthermore set $V(x) = \bigcap\{G'_y : x \in G_y\} \cap G_x$. First we want to show that $V(x)$ is a neighborhood at x : Otherwise x cannot have a smallest neighborhood and $\{G'_y : x \in G_y\}$ is a neighborhood base at x, since $\mathcal{G}$ is an ortho-pair-base. Therefore there is G_y such that $x \in G_y \subseteq G'_y \subset G_x$. Then $\{x, y\} \subseteq G_x \subseteq G'_x \subseteq S(x)$ and $\{x, y\} \subseteq G_y \subseteq G'_y \subseteq S(y)$. Thus $S(x) = S(y)$ by unsymmetry of S. Furthermore we have $(G_x, G'_x) < (G_y, G'_y)$ or $(G_x, G'_x) > (G_y, G'_y)$ which contradicts the definition of (G_y, G'_y) resp. (G_x, G'_x). We conclude that V is a neighbornet of X. Suppose that $x \in X$ and $y \in V(x)$. Then $y \in G_x$. Thus $V(y) \subseteq G'_x \subseteq S(x)$. We have shown that $V^2 \subseteq S$. □

Recall that for a neighbornet V of a topological space X the neighbornet V^+ is defined as follows: $V^+(x) = \bigcap\{V(G) : G \text{ is a neighborhood at } x\}$ whenever $x \in X$ (see [19]). Note that $V^+ \subseteq V^2$.

Proposition 2.2. *Let X be a topological T_1-space with an ortho-pair-base. Then for each neighbornet U of X, U^+ contains an unsymmetric neighbornet S of X.*

Proof. Suppose that $\mathcal{G}_0 = \{(G_\alpha, G'_\alpha) : \alpha < \delta\} \cup \{(\{x\},\{x\}) : x$ is isolated in $X\}$ is an ortho-pair-base for X, where we can assume that each G'_α is not a singleton. Set $H_0 = X$ and define inductively, given an ordinal α, $H_{\alpha+1} = \operatorname{int}\{x \in H_\alpha : \bigcup\{G' : x \in G, (G,G') \in \mathcal{G}_\alpha\} \nsubseteq U(x)\}$ and $\mathcal{G}_{\alpha+1} = \{(G,G') \in \mathcal{G}_\alpha : G' \subseteq H_{\alpha+1}\} \setminus \{(G,G') \in \mathcal{G}_\alpha : G'_\alpha \subseteq G'\}$ where the second set of that definition is assumed to be empty if G'_α is undefined; furthermore for a limit ordinal α set $\mathcal{G}_\alpha = \bigcap_{\beta<\alpha} \mathcal{G}_\beta$ and $H_\alpha = \bigcap_{\beta<\alpha} H_\beta$.

Clearly, for each α, $H_{\alpha+1}$ is open and the transfinite sequence $(H_\alpha)_\alpha$ is decreasing. Note that certainly $H_{\delta+1} = \varnothing$. So we assume that the induction stops at the first ordinal ϵ such that $H_\epsilon = \varnothing$. Observe also that $\bigcup\{G' : (G,G') \in \mathcal{G}_\alpha\} \subseteq H_\alpha$ for each limit ordinal α. We want to show next by induction on α that for each $x \in H_\alpha$ we have that $\mathcal{G}_\alpha$ contains a neighborhood pair-base at x : If this condition is satisfied for some α, then clearly it is also fulfilled for $\alpha + 1$, since no G'_α is a singleton. For a limit ordinal α we argue as follows: Suppose that $x \in \bigcap_{\beta<\alpha} H_\beta$. Since $x \in H_{\beta+1}$ whenever $\beta < \alpha$, we have $\bigcup\{G' : x \in G, (G,G') \in \mathcal{G}_\beta\} \nsubseteq U(x)$ whenever $\beta < \alpha$. Consequently for each $\beta < \alpha$ there exists $(E_\beta, E'_\beta) \in \mathcal{G}_\beta$ such that $x \in E_\beta$ and $E'_\beta \nsubseteq U(x)$. Thus $x \in \operatorname{int} \bigcap_{\beta<\alpha} E'_\beta$ by definition of the characteristic property of the ortho-pair-base $\mathcal{G}_0$. Let $(L, L') \in \mathcal{G}_0$ be such that $x \in L$ and $L' \subseteq \operatorname{int} \bigcap_{\beta<\alpha} E'_\beta$. Suppose that $(L, L') \notin \mathcal{G}_\alpha$. Then there is some minimal $\beta < \alpha$ such that $(L, L') \notin \mathcal{G}_\beta$. Note that β necessarily is a successor ordinal and so $(L, L') \in \mathcal{G}_{\beta-1}$. Then $L' \nsubseteq H_\beta$ or $G'_{\beta-1} \subseteq L'$. Therefore $E'_\beta \nsubseteq H_\beta$ or $G'_{\beta-1} \subseteq E'_\beta$. Thus $(E_\beta, E'_\beta) \notin \mathcal{G}_\beta$ —a contradiction. We conclude that $\{(L, L') : x \in L, (L, L') \in \mathcal{G}_\alpha\}$ is a neighborhood pair-base at x. In particular, since $(L, L') \in \cap_{\beta<\alpha}\mathcal{G}_\beta$, we deduce that $x \in L' \subseteq \cap_{\beta<\alpha} H_\beta = H_\alpha$. So H_α is also open if α is a limit ordinal.

Let $x \in X$ and let α_x be the first ordinal α such that $x \notin H_\alpha$. Note that α_x necessarily is a successor ordinal and so $x \in H_{\alpha_x - 1}$. Let (G_x, G'_x) be the first element of $\mathcal{G}_{\alpha_x-1}$ with respect to the well-ordering of $\mathcal{G}_0$ such that $x \in G_x$. Set $S(x) = G_x$. Then the neighbornet $S = \bigcup_{x\in X}(\{x\} \times S(x))$ is unsymmetric: If $x, y \in X$ and $\{x, y\} \subseteq S(x) \cap S(y)$, then $\alpha_x = \alpha_y$. Otherwise suppose for instance that $\alpha_x < \alpha_y$. Then $x \notin H_{\alpha_y-1}$, but $x \in G_y$ where $(G_y, G'_y) \in \mathcal{G}_{\alpha_y-1}$ and hence $x \in G'_y \subseteq H_{\alpha_y-1}$ —a contradiction. Therefore we conclude that $\alpha_x = \alpha_y$ and so $S(x) = S(y)$.

Let $x \in X$ and let G be an arbitrary neighborhood of x. By definition of H_{α_x}, there is $y \in S(x) \cap G \cap H_{\alpha_x-1}$ such that $\bigcup\{G' : y \in G, (G,G') \in \mathcal{G}_{\alpha_x-1}\} \subseteq U(y) \subseteq U(G)$. Thus $S(x) = G_x \subseteq G'_x \subseteq U(G)$. Hence we have shown that $S \subseteq U^+$. □

Remark 2.3. *Combining the two preceding propositions we obtain the result due to Kofner that in a T_1-space X with an ortho-pair-base for each neighbornet U of X there is a neighbornet V of X such that $V^2 \subseteq U^+$. As Kofner observed in* [19, Proposition 3], *the latter result implies that for each neighbornet U of X, U^+ is a normal neighbornet (compare* [19, Theorem 1]*) so that in particular each γ-space with an ortho-pair-base is quasi-metrizable* [19, Theorem 2]. *(Recall that a T_1-space X is a γ-space provided that it possesses a sequence $(V_n)_{n\in\mathbf{N}}$ of* *neighbornets such that* $\{V_n^2(x) : n \in \mathbf{N}\}$ *is a neighborhood base at* x *whenever* $x \in X$*.) Obviously it also follows from these results that in a topological space with an ortho-pair-base for each unsymmetric neighbornet* U *there exists an unsymmetric neighbornet* V *such that* $V^2 \subseteq U$*. Kofner noted in* [19, p. 1440] *that each developable* γ*-space possesses an ortho-pair-base. Next we want to apply the preceding results to our discussion of the* λ*-base property.*

Proposition 2.4. *Let* X *be a* T_1*-space possessing a* λ*-base and having the property that for each unsymmetric neighbornet* U *there exists an unsymmetric neighbornet* V *such that* $V^2 \subseteq U$*. Then* X *admits a left* K*-complete quasi-metric.*

Proof. Since X has a base of countable order [37] and thus a primitive base [39, Theorem 4.1], X possesses a sequence $(H_n)_{n\in\mathbf{N}}$ of unsymmetric neighbornets such that $\{H_n(x) : n \in \mathbf{N}\}$ is a neighborhood base at x whenever $x \in X$ (see [8, p. 147]). Let $(\mathcal{B}_n)_{n\in\mathbf{N}}$ be a λ-base sequence of X. We can suppose that each base $\mathcal{B}_n$ is well-ordered by $\leq_n$. Inductively we shall define unsymmetric neighbornets V_n and B_n such that $V_{n+1}^2 \subseteq H_n \cap B_n$ and $B_n \subseteq V_n$ whenever $n \in \mathbf{N}$. Set $V_1(x) = X$ whenever $x \in X$. Suppose now that, for some $n \in \mathbf{N}$, the unsymmetric neighbornet V_n is defined. Then for each $x \in X$ we find the first element $B \in \mathcal{B}_n$ such that $x \in B \subseteq V_n(x)$ and set $B_n(x) = B$. Similarly as above, note first that the neighbornet B_n is unsymmetric: If $x, y \in X$ and $x, y \in B_n(x) \cap B_n(y)$, then $V_n(x) = V_n(y)$ by unsymmetry of V_n. By definition of B_n it follows that $B_n(x) = B_n(y)$.

By our assumption on unsymmetric neighbornets of X we can find an unsymmetric neighbornet V_{n+1} of X such that $V_{n+1}^2 \subseteq H_n \cap B_n$, since $H_n \cap B_n$ is unsymmetric. The induction having carried out, we note that $B_{n+1}^2 \subseteq B_n$ and $B_{n+1} \subseteq H_n$ whenever $n \in \mathbf{N}$. Then $\{B_n : n \in \omega\}$ is a base for a compatible quasi-metrizable quasi-uniformity $\mathcal{V}$ on X. Let d be a quasi-metric on X inducing $\mathcal{V}$ and let $(x_n)_{n\in\mathbf{N}}$ be a left K-Cauchy sequence in (X, d). There is a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ in $\mathbf{N}$ such that for each $k \in \mathbf{N}$, $(x_n, x_m) \in B_k$ whenever $n_k \leq n \leq m$ and $n, m \in \mathbf{N}$. For each $k \in \mathbf{N} \setminus \{1\}$ we have $x_{n_{k+1}} \in B_k(x_{n_k})$ and thus $B_k(x_{n_{k+1}}) \subseteq B_{k-1}(x_{n_k})$. Since $B_k(x_{n_{k+1}}) \in \mathcal{B}_k$ whenever $k \in \mathbf{N}$ we conclude by the λ-base property that $\{B_k(x_{n_{k+1}}) : k \in \mathbf{N}\}$ and thus $(x_n)_{n\in\mathbf{N}}$ converges to some $x \in X$ (compare [34, Lemma 1]). We have shown that (X, d) is left K-complete. □

Corollary 2.5. *Each* T_1*-space with an ortho-pair-base that also possesses a* λ*-base admits a left* K*-complete quasi-metric.*

In particular we conclude that a Moore space admits a left K-complete quasi-metric if and only if it is a complete Aronszajn quasi-metrizable space (compare [33, Theorem 1]). Moore spaces that are complete Aronszajn spaces are also called semicomplete Moore spaces in the literature [30]. The Tychonoff example due to [4, Example 2.9] shows that a quasi-metrizable semicomplete Moore space need not be Čech complete. Quasi-metrizability of this space is clear, since it is a metacompact Moore space (see [9, Theorem 7.26]). Moreover it has a λ-base, because it is locally completely metrizable (compare [4, Proposition 2.2]).

Observe that this example answers negatively another question of Romaguera [33, Question 2], since each sequentially complete quasi-metric Tychonoff space is Čech complete [22, Proposition 4]. Let us finally state explicitly the two questions discussed in this section.

Problem 2.6. *Does each quasi-metrizable space with a λ-base admit a left K-complete quasi-metric?*

Problem 2.7. *Suppose that X admits a left K-complete quasi-metric and $f : X \to Y$ is an open continuous surjection onto a quasi-metric space Y. Does Y admit a left K-complete quasi-metric? (As mentioned above, these conditions imply that Y possesses a λ-base* [37, Theorem 1]. *Observe also that Theorem 8 of* [37] *asserts that a regular T_0-space has a λ-base if and only if it is an open continuous image of a completely metrizable space.)*

3. Preservation of completeness properties under uniformly open mappings

In this section we shall show that as in the classical, symmetric case better results than in Section 2 can be achieved if we assume that the mappings are uniformly open with respect to some given (quasi-)uniform structures on the spaces under consideration. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces. A (multi-valued) mapping $F : X \to Y$ is called *uniformly open* provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $V(F(x)) \subseteq F(U(x))$ whenever $x \in X$ (compare [5]). It is known that in the area of uniform (Hausdorff) spaces each open continuous mapping with compact domain is uniformly open [7, Proposition 2.2]. In fact the following more general result holds.

Proposition 3.1. *Let $(X,\mathcal{U})$ be a compact uniform space and let the mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be open and continuous where $(Y,\mathcal{V})$ is a quasi-uniform space. Then f is uniformly open.*

Proof. Let $U \in \mathcal{U}$. There is $P \in \mathcal{U}$ such that $P^2 \subseteq U$. Since f is open, for each $a \in X$ we find $W_a \in \mathcal{V}$ such that $W_a^2(f(a)) \subseteq f(P(a))$. By continuity of f and since $\mathcal{U}$ is a uniformity, we can consider the open cover $\{\text{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) : a \in X\}$ of X. Since X is compact, there is a finite subset F of X such that $\bigcup_{a\in F} \text{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) = X$. Set $W = \bigcap_{a\in F} W_a$ and note that $W \in \mathcal{V}$. Consider $x \in X$. There is $b \in F$ such that $x \in P^{-1}(b) \cap f^{-1}W_b(f(b))$. Therefore $f(x) \in W_b(f(b))$ and $W(f(x)) \subseteq W_b^2(f(b)) \subseteq fP(b) \subseteq fP^2(x) \subseteq fU(x)$. We have shown that f is uniformly open. □

Applying the preceding result to the identity mapping on a compact Hausdorff space, we draw the following conclusion.

Corollary 3.2. [9, Proposition 1.47] *The uniformity is the coarsest quasi-uniformity on a compact Hausdorff space.*

The identity mapping on a topological space X admitting two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U}$ is not contained in $\mathcal{V}$ also shows that the conclusion of Proposition 3.1 can only hold under some strong conditions.

The following classical result from Kelley's book [15, p. 203] is well known: Let f be a uniformly open continuous mapping from a complete pseudo-metrizable space into a uniform Hausdorff space. Then the range of the mapping f is complete. On the other hand, it is known that if G is a topological group whose left uniformity is complete and N is a closed normal subgroup, then the left uniformity of the quotient group G/N need not be complete, although the quotient mapping is continuous and uniformly open (compare [31, p. 195] and [27]). Such examples show that completeness of the domain space is not sufficient to generalize the afore-mentioned result from Kelley's book to uniform spaces.

In order to extend our investigations on quasi-metric spaces from Section 2 to general quasi-uniform spaces, we recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *left K-Cauchy* provided that for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ whenever $x \in F$. A quasi-uniform space $(X,\mathcal{U})$ is called *left K-complete* provided that each left K-Cauchy filter converges (compare [34]). The negative uniform result mentioned above however suggests that in an arbitrary quasi-uniform space $(X,\mathcal{U})$ we should consider a property stronger than left K-completeness, for instance, that each costable filter has a $\tau(\mathcal{U})$-cluster point, where a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *costable* provided that for each $U \in \mathcal{U}$ we have $\bigcap_{F\in\mathcal{F}} U^{-1}(F) \in \mathcal{F}$. Costable filters characterize hereditary precompactness in the sense that a quasi-uniform space $(X,\mathcal{U})$ is hereditarily precompact if and only if each filter on $(X,\mathcal{U})$ is costable (see e.g. [13, Proposition 2.5]). An ultrafilter on a quasi-uniform space is costable if and only if it is a left K-Cauchy filter [34, Proposition 1]. Costable filters were called *Császár filters* by Pérez-Peñalver and Romaguera in [29]. They said that a quasi-uniform space $(X,\mathcal{U})$ is *Császár complete* provided that each costable filter of $(X,\mathcal{U})$ has a $\tau(\mathcal{U}^s)$-cluster point. The latter conditions strengthens the well-known property of Smyth completeness, which means that each left K-Cauchy filter has a $\tau(\mathcal{U}^s)$-cluster point (equivalently, a $\tau(\mathcal{U}^s)$-limit point). Pérez-Peñalver and Romaguera also remarked that for any topological space X the well-monotone quasi-uniformity $\mathcal{W}_X$ has the property that each costable filter on $(X,\mathcal{W}_X)$ has a cluster point [29, Proposition 2]. It was noted (compare [26, p. 169], [32]) that for a quasi-pseudometric space (X,d), each costable filter of the induced quasi-uniform space $(X,\mathcal{U}_d)$ clusters if and only if each left K-Cauchy sequence (resp. each left K-Cauchy filter) converges. So for quasi-pseudometric spaces the property considered in the following is indeed equivalent to left K-completeness. For uniform spaces the property under consideration is equivalent to supercompleteness. A uniform space X is called *supercomplete* if each stable filter has a cluster point [2, 10]. For instance that condition is satisfied by a complete bilateral uniformity of a topological group of pointwise countable type [35]. It is well known that supercompleteness characterizes completeness of the Hausdorff uniformity on the hyperspace of nonempty closed subsets (equivalently, nonempty subsets) of a uniform space. On the other hand, for a quasi-uniform space $(X,\mathcal{U})$ the condition that each costable filter clusters in $(X,\mathcal{U})$ is only necessary, but not sufficient that its Hausdorff quasi-uniformity (on the collection of nonempty sets) is left K-complete (see [25]).

Proposition 3.3. *Let f be a uniformly open continuous mapping from a quasi-uniform space $(X,\mathcal{U})$ in which each costable filter $\mathcal{F}$ has a cluster point onto a quasi-uniform space $(Y,\mathcal{V})$. Then each costable filter on $(Y,\mathcal{V})$ has a cluster point.*

Proof. Let $\mathcal{F}$ be a costable filter on $(Y,\mathcal{V})$ and fix $U\in\mathcal{U}$. Since f is uniformly open, there is $V\in\mathcal{V}$ such that $V(f(x))\subseteq f(U(x))$ whenever $x\in X$. Because the filter $\mathcal{F}$ is costable in $(Y,\mathcal{V})$, there is $F_0\in\mathcal{F}$ such that $F_0\subseteq\bigcap_{F\in\mathcal{F}}V^{-1}(F)$. We want to show that $f^{-1}(F_0)\subseteq\bigcap_{F\in\mathcal{F}}U^{-1}(f^{-1}(F))$: Let $F\in\mathcal{F}$ and $a\in f^{-1}(F_0)$. Hence $f(a)=f_0$ for some $f_0\in F_0$. Thus $f_0\in V^{-1}(e)$ for some $e\in F$. Then $e\in V(f_0)\subseteq f(U(a))$. Therefore $e=f(c)$ for some $c\in U(a)$. It follows that $a\in U^{-1}(c)$ and $c\in f^{-1}(F)$. We have shown that $a\in U^{-1}(f^{-1}(F))$. We conclude that $f^{-1}(F_0)\subseteq\bigcap_{F\in\mathcal{F}}U^{-1}(f^{-1}(F))$ and $f^{-1}\mathcal{F}=\text{fil}\{f^{-1}(F):F\in\mathcal{F}\}$ is costable in $(X,\mathcal{U})$. Suppose now that x is a cluster point of $f^{-1}\mathcal{F}$. By continuity of f, $f(x)$ is a cluster point of $\mathcal{F}$. □

Corollary 3.4. *A uniform space that is the image of a supercomplete uniform space under a uniformly open continuous mapping is supercomplete.*

Since in a quasi-uniform space each left K-Cauchy filter is costable and converges to its cluster points (see [34]), the next result is a consequence of the preceding proposition and the observation about quasi-pseudometric spaces mentioned above.

Corollary 3.5. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ be a uniformly open continuous surjection. If $\mathcal{U}$ is quasi-pseudometrizable and left K-complete, then $\mathcal{V}$ is left K-complete.*

Because uniformly continuous mappings between quasi-uniform spaces are continuous with respect to the associated supremum uniformities, the following corollary is also readily verified.

Corollary 3.6. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f:(X,\mathcal{U})\to(Y,\mathcal{V})$ a uniformly open uniformly continuous surjection. If $\mathcal{U}$ is Császár complete, then $\mathcal{V}$ is Császár complete.*

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a *weakly Cauchy filter* or *Corson filter* provided that $\bigcap_{F\in\mathcal{F}}U^{-1}(F)\neq\varnothing$ whenever $U\in\mathcal{U}$ (see e.g. [29]). Obviously, each costable filter is weakly Cauchy. The property (compare [9, Proposition 5.32]) that each weakly Cauchy filter has a cluster point is often called *cofinal completeness* and even in metric spaces is strictly stronger than completeness (see e.g. [2, Example 1]). It is known that each uniformly locally compact and each paracompact fine uniform space is cofinally complete (e.g. [2, Corollaries 4 and 5]). In [36] it is shown that a (Tychonoff) topological group is locally compact if and only if it is of pointwise countable type and its left uniformity is cofinally complete. The following strengthening of Császár completeness was considered in [29]. A quasi-uniform space $(X,\mathcal{U})$ is called

Corson complete provided that each weakly Cauchy filter has a $\tau(\mathcal{U}^s)$-cluster point. As we show next, these two completeness properties are preserved under uniformly open uniformly continuous surjections, too.

Proposition 3.7. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ a uniformly open continuous surjection. If X is cofinally complete, then Y is cofinally complete.*

Proof. It will suffice to show that $f^{-1}\mathcal{F}$ is a weakly Cauchy filter on $(X,\mathcal{U})$ provided that f is uniformly open and $\mathcal{F}$ is a weakly Cauchy filter on $(Y,\mathcal{V})$: So suppose that $\mathcal{F}$ is a weakly Cauchy filter on $(Y,\mathcal{V})$. Let $U \in \mathcal{U}$. By uniform openness of f, there is $V \in \mathcal{V}$ such that $V(f(x)) \subseteq f(U(x))$ whenever $x \in X$. By our assumption, there is $y \in Y$ such that $V(y) \cap F \neq \varnothing$ whenever $F \in \mathcal{F}$. Let $x \in X$ be such that $y = f(x)$. Then $fU(x) \cap F \neq \varnothing$ whenever $F \in \mathcal{F}$. Thus $U(x) \cap f^{-1}F \neq \varnothing$ whenever $F \in \mathcal{F}$. Therefore $f^{-1}\mathcal{F} = \text{fil}\{f^{-1}F : F \in \mathcal{F}\}$ is a weakly Cauchy filter. □

Corollary 3.8. *Let $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ be a uniformly open uniformly continuous mapping from a Corson complete quasi-uniform space $(X,\mathcal{U})$ onto a quasi-uniform space $(Y,\mathcal{V})$. Then $(Y,\mathcal{V})$ is Corson complete.*

Our final proposition in this section applies Corollary 3.5 to the question considered in Section 2.

Proposition 3.9. *Suppose that $f : X \to Y$ is an open continuous mapping from a topological space X onto a T_1-space Y. If X admits a left K-complete quasi-metric d such that all fibers of f are precompact in (X,d^{-1}) then Y admits a left K-complete quasi-metric. In particular, a T_1-image of a completely metrizable space under an open compact mapping admits a left K-complete quasi-metric.*

Proof. We shall work with the quasi-metric quasi-uniformity $\mathcal{U}_d = \text{fil}\{B_{2^{-n}} : n \in \mathbf{N}\}$ on X. For each $y \in Y$ and $n \in \mathbf{N}$ set $V_n(y) = \bigcap_{x \in f^{-1}\{y\}} f(B_{2^{-n}}(x))$ whenever $y \in Y$. Then $\{V_n : n \in \mathbf{N}\}$ is a base for a quasi-metrizable quasi-uniformity $\mathcal{V}$ on Y, because $V_{n+1}^2 \subseteq V_n$ whenever $n \in \mathbf{N}$ and $\bigcap_{n \in \mathbf{N}} V_n = \{(y,y) : y \in Y\}$. Since the fibers are precompact in (X,d^{-1}), we see that $\mathcal{V}$ is compatible: By our assumption for each $y \in Y$ and $U \in \mathcal{U}_d$ there exists a finite subset $F \subseteq f^{-1}\{y\}$ such that $f^{-1}\{y\} \subseteq \bigcup_{x \in F} U^{-1}(x)$ and thus for each $x' \in f^{-1}\{y\}$ there is $x \in F$ such that $x' \in U^{-1}(x)$ and so $f(U(x)) \subseteq f(U^2(x'))$. Therefore $\bigcap_{x \in F} f(U(x)) \subseteq \bigcap_{x' \in f^{-1}\{y\}} f(U^2(x'))$. Since $\bigcap_{x \in F} f(U(x))$ is a neighborhood of y and f is continuous, we deduce that $\mathcal{V}$ is compatible. Since $f:(X,\mathcal{U}_d) \to (Y,\mathcal{V})$ is uniformly open by definition of $\mathcal{V}$, we conclude that $\mathcal{V}$ is left K-complete by Corollary 3.5. □

4. Almost uniformly open mappings

In this article a (multi-valued) mapping $F : X \to Y$ between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called *almost uniformly open* provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $V(F(x)) \subseteq \text{cl}_{\tau(\mathcal{V}^{-1})} F(U(x))$. Note that this definition yields the usual concept of almost uniform openness for mappings between uniform and metric spaces. Extending classical work on metric spaces (see [15, p. 202]) Dektjarev [6] proved the following result for supercomplete uniform spaces: Let F be an almost uniformly open multi-valued mapping with closed graph from the supercomplete uniform space X into an arbitrary uniform space Y. Then, for any entourages U and V in X and any point $x_0 \in X$, the inclusion $\overline{FU(x_0)} \subseteq FVU(x_0)$ is valid.

In this section we want to address the problem under which conditions an almost uniformly open mapping between quasi-uniform spaces is uniformly open. To this end we first recall that a quasi-uniform space $(X, \mathcal{U}^{-1})$ is called *right K-complete* provided that each left K-Cauchy filter on $(X, \mathcal{U})$ converges with respect to the topology $\tau(\mathcal{U}^{-1})$ (compare [34]). In the following we shall consider a stronger condition and further variant of the uniform property of supercompleteness, namely the condition that each costable filter on the quasi-uniform space $(X, \mathcal{U})$ has a $\tau(\mathcal{U}^{-1})$-cluster point. The latter condition was already studied to some extent by Künzi and Ryser [26, Proposition 6] where it was shown to be equivalent to right K-completeness of the Hausdorff quasi-uniformity transmitted by $\mathcal{U}^{-1}$ onto the collection of nonempty subsets of X. We also recall that a quasi-metric space (X, d^{-1}) is called *right K-sequentially complete* provided that each left K-Cauchy sequence of (X, d) converges in $(X, \tau(d^{-1}))$. It is known that right K-sequential completeness (for non-R_1-spaces) can be strictly weaker than right K-completeness of the induced quasi-uniformity in quasi-metric spaces [1, Remark 2]. This complication suggests that we should first establish a version of Dektjarev's result for quasi-metric spaces and only afterwards consider the more general quasi-uniform case. We remark that Khanh has already obtained a quantitative version of our next proposition in [16, Theorem 2]. On the other hand, Cao and Reilly [3, Lemma 5.3] gave some bitopological version of that result. Related to Khanh's studies further investigations in quasi-uniform spaces were conducted by Chou and Penot [5].

Proposition 4.1. *(compare* [16]*) Each almost uniformly open mapping $f : X \to Y$ from a quasi-metric space (X, d) into a quasi-metric space (Y, d') such that the graph of f is $\tau(d^{-1}) \times \tau((d')^{-1})$-closed and (X, d^{-1}) is right K-sequentially complete is uniformly open.*

Proof. Let $\mathcal{U}$ resp. $\mathcal{V}$ be the quasi-metric quasi-uniformities on (X, d) resp. (Y, d') generated by the standard bases $\{U_\epsilon : \epsilon > 0\}$ resp. $\{V_\epsilon : \epsilon > 0\}$. By our assumption on f for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $V(f(x)) \subseteq \mathrm{cl}_{\tau(\mathcal{V}^{-1})} fU(x)$ whenever $x \in X$. Hence it suffices to show that $\mathrm{cl}_{\tau(\mathcal{V}^{-1})} fU_\epsilon(x) \subseteq fU_{\epsilon+\delta}(x)$ whenever $\epsilon, \delta > 0$ and $x \in X$.

Fix $\epsilon, \delta > 0$. For each $n \in \mathbf{N}$, set $\epsilon_n = \frac{\delta}{2^n}$ and choose $\delta_n \leq \frac{1}{n}$ such that $V_{\delta_n}(f(x)) \subseteq \mathrm{cl}_{\tau(\mathcal{V}^{-1})} fU_{\epsilon_n}(x)$ whenever $x \in X$. Fix $x \in X$ and let $y \in \mathrm{cl}_{\tau(\mathcal{V}^{-1})} fU_\epsilon(x)$. Find $x_1 \in U_\epsilon(x)$ such that $(f(x_1), y) \in V_{\delta_1}$. Inductively we define a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that $(f(x_n), y) \in V_{\delta_n}$ and $(x_n, x_{n+1}) \in U_{\epsilon_n}$ whenever $n \in \mathbf{N}$: Suppose that x_n was chosen for some $n \in \mathbf{N}$ such that the induction hypothesis is satisfied. Therefore we have $y \in V_{\delta_n}(f(x_n)) \subseteq \mathrm{cl}_{\tau(\mathcal{V}^{-1})} f(U_{\epsilon_n}(x_n))$. Hence we find $x_{n+1} \in U_{\epsilon_n}(x_n)$ such that $f(x_{n+1}) \in V^{-1}_{\delta_{n+1}}(y)$. This completes the induction. It follows that $(x_n)_{n\in\mathbf{N}}$ is a left K-Cauchy sequence in (X,d). By our assumption on completeness of X there is $x'\in X$ such that $(x_n)_{n\in\mathbf{N}}$ converges to x' in $(X,\tau(\mathcal{U}^{-1}))$. We conclude that $d(x,x')<\epsilon+\delta$, because $d(x_n,x')\to 0$ and thus $d(x_1,x')\leq\delta$ by the triangle inequality. Consequently $x'\in U_{\epsilon+\delta}(x)$. Since the graph of f is $\tau(d^{-1})\times\tau((d')^{-1})$-closed and $d'(f(x_n),y)\to 0$, we see that $y=f(x')$. Thus $\mathrm{cl}_{\tau(\mathcal{V}^{-1})}fU_\epsilon(x)\subseteq fU_{\epsilon+\delta}(x)$. We have shown that f is uniformly open. □

We shall now give a version of Dektjarev's argument for quasi-uniform spaces.

Proposition 4.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space such that each costable filter on $(X,\mathcal{U})$ has a $\tau(\mathcal{U}^{-1})$-cluster point and let F be an almost uniformly open multi-valued mapping from $(X,\mathcal{U})$ into an arbitrary quasi-uniform space $(Y,\mathcal{V})$. Suppose that the graph of F is $\tau(\mathcal{U}^{-1})\times\tau(\mathcal{V}^{-1})$-closed. Then for any entourages U und W in $\mathcal{U}$ and any point $x_0\in X$, we have $\mathrm{cl}_{\tau(\mathcal{V}^{-1})}FU(x_0)\subseteq FWU(x_0)$, In particular F is uniformly open.*

Proof. Suppose that $\{U_i : i\in I\}$ is a base for $\mathcal{U}$ and $\{V_i : i\in I\}$ is a base for $\mathcal{V}$. With every entourage P of $\mathcal{U}$, we associate a sequence of entourages $(P_n)_{n\in\mathbf{N}}$ such that $P_1^2\subseteq P$ and $P_{n+1}^2\subseteq P_n$ whenever $n\in\mathbf{N}$. By our assumption on F, we can suppose that for each $U\in\mathcal{U}$ there is $U_F\in\mathcal{V}$ such that $U_FF(z)\subseteq\mathrm{cl}_{\tau(\mathcal{V}^{-1})}FU(z)$ whenever $z\in X$. Fix now $U,W\in\mathcal{U}$. Without loss of generality we assume that $(U_i)_n\subseteq W_{n+1}$ whenever $i\in I$ and $n\in\mathbf{N}$. Let $x_0\in X$ and $y\in\mathrm{cl}_{\tau(\mathcal{V}^{-1})}FU(x_0)$. Furthermore let D be the collection of nonempty finite subsets of I partially ordered by inclusion and for any $\nu\in D$ denote the number of elements of ν by $|\nu|$. We shall construct for each $\mu\in D$ a nonempty set $B_\mu\subseteq W_1U(x_0)$ such that $B_\nu\subseteq(\bigcap_{i\in\nu}(U_i)_{|\nu|})^{-1}(B_\mu)$ whenever $\nu\in D$ and $\nu\subset\mu$.

The sets B_ν are constructed by induction on $|\nu|$. For each $i\in I$, set $B_i=\{x\in U(x_0) : V_i^{-1}(y)\cap(((U_i)_1)_F)^{-1}(y)\cap F(x)\neq\varnothing\}$. Furthermore for each $\mu\in D$ with $|\mu|\geq 2$ set $B_\mu=\{x\in\bigcup_{\nu\subset\mu}(\bigcap_{i\in\nu}(U_i)_{|\nu|})(B_\nu) : (\cap_{i\in\mu}V_i^{-1}(y))\cap((\bigcap_{i\in\mu}(U_i)_{|\mu|})_F)^{-1}(y)\cap F(x)\neq\varnothing\}$. We shall verify next that the sets B_μ $(\mu\in D)$ satisfy the stated conditions: Since $y\in\mathrm{cl}_{\tau(\mathcal{V}^{-1})}FU(x_0)$, there is a net $(z_\delta)_{\delta\in E}$ in $FU(x_0)$ converging to y in $(Y,\mathcal{V}^{-1})$. For each $\delta\in E$ choose $u_\delta\in U(x_0)$ such that $(u_\delta,z_\delta)\in F$. We conclude that for each $i\in I$, $u_\delta\in B_i$ eventually, and, thus, also, for each $\mu\in D$, we have $u_\delta\in B_\mu$ eventually. Hence each B_μ $(\mu\in D)$ is nonempty. For all $i\in I$, the inclusion $B_i\subseteq U(x_0)$ holds by definition. Let $|\mu|=k\geq 2$. Inductively we can assume that for all ν for which $|\nu|<k$, the inclusion $B_\nu\subseteq W_{|\nu|}W_{|\nu|-1}\ldots W_2U(x_0)$ is satisfied. (In particular, we have $B_\nu\subseteq U(x_0)$ for $|\nu|=1$.) Then, by definition, $B_\mu\subseteq\bigcup_{\nu\subset\mu}(\bigcap_{i\in\nu}(U_i)_{|\nu|})(B_\nu)\subseteq\bigcup_{\nu\subset\mu}W_{|\nu|+1}(B_\nu)\subseteq\bigcup_{\nu\subset\mu}W_{|\nu|+1}W_{|\nu|}\ldots W_2U(x_0)=W_{|\mu|}\ldots W_2U(x_0)$. Hence $B_\mu\subseteq W_1U(x_0)$ whenever $\mu\in D$. Consider now $\nu,\mu\in D$ such that $\nu\subset\mu$ and $x\in B_\nu$. From $((\cap_{i\in\nu}(U_i)_{|\nu|})_F)^{-1}(y)\cap F(x)\neq\varnothing$, that is $y\in(\bigcap_{i\in\nu}(U_i)_{|\nu|})_F(F(x))\subseteq\mathrm{cl}_{\tau(\mathcal{V}^{-1})}F(\cap_{i\in\nu}(U_i)_{|\nu|})(x)$, we see that there exists $x'\in(\cap_{i\in\nu}(U_i)_{|\nu|})(x)$ such that $(\cap_{i\in\mu}V_i^{-1}(y))\cap((\bigcap_{i\in\mu}(U_i)_{|\mu|})_F)^{-1}(y)\cap F(x')\neq\varnothing$. Therefore $x'\in B_\mu$ by definition of B_μ. Furthermore we deduce that

$x \in (\cap_{i\in\nu}(U_i)_{|\nu|})^{-1}(x')$, that is $B_\nu \subseteq (\bigcap_{i\in\nu}(U_i)_{|\nu|})^{-1}(B_\mu)$. This concludes the verification of the stated conditions.

For each $\mu \in D$ set $C_\mu = \bigcup_{\mu\subseteq\nu} B_\nu$. Clearly $\{C_\mu : \mu \in D\}$ is a filterbase on X. We shall show that the generated filter $\mathcal{F}$ is costable in $(X,\mathcal{U})$. Let $H \in \mathcal{U}$ and $\mu \in D$. There is $i \in I$ such that $U_i \subseteq H$. Consider $x \in C_{\{i\}}$. Consequently $x \in B_\nu$ for some $\nu \in D$ such that $i \in \nu$. Note that $\epsilon = \nu \cup \mu \in D$. Then $x \in B_\nu \subseteq (\bigcap_{j\in\nu}(U_j)_{|\nu|})^{-1}(B_\epsilon) \subseteq U_i^{-1}(B_\epsilon) \subseteq H^{-1}(B_\epsilon) \subseteq H^{-1}(C_\mu)$. Hence we have shown that $C_{\{i\}} \subseteq \bigcap_{\mu\in D} H^{-1}(C_\mu)$. Thus $\mathcal{F}$ is costable in $(X,\mathcal{U})$. Observe next that the set of cluster points of $\mathcal{F}$ in $(X,\mathcal{U}^{-1})$ belongs to $WU(x_0)$, since each $C_\mu \subseteq W_1U(x_0)$ $(\mu \in D)$.

By our assumption there exists a $\tau(\mathcal{U}^{-1})$-cluster point x of $\mathcal{F}$. Consider arbitrary $i,k \in I$. Choose $\mu \in D$ such that $\{i,k\} \subseteq \mu$ and $U_i^{-1}(x) \cap B_\mu \neq \varnothing$. Find $x' \in U_i^{-1}(x) \cap B_\mu$. Then $V_k^{-1}(y) \cap F(x') \neq \varnothing$ by definition of B_μ. We conclude that $(U_i^{-1}(x) \times V_k^{-1}(y)) \cap F \neq \varnothing$. Thus $(x,y) \in F$ by closedness of F with respect to the topology $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{V}^{-1})$. We have shown that $y \in F(x) \subseteq FWU(x_0)$. It follows that F is uniformly open. □

Corollary 4.3. *(compare* [28]*) An almost uniformly open mapping with a closed graph from a supercomplete uniform space into an arbitrary uniform space is uniformly open. In particular, an almost uniformly open continuous mapping from a supercomplete uniform space into a uniform Hausdorff space is uniformly open.*

Corollary 4.4. *Let $(X,\mathcal{U})$ be a Császár complete quasi-uniform space and $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ an almost uniformly open uniformly continuous mapping onto a quasi-uniform T_1-space $(Y,\mathcal{V})$. Then f is uniformly open and $(Y,\mathcal{V})$ is Császár complete.*

Proof. Only the final paragraph of the proof of Proposition 4.2 has to be modified. This time $\mathcal{F}$ has a $\tau(\mathcal{U}^s)$-cluster point x in X. Let $k \in I$. By continuity of f there is $i \in I$ such that $f(U_i(x)) \subseteq V_k(f(x))$. Find $\mu \in D$ such that $\{i,k\} \subseteq \mu$ and there is $x' \in U_i(x) \cap U_i^{-1}(x) \cap B_\mu$. Thus $f(x') \in V_k(f(x))$; furthermore $f(x') \in V_k^{-1}(y)$ by definition of B_μ. Consequently $(f(x),y) \in \cap\mathcal{V}$ and thus $y = f(x) \in fWU(x_0)$. We conclude that f is uniformly open. The second assertion is a consequence of Corollary 3.6. □

5. Preservation of completeness under closed mappings

We finish this article with three results on closed continuous mappings between quasi-metrizable spaces. Let us recall that Kofner [18] has shown that each first-countable closed continuous image of a quasi-metrizable space is quasi-metrizable. His techniques can be modified to yield the following two results.

Proposition 5.1. *The image of a left K-complete quasi-metric space under a perfect continuous mapping admits a left K-complete quasi-metric.*

Proof. Let $f : X \to Y$ be a perfect continuous mapping from a left K-complete quasi-metric space (X,d) onto a topological space Y. For each $y \in Y$ and $n \in \mathbf{N}$

set $V_n(y) = \{y' \in Y : f^{-1}\{y'\} \subseteq B_{2^{-n}}(f^{-1}\{y\})\}$. Then clearly, by the assumption made on f, $\{V_n : n \in \mathbf{N}\}$ is a base of a compatible quasi-metrizable quasi-uniformity $\mathcal{V}$ on Y (see [17, Theorem 2]). Let e be a quasi-metric on Y inducing $\mathcal{V}$. Furthermore let $(y_n)_{n\in\mathbf{N}}$ be a left K-Cauchy sequence in (Y, e). There is a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ in $\mathbf{N}$ such that $(y_{n_k}, y_p) \in V_k$ whenever $p \in \mathbf{N}$ and $p \geq n_k$. Hence $f^{-1}\{y_{n_{k+1}}\} \subseteq B_{2^{-k}}(f^{-1}\{y_{n_k}\})$ whenever $k \in \mathbf{N}$. By compactness of the fibers of f, we find finite subsets F_{n_k} of $f^{-1}\{y_{n_k}\}$ such that $f^{-1}\{y_{n_k}\} \subseteq B_{2^{-k}}(F_{n_k})$ and therefore $F_{n_{k+1}} \subseteq f^{-1}\{y_{n_{k+1}}\} \subseteq B_{2^{-(k-1)}}(F_{n_k})$ whenever $k \in \mathbf{N}$. By König's Lemma [21] applied to the sequence of finite sets $(F_{n_k})_{k\in\mathbf{N}}$ we see that there exists a sequence $(y'_{n_k})_{k\in\mathbf{N}}$ such that $y'_{n_k} \in F_{n_k}$ and $d(y'_{n_k}, y'_{n_{k+1}}) < 2^{-(k-1)}$ whenever $k \in \mathbf{N}$. Thus by left K-completeness of (X, d) we can find $x \in X$ such that the left K-Cauchy sequence $(y'_{n_k})_{k\in\mathbf{N}}$ converges to x. Therefore by continuity of f, the sequence $(y_{n_k})_{k\in\mathbf{N}}$ and hence by [34, Lemma 1] the sequence $(y_n)_{n\in\mathbf{N}}$ converges to $f(x)$. Hence (Y, e) is left K-complete. We conclude that the topological property of admitting a left K-complete quasi-metric is preserved under perfect continuous surjections. □

Proposition 5.2. *A first-countable image Y of a right K-sequentially complete quasi-metric space (X, d) under a closed continuous mapping f admits a right K-sequentially complete quasi-metric.*

Proof. For any $y \in Y$, let $\{V_n(y) : n \in \mathbf{N}\}$ be a decreasing basic sequence of open neighborhoods at y. Set $W_n(y) = \{z \in Y : f^{-1}\{z\} \subseteq B_{2^{-n}}(f^{-1}\{y\}) \cap f^{-1}V_n(y)\}$ whenever $y \in Y$ and $n \in \mathbf{N}$. Furthermore set $\widehat{W}_n = \bigcup\{W_{k_p} \circ \ldots \circ W_{k_1} : 2^{-k_1} + \ldots + 2^{-k_p} \leq 2^{-n}$ and $k_1, \ldots, k_p, p \in \mathbf{N}\}$ whenever $n \in \mathbf{N}$. Note that $\widehat{W}^2_{n+1} \subseteq \widehat{W}_n$ whenever $n \in \mathbf{N}$. Kofner's argument [18, p. 334] shows that the quasi-metrizable quasi-uniformity $\mathcal{W}$ generated by $\{\widehat{W}_n : n \in \mathbf{N}\}$ is compatible on Y. Note that if $a, b \in Y$, $s \in \mathbf{N}$ and $a \in W_s(b)$ we can find for any $a' \in f^{-1}\{a\}$ some $b' \in f^{-1}\{b\}$ such that $a' \in B_{2^{-s}}(b')$. Let e be a quasi-metric on Y inducing $\mathcal{W}$. It suffices to show that e is right K-sequentially complete.

Let $(y_n)_{n\in\mathbf{N}}$ be a left K-Cauchy sequence in (Y, e^{-1}). For each $k \in \mathbf{N}$ there is a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ in $\mathbf{N}$ such that $(y_l, y_{n_k}) \in \widehat{W}_k$ whenever $l \in \mathbf{N}$ and $l \geq n_k$. In particular $(y_{n_{k+1}}, y_{n_k}) \in \widehat{W}_k$ whenever $k \in \mathbf{N}$. It follows that for each $k \in \mathbf{N}$ there are $p \in \mathbf{N}$, $s_1, \ldots, s_p \in \mathbf{N}$ and $a_1, \ldots, a_{p-1} \in Y$ such that $2^{-s_1} + \ldots + 2^{-s_p} \leq 2^{-k}$ and $(y_{n_{k+1}}, a_1) \in W_{s_1}, \ldots, (a_{p-1}, y_{n_k}) \in W_{s_p}$. (In particular, $(y_{n_{k+1}}, y_{n_k}) \in W_{s_1}$ if $p = 1$.) Choose $y'_{n_1} \in X$ such that $y_{n_1} = f(y'_{n_1})$. Inductively over $k \in \mathbf{N}$ we can find points $a'_{p-1}, \ldots, a'_1$ and $y'_{n_{k+1}} \in X$ such that $f(y'_{n_{k+1}}) = y_{n_{k+1}}$, for each $i = 1, \ldots, p-1$ we have $f(a'_i) = a_i$ and $(y'_{n_{k+1}}, a'_1) \in B_{2^{-s_1}}, \ldots, (a'_{p-1}, y'_{n_k}) \in B_{2^{-s_p}}$. Thus for each $k \in \mathbf{N}$, $(y'_{n_{k+1}}, y'_{n_k}) \in B_{2^{-(k-1)}}$. We conclude that $(y'_{n_k})_{k\in\mathbf{N}}$ is a left K-Cauchy sequence in (X, d^{-1}) and converges to some x in (X, d). Then the sequence $(y_{n_k})_{k\in\mathbf{N}}$ converges to $f(x)$ by continuity of f. Since $(y_n)_{n\in\mathbf{N}}$ is a left K-Cauchy sequence in (Y, e^{-1}), it also converges to $f(x)$ in (Y, e) (see [34, Lemma 1]). We deduce that Y admits a right K-sequentially complete quasi-metric. □

Problem 5.3. *Does a first-countable image of a left K-complete quasi-metric space under a closed continuous mapping admit a left K-complete quasi-metric?*

Finally we would like to mention that it is well known that under appropriate hypotheses preimages of quasi-uniform spaces which possess some kind of completeness property also satisfy that type of completeness condition (see e.g. [26, Proposition 7]). We finish this article with another such result. (It is known on the other hand that the property of quasi-metrizability behaves rather badly under preimages (compare [23]).)

Proposition 5.4. *Let $f : X \to Y$ be a closed continuous mapping from a quasi-metric space (X, d) such that all fibers are left K-complete onto a left K-complete quasi-metric space (Y, d'). Then X admits a left K-complete quasi-metric. (The analogous result also holds for right K-sequential completeness instead of left K-completeness.)*

Proof. For each $n \in \mathbf{N}$ set $V_n = \{(x, y) \in X \times X : d'(f(x), f(y)) < 2^{-n}$ and $d(x, y) < 2^{-n}\}$. Let e be a quasi-metric on X inducing the (compatible) quasi-uniformity generated by $\{V_n : n \in \mathbf{N}\}$. Furthermore let $(x_n)_{n\in\mathbf{N}}$ be a left K-Cauchy sequence in (X, e). Note first that the left K-Cauchy sequence $(f(x_n))_{n\in\mathbf{N}}$ converges to some $y \in Y$. By our assumption on the fibers, $(x_n)_{n\in\mathbf{N}}$ has a cluster point and thus, by [34, Lemma 1], converges provided that $(f(x_n))_{n\in\mathbf{N}}$ has a constant subsequence. So let us assume that this is not the case. In particular we can suppose that $f(x_n) \neq y$ for n larger than some $n_0 \in \mathbf{N}$. By closedness of f, we deduce that $y \in f(\mathrm{cl}_{\tau(d)}\{x_n : n > n_0, n \in \mathbf{N}\})$. Choose $x \in \mathrm{cl}_{\tau(d)}\{x_n : n > n_0, n \in \mathbf{N}\}$ such that $f(x) = y$. Then evidently x is a cluster point and thus by [34, Lemma 1] a limit point of the sequence $(x_n)_{n\in\mathbf{N}}$. We conclude that (X, e) is left K-complete. A similar argument establishes the second assertion. □

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