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Separation axioms in topological preordered spaces and the existence of continuous order-preserving functions

G. Bosi, R. Isler

Abstract. We characterize the existence of a real continuous order-preserving function on a topological preordered space, under the hypotheses that the topological space is normal and the preorder satisfies a strong continuity assumption, called *IC-continuity*. Under the same continuity assumption concerning the preorder, we present a sufficient condition for the existence of a continuous order-preserving function in case that the topological space is completely regular.

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1. Introduction

McCartan [**?**] introduced a natural continuity hypothesis on a preorder $\preceq$ on a topological space (X, τ). Such an assumption, which is called *IC-continuity* throughout this paper, is stronger than the usual hypothesis according to which all the *lower* and *upper sections* are closed. Separation axioms in ordered topological spaces were studied, in connection with suitable continuity assumptions, by Burgess and Fitzpatrick [**?**] and later by Künzi [**?**]. On the other hand, the existence of a real continuous order-preserving function on a *normally preordered topological space* was characterized by Mehta [**?**].

In this paper we are concerned with the existence of a real continuous order-preserving function f on a *topological preordered space* $(X, \tau, \preceq)$ in case that (X, τ) is either a normal or a completely regular space, and the preorder $\preceq$ is *IC*-continuous. Such a problem was already faced by Bosi and Isler [**?**]. We recall that a full characterization of the existence of a real continuous order-preserving function on a topological preordered space was provided by Herden [**?**], [**?**] (see also Mehta [**?**] for an excellent review), who introduced the notion of a *separable system*. Such a concept appears as a generalization of the concept of a *decreasing scale*, which was used by Burgess and Fitzpatrick [**?**] and seems more suitable to our aims.

2. Definitions and preliminary considerations

Given a *preorder* $\precsim$ on an arbitrary set X (i.e., a binary relation on X which is *reflexive* and *transitive*), denote by $\prec$ and $\sim$ the *asymmetric part* and respectively the *symmetric part* of $\precsim$. If $(X, \precsim)$ is a preordered set, and τ is a topology on X, then the triplet $(X, \tau, \precsim)$ will be referred to as a *topological preordered space*.

A subset A of a set X endowed with a preorder $\precsim$ is said to be *decreasing* (respectively, *increasing*) if $(x \in A) \wedge (y \precsim x) \Rightarrow y \in A$ (respectively, $(x \in A) \wedge (x \precsim y) \Rightarrow y \in A$).

If A is any subset of a set X endowed with a preorder $\precsim$, then denote by $d(A)$ (respectively by $i(A)$) the intersection of all the decreasing (respectively, increasing) subsets of X containing A.

Given a topological preordered space $(X, \tau, \precsim)$, we say that $\precsim$ is

(i) *continuous* if $d(x) = d(\{x\})$ and $i(x) = i(\{x\})$ are closed sets for every $x \in X$,
(ii) *I-continuous* if $d(A)$ and $i(A)$ are open sets for every open subset A of X,
(iii) *C-continuous* if $d(A)$ and $i(A)$ are closed sets for every closed subset A of X,
(iv) *IC-continuous* if $\precsim$ is both I-continuous and C-continuous.

Definitions (ii) and (iii) were introduced by McCartan [**?**]. The previous terminology is similar to the terminology adopted by Künzi [**?**]. Obviously, given a topological preordered space $(X, \tau, \precsim)$, if (X, τ) is a T_1 space, and $\precsim$ is C-continuous, then $\precsim$ is continuous. So, if the T_1 separation axiom holds, the concept of C-continuity is stronger than the concept of continuity of a preorder on a topological space.

From Nachbin [**?**], a topological preordered space $(X, \tau, \precsim)$ is said to be *normally preordered* if, given a closed decreasing set F_0 and a closed increasing set F_1 with $F_0 \cap F_1 = \varnothing$, there exist an open decreasing set A_0 containing F_0, and an open increasing set A_1 containing F_1 such that $A_0 \cap A_1 = \varnothing$.

It is easily seen that a topological preordered space $(X, \tau, \precsim)$ is normally preordered if (X, τ) is normal and $\precsim$ is IC-continuous. Indeed, given a closed decreasing set F_0 and a closed increasing set F_1 with $F_0 \cap F_1 = \varnothing$, from normality of (X, τ) there exist an open set A'_0 containing F_0, and an open set A'_1 containing F_1 such that $A'_0 \cap A'_1 = \varnothing$, and from IC-continuity of the preorder $\precsim$ we have that $A_0 = d(A'_0) \setminus i\left(\overline{A'_1} \setminus d(A'_0)\right)$ is an open decreasing set containing F_0, $A_1 = i(A'_1) \setminus d\left(\overline{A'_0} \setminus i(A'_1)\right)$ is an open increasing set containing F_1, and $A_0 \cap A_1 = \varnothing$.

From Herden [**?**], a topological preordered space $(X, \tau, \precsim)$ is *Nachbin separable* if there exists a countable family $\{A_n, B_n\}_{n \in \mathbf{N}}$ of pairs of closed disjoint subsets of X such that A_n is decreasing, B_n is increasing, and $\{(x,y) \in X \times X : x \prec y\} \subseteq \bigcup_{n \in \mathbf{N}} (A_n \times B_n)$.

From Burgess and Fitzpatrick [?], given a topological preordered space $(X, \tau, \preceq)$, a family $\mathcal{G} = \{G_r : r \in \mathcal{S}\}$ of open decreasing subsets of X is said to be a *decreasing scale* in $(X, \tau, \preceq)$ if the following conditions are satisfied:

(i) $\mathcal{S}$ is a dense subset of $[0,1]$ such that $1 \in \mathcal{S}$ and $G_1 = X$, and
(ii) for every $r_1, r_2 \in \mathcal{S}$ with $r_1 < r_2$, it is $\overline{G}_{r_1} \subseteq G_{r_2}$.

Observe that any decreasing scale $\mathcal{G}$ in a topological preordered space $(X, \tau, \preceq)$ is a *linear separable system* in Herden's terminology (see Herden [?]).

If $(X, \preceq)$ is a preordered set, then a real function f on X is said to be

(i) *increasing* if, for every $x, y \in X$, $[x \preceq y \Rightarrow f(x) \leq f(y)]$,
(ii) *order-preserving* if it is increasing and, for every $x, y \in X$, $[x \prec y \Rightarrow f(x) < f(y)]$.

It is well known that, if there exists a continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$, then $(X, \tau, \preceq)$ is Nachbin separable.

Finally, we recall that, given a topological space (X, τ), a family $\mathcal{G} = \{G_r : r \in \mathcal{S}\}$ of open subsets of X is said to be a *scale* in (X, τ) if $\mathcal{G}$ is a (decreasing) scale in $(X, \tau, =)$.

3. Existence of Continuous Order-Preserving Functions

Our first aim is to characterize the existence of a real continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$ with $\preceq$ *IC*-continuous and (X, τ) normal.

Theorem 3.1. *Let $(X, \tau, \preceq)$ be a topological preordered space, with $\preceq$ IC-continuous and (X, τ) normal. Then the following conditions are equivalent:*

(i) *There exists a real continuous order-preserving function f on the space $(X, \tau, \preceq)$ with values in $[0,1]$;*
(ii) *$(X, \tau, \preceq)$ is Nachbin separable;*
(iii) *There exists a countable family $\{A'_n, B'_n\}_{n \in \mathbf{N}}$ of pairs of closed disjoint subsets of X such that*

$$\{(x,y) \in X \times X : x \prec y\} \subseteq \bigcup_{n \in \mathbf{N}} (A'_n \times B'_n)$$

and, for every $n \in \mathbf{N}$, if $a'_n \in A'_n$, $b'_n \in B'_n$, then $b'_n \notin d(a'_n)$.

Proof. (i) $\Longrightarrow$ (ii) From considerations above, $(X, \tau, \preceq)$ is normally preordered, and therefore the implication follows from Herden [?, Corollary 4.2].

(ii) $\Longrightarrow$ (iii) Just observe that any countable family $\{A'_n, B'_n\}_{n \in \mathbf{N}}$ satisfying the condition of Nachbin separability also verifies condition (iii).

(iii) $\Longrightarrow$ (i) Assume that condition (iii) holds, and let $\{A'_n, B'_n\}_{n \in \mathbf{N}}$ be a countable family of closed disjoint subsets of X with the indicated property. Define, for every $n \in \mathbf{N}$, $A_n = d(A'_n)$, $B_n = i(B'_n)$. Since $\preceq$ is *C*-continuous, A_n and B_n are closed subsets of X for every $n \in \mathbf{N}$. Further, A_n and B_n are disjoint sets for every $n \in \mathbf{N}$ (otherwise, for some $n \in \mathbf{N}$ there exist $x \in X$, $a'_n \in A'_n$ and $b'_n \in B'_n$ such that $b'_n \preceq x \preceq a'_n$, and therefore $b'_n \in d(a'_n)$). Hence, $(X, \tau, \preceq)$ is

Nachbin separable. Since $\preceq$ is also I-continuous, it has been already observed that $(X,\,\tau,\,\preceq)$ is normally preordered. Hence, from Mehta [**?**, Theorem 1] there exists a continuous order-preserving function f on $(X,\,\tau,\,\preceq)$ with values in $[0,1]$. □

In the sequel, a compact space is a compact-T_2 space, as in Engelking [**?**].

Corollary 3.2. *Let $(X,\tau,\preceq)$ be a topological preordered space, with $\preceq$ IC-continuous and (X,τ) compact. Then the following conditions are equivalent:*

(i) *There exists a real continuous order-preserving function f on the space $(X,\tau,\preceq)$ with values in $[0,1]$;*

(ii) *There exists a countable family $\{A_n,B_n\}_{n\in\mathbf{N}}$ of pairs of compact disjoint subsets of X such that A_n is decreasing, B_n is increasing, and*

$$\{(x,y)\in X\times X : x\prec y\}\subseteq\bigcup_{n\in\mathbf{N}}(A_n\times B_n)\,;$$

(iii) *There exists a countable family $\{A'_n,B'_n\}_{n\in\mathbf{N}}$ of pairs of compact disjoint subsets of X such that*

$$\{(x,y)\in X\times X : x\prec y\}\subseteq\bigcup_{n\in\mathbf{N}}\left(A'_n\times B'_n\right)$$

and, for every $n\in\mathbf{N}$, if $a'_n\in A'_n$, $b'_n\in B'_n$, then $b'_n\notin d(a'_n)$.

Proof. Observe that any compact space (X,τ) is normal. Further, it is well known that, given a compact space, any closed subspace is compact, as well as any compact subspace is closed. Then the thesis follows from Theorem **??**. □

In the following theorem we provide a sufficient condition for the existence of a continuous order-preserving function f on a topological preordered space $(X,\tau,\preceq)$, with (X,τ) completely regular, and $\preceq$ IC-continuous.

Theorem 3.3. *Let $(X,\tau,\preceq)$ be a topological preordered space, with $\preceq$ IC-continuous and (X,τ) completely regular. There exists a real continuous order-preserving function f on $(X,\tau,\preceq)$ with values in $[0,1]$ if the following condition is verified:*

(i) *There exists a countable family $\{A'_n,B'_n\}_{n\in\mathbf{N}}$ of pairs of disjoint subsets of X, with A'_n compact and decreasing and B'_n closed for every $n\in\mathbf{N}$, such that*

$$\{(x,y)\in X\times X : x\prec y\}\subseteq\bigcup_{n\in\mathbf{N}}\left(A'_n\times B'_n\right).$$

Proof. Let $\{A'_n,B'_n\}_{n\in\mathbf{N}}$ be a countable family of pairs of subsets of X satisfying condition (i) above. From C-continuity of $\preceq$, $i(B'_n)$ is closed and increasing for every $n\in\mathbf{N}$. Further, it is clear that A'_n and $i(B'_n)$ are disjoint subsets of X for every $n\in\mathbf{N}$. Since (X,τ) is completely regular, for every $n\in\mathbf{N}$ there exists a continuous function $f_n : X\to[0,1]$ such that $f_n(x)=0$ on A'_n and $f_n(x)=1$ on $i(B'_n)$ (see e.g. Engelking [**?**, Theorem 3.1.7]). Hence, for every $n\in\mathbf{N}$ there exists a scale $\mathcal{G}'_n=\{G'_{nr} : r\in\mathcal{S}_n\}$ such that $A'_n\subseteq G'_{nr}\subseteq X\setminus i(B'_n)$ for every $r\in\mathcal{S}_n\setminus\{1\}$. Since $\preceq$ is IC-continuous, $\mathcal{G}_n=\{d(G'_{nr}) : r\in\mathcal{S}_n\}$ is a decreasing scale in $(X, \tau, \preceq)$ for every $n \in \mathbf{N}$ (see Burgess and Fitzpatrick [**?**, Lemma 6.1]). Define, for every $n \in \mathbf{N}$, a real function $f_n : X \to [0,1]$ by $f_n(x) = \inf\{r \in \mathcal{S}_n : x \in d(G'_{nr})\}$. Since it is $f_n(x) = \inf\{r \in \mathcal{S}_n : x \in \overline{d(G'_{nr})}\}$, it is easy to show that f_n is continuous. Further, f_n is increasing, since for every $x, y \in X$ such that $x \preceq y$, $\{r \in \mathcal{S}_n : y \in d(G'_{nr})\} \subseteq \{r \in \mathcal{S}_n : x \in d(G'_{nr})\}$, and therefore $f_n(x) \leq f_n(y)$ from the definition of f_n. From condition (i), for every $x, y \in X$ with $x \prec y$, there exists $n \in \mathbf{N}$ such that $f_n(x) = 0$ and $f_n(y) = 1$ (see Burgess and Fitzpatrick [**?**, Theorem 4.1]). Define $f = \sum_{n \in \mathbf{N}} 2^{-n} f_n$. It is immediate to observe that f is a real continuous order-preserving function on $(X, \tau, \preceq)$. □

Remark 3.4. It is well known that any compact space is completely regular (see e.g. Engelking [**?**, Theorem 3.3.1]). So, the situation considered in Theorem **??** is the most general among those considered in the paper. In the particular case when (X, τ) is compact, condition (i) of Theorem **??** is equivalent to conditions (ii) and (iii) of Corollary **??**.

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G. Bosi

Dipartimento di Matematica Applicata

Università di Trieste

Piazzale Europa 1, 34127 Trieste

Italy

E-mail address: giannibo@econ.univ.trieste.it

R. Isler
Dipartimento di Matematica Applicata
Università di Trieste
Piazzale Europa 1, 34127 Trieste
Italy