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An operation on topological spaces

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Abstract. A (binary) product operation on a topological space X is considered. The only restrictions are that some element e of X is a left and a right identity with respect to this multiplication, and that certain natural continuity requirements are satisfied. The operation is called *diagonalization* (of X). Two problems are considered: 1. When a topological space X admits such an operation, that is, when X is diagonalizable? 2. What are necessary conditions for diagonalizablity of a space (at a given point)? A progress is made in the article on both questions. In particular, it is shown that certain deep results about the topological structure of compact topological groups can be extended to diagonalizable compact spaces. The notion of a Moscow space is instrumental in our study.

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1. Diagonalizable spaces

In this article we build upon some ideas and techniques from [5], showing that they are applicable in a much more general setting. The key new idea is materialized below in a new notion of a diagonalizable space, which turns out to be a very broad generalization of the notion of a semitopological semigroup with identity. It also generalizes the notion of a Mal′tsev space. Diagonalizablity is preserved by retracts and by products. Thus, a diagonalizable space need not be homogeneous. Moreover, every zero-dimensional first countable space is diagonalizable. However, despite its very general nature, diagonalizablity turns out to be so strong a property, that we are able to extend some important theorems about compact topological groups to compact diagonalizable spaces. These results involve Stone-Čech compactifications, C-embeddings, and products; in particular, they extend the classical results of I. Glicksberg [10] and E. van Douwen [8] (see also [12]). A central role in what follows also belongs to the notion of a Moscow space, which was recently shown to have delicate applications in topological algebra.

A topological space X will be called *diagonalizable at a point* $e \in X$ if there exists a mapping ϕ of the square $X \times X$ in X satisfying the following two conditions:

1) $\phi(x,e) = \phi(e,x)$, for each $x \in X$;
2) For each $a \in X$, the mappings ρ_a and λ_a of the space X into itself, defined by the formulas $\rho_a(x) = \phi(x,a)$ and $\lambda_a(x) = \phi(a,x)$ for each $x \in X$, are continuous at $x = e$.

The mapping ϕ in this case is called *a diagonalizing mapping (at e)*, or *a diagonalization* of X at e, and the mappings ρ_a and λ_a are called, respectively, *the right action* and *the left action* by a on X, corresponding to the product operation ϕ. If in the definition above the mapping ϕ can be chosen to be jointly continuous at (e,a) and (a,e) for each $a \in X$, we say that X is *continuously diagonalizable at e*. Clearly, every space X is diagonalizable at every isolated point of X. If X is (continuously) diagonalizable at every point $e \in X$, then X is called (continuously) diagonalizable.

A space X with a fixed separately (jointly) continuous mapping $\phi : X \times X \to X$ and a fixed point $e \in X$ will be called a *semitopoid (a topoid)* with identity e if ϕ is a diagonalization of X at e, that is, if $\phi(x,e) = \phi(e,x) = x$, for each $x \in X$. The next assertion is obvious.

Proposition 1.1. *If a space X is (continuously) diagonalizable at some point a of X, and X is homogeneous, then X is (continuously) diagonalizable.*

Example 1.2.

1) Every topological, and even every paratopological, group G is continuously diagonalizable: as a continuous diagonalization mapping ϕ at the neutral element e of G we can take just the product operation: $\phi(x,y) = xy$, for each (x,y) in $G \times G$. It remains to refer to homogeneity of G.

   Therefore, Sorgenfrey line is continuously diagonalizable, since it is a paratopological group.
2) Every semitopological group G, that is, a group G with a topology such that the product operation in G is separately continuous (with respect to each argument), is diagonalizable at the neutral element e by the product operation. Since every semitopological group G is a homogeneous space, it follows from Proposition 1.1 that *every semitopological group is diagonalizable.*
3) Let X be *a Mal′tsev space*, that is, a space with a continuous mapping f of the cube $X \times X \times X$ in X such that $f(x,y,y) = f(y,y,x) = x$ for all x and y in X (such f is called a continuous *antimixer* on X). Then X is continuously diagonalizable. Indeed, fix any e in X, and define a mapping ϕ of $X \times X$ in X by the formula:

$$\phi(x,y) = f(x,e,y).$$

Clearly, ϕ is continuous, and $\phi(e,x) = f(e,e,x) = x = f(x,e,e) = \phi(x,e)$. Thus, ϕ is a continuous diagonalization mapping at e, and X is continuously diagonalizable.

4) Similarly, every space with a separately continuous antimixer is diagonalizable.
5) Every retract of a topological group is a Mal′'tsev space (see [13]). Therefore, every retract of a topological group is continuously diagonalizable. In particular, every absolute retract is continuously diagonalizable. Hence, every Tychonoff cube is continuously diagonalizable. Notice that a Mal′tsev space, unlike a topological group, need not be homogeneous (consider, for example, the closed unit interval). Thus, diagonalizable spaces do not have to be homogeneous.

   Several simple statements below demonstrate that the class of diagonalizable spaces is much larger than the classes of semitopological groups or Mal′tsev spaces.

Proposition 1.3. *Every linearly ordered topological space X with the smallest element e is continuously diagonalizable at e.*

Proof. Let $<$ be a linear ordering on X, generating the topology of X, such that e is the smallest element of X. For arbitrary $(x,y) \in X \times X$, put $\phi(x,y) = max\{x,y\}$. Clearly, ϕ is a continuous diagonalizing mapping at e. □

Theorem 1.4. *Every linearly ordered compact space X is continuously diagonalizable at least at one point.*

Proof. Indeed, every compact space X, the topology of which is generated by a linear ordering $<$, has the smallest element with respect to $<$ [9]. □

Corollary 1.5. *Every homogeneous linearly ordered compact space is continuously diagonalizable.*

Proof. This follows from Theorem 1.4 and Proposition 1.1. □

Example 1.6. The "double arrow" space is continuously diagonalizable, since it is compact, homogeneous, and linearly ordered.

The conclusion in Theorem 1.4 can be considerably strengthened if we assume that the topology of X is generated by a well ordering. Indeed, we have the following

Theorem 1.7. *If X is a topological space, the topology of which is generated by a well ordering $<$, then X is continuously diagonalizable.*

Proof. Assume that e is any point of X. Put $Y = \{x \in X : x \leq e\}$ and $Z = \{x \in X : e < x\}$. Then Y and Z are open and closed subsets of X, the space Y is linearly ordered, and e is the last element of Y. From Proposition 1.3 it follows that Y is continuously diagonalizable at e (consider the reverse ordering of Y).

It remains to apply the next Lemma:

Lemma 1.8. *Assume that Y is an open and closed subspace of X, $e \in Y$, and Y is (continuously) diagonalizable at e. Then X is (continuously) diagonalizable at e.*

Proof. Put $Z = X \setminus Y$. The sets $Y \times Y$, $Y \times Z$, $Z \times Y$, and $Z \times Z$ are pairwise disjoint and open and closed in $X \times X$. Together they cover $X \times X$.

Let us fix a (continuous) diagonalizing mapping ψ for the space Y at e. Then we define a diagonalizing mapping ϕ for X at e as follows.

If $(x, y) \in Y \times Y$, we put $\phi(x, y) = \psi(x, y)$.

If $(x, y) \in Y \times Z$, we put $\phi(x, y) = y$.

If $(x, y) \in Z \times Y$, we put $\phi(x, y) = x$.

If $(x, y) \in Z \times Z$, we put $\phi(x, y) = e$.

Clearly, ϕ is a (continuous) diagonalizing mapping for X at e. □

Let us call a space X *continuously homogeneous* if there exist a point $e \in X$ and a continuous mapping h of X into the space $H_p(X)$ of homeomorphisms of X onto itself in the topology of pointwise convergence satisfying the following conditions:

(1) $h_x(e) = x$, for each $x \in X$, where $h_x = h(x)$; and
(2) $h_e = h(e)$ is the identity mapping of X onto itself.

Such a mapping h will be called *a homogeneity C_p-structure* on X at e.

Clearly, every continuously homogeneous space is homogeneous.

Proposition 1.9. *Every continuously homogeneous space X is diagonalizable.*

Proof. Let $h : X \to H_p(X)$ be a homogeneity C_p-structure on X at a point $e \in X$. Put $\phi(x, y) = h_x(y)$, for each $(x, y) \in X \times X$, where $h_x = h(x)$. It is easily verified that the mapping ϕ is separately continuous. We also have: $\phi(e, x) = h_e(x) = x$, since h_e is the identity mapping, and $\phi(x, e) = h_x(e) = x$, by the other property of homogeneity C_p-structure. Thus, X is diagonalizable at e. Since X is homogeneous, it follows that X is diagonalizable. □

Theorem 1.10. *Every retract of a (continuously) diagonalizable space is (continuously) diagonalizable.*

Proof. Assume that X is a (continuously) diagonalizable space, Y a subspace of X, and r a retraction of X onto Y. Take any point e in Y, and fix a diagonalizing mapping $\phi : X \times X \to X$ at e.

Define a mapping ϕ_r of $Y \times Y$ in Y by the formula $\phi_r = r\phi(y, z)$, for every (y, z) in $Y \times Y$. Clearly, if ϕ is (separately) continuous, then ϕ_r is also (separately) continuous.

Take any $y \in Y$. Then

$$\phi_r(y, e) = r\phi(y, e) = r(y) = y$$

and

$$\phi_r(e, y) = r\phi(e, y) = r(y) = y,$$

since $y \in Y$ and r is a retraction of X onto Y. Therefore, Y is (continuously) diagonalizable at e. □

Remark 1.11. Notice, that a space X is (continuously) diagonalizable at $e \in X$ if and only if there exists a (continuous) separately continuous mapping δ of the product space $X \times X$ onto the diagonal $\Delta_X = \{(x,x) : x \in X\}$ such that $\delta(x,e) = \delta(e,x) = (x,x)$, for each x in X. This obvious observation explains the name "diagonalizable space".

There is another curious result on diagonalizablity involving retractions. Observe that for any $e \in X$ the subspaces $\{e\} \times X$ and $X \times \{e\}$ are retracts of $X \times X$ (under the obvious projections). Now let us ask the following question: when the subspace $(X \times \{e\}) \cup (\{e\} \times X)$ is a retract of $X \times X$? If this is the case, we will call the space X *crosslike* at $e \in X$. If $(X \times \{e\}) \cup (\{e\} \times X)$ is a retract of $X \times X$ under a separately continuous retraction, then X will be said to be *weakly crosslike* (at e). For example, the closed unit interval and the real line are crosslike spaces.

Proposition 1.12. *If a space X is weakly crosslike at $e \in X$, then X is diagonalizable at e.*

Proof. Fix a separately continuous retraction r of $X \times X$ onto the subspace $(X \times \{e\}) \cup (\{e\} \times X)$. For each $x \in X$, put $f(e,x) = f(x,e) = x$. Then f is a continuous mapping of $(X \times \{e\}) \cup (\{e\} \times X)$ onto X. Clearly, the mapping $\phi = f \circ r$ is a diagonalization of X at e. $\square$

Similarly, the next assertion is proved:

Proposition 1.13. *If a space X is crosslike at $e \in X$, then X is continuously diagonalizable at e.*

A space X is called *zero-dimensional* at a point $e \in X$ if there exists a base of X at e consisting of open and closed sets (notation: $ind(e,X) = 0$).

Theorem 1.14. *If a space X is zero-dimensional at a point $e \in X$, and e is a G_δ in X, then X is crosslike at e, and, hence, continuously diagonalizable at e.*

Proof. We can fix a countable family $\{V_n : n \in \omega\}$ of open and closed neighborhoods of e in X such that $V_{n+1} \subset V_n$, for each $n \in \omega$, and $\{e\} = \cap\{V_n : n \in \omega\}$. Put $W_n = V_n \times (X \setminus \overline{V_{n+1}})$, $U_n = (X \setminus \overline{V_n}) \times V_n$, $W = \cup\{W_n : n \in \omega\}$, and $U = \cup\{U_n : n \in \omega\}$. Obviously, the sets U and W are open in $X \times X$, and $(\{e\} \times (X \setminus \{e\}) \subset W$, $(X \setminus \{e\}) \times \{e\}) \subset U$.

It is easy to check that U and W are disjoint, and that they are closed in $(X \times X) \setminus \{(e,e)\}$. Therefore, the set $K = (X \times X) \setminus (U \cup W \cup \{e\})$ is open in $X \times X$. For each $(x,y) \in W$, put $r(x,y) = (e,y)$. For each $(x,y) \in U$, put $r(x,y) = (x,e)$. For each $(x,y) \in (X \times X) \setminus (U \cup W)$, put $r(x,y) = (e,e)$.

Clearly, r is a continuous retraction of $X \times X$ onto the cross $(X \times \{e\}) \cup (\{e\} \times X)$ at e. Hence, X is crosslike at e, and, by Proposition 1.13, X is continuously diagonalizable at e. $\square$

The same idea leads to one more elementary result in the same direction.

Recall that, for a non-empty space X, the equality $\mathrm{Ind}(X) = 0$ signifies that for any two disjoint closed subsets P and F in X there exists an open and closed subset W such that $P \subset W$ and $F \cap W = \varnothing$ (see [9]).

Proposition 1.15. *Let X be a space and e a point in X such that the subspace $Z = (X \times X) \setminus \{(e,e)\}$ of $X \times X$ satisfies the condition* $\mathrm{Ind}(Z) = 0$. *Then the space X is crosslike at e.*

Proof. Put $P = \{e\} \times X$ and $F = X \times \{e\}$. Then P and F are disjoint closed subsets of Z. Since $\mathrm{Ind}(Z) = 0$, there exists an open and closed subset W of Z such that $P \subset W$ and $F \cap W = \varnothing$.

Now we define a mapping r of $X \times X$ in $(\{e\} \times X) \cup (X \times \{e\})$ as follows. If $(x,y) \in W$, let $r(x,y) = (e,y)$. If $(x,y) \in Z \setminus W$, let $r(x,y) = (x,e)$. Finally, we put $r(e,e) = (e,e)$. Clearly, the restriction of r to W is continuous, since it is the restriction of the projection mapping of $X \times X$. Similarly, the restriction of r to $Z \setminus W$ is continuous. Therefore, r is continuous at all points of Z. Since Z is open in $X \times X$, to see that the mapping r is continuous, we only have to check its continuity at the point (e,e). However, the continuity of r at (e,e) is also obvious.

Finally, we observe that r is the identity mapping on $(\{e\} \times X) \cup (X \times \{e\})$. Thus, $(\{e\} \times X) \cup (X \times \{e\})$ is a retract of $X \times X$. □

Theorem 1.14 shows how much more general is the diagonalizablity assumption, than the assumption that the space has a (separately) continuous antimixer. Indeed, according to [15], every compact space with a separately continuous antimixer is a Dugundji compactum, and it is well known that every first countable Dugundji compactum is metrizable. We also see from Theorem 1.14 that diagonalizablity of a compact space does not impose any homogeneity restrictions on the space. In that the diagonalizablity differs drastically from the assumptions that X is a paratopological group or a semitopological group.

Example 1.16. Let X be a space, e a point of X, and $\mathcal{P}_e(X)$ the space of all closed subsets of X containing e, in the Vietoris topology. Put $Z = \mathcal{P}_e(X)$, and define a mapping $\phi : Z \times Z \to Z$ by the rule: $\phi(A,B) = A \cup B$, for any $(A,B) \in Z \times Z$.

It is easily verified that the mapping ϕ is continuous. It is also clear that $\phi(E,A) = A = \phi(A,E)$, for each $A \in Z$, where $E = \{e\}$. Therefore, ϕ is a continuous diagonalization of the space $Z = \mathcal{P}_e(X)$ at the point $E = \{e\} \in \mathcal{P}_e(X)$.

Since there is no reason to believe that the space $\mathcal{P}_e$ should be diagonalizable at every point, the above conclusion suggests that the space $\mathcal{P}_e$ normally can be expected to be not homogeneous and provides some means for proving that.

Though the proof of the next statement is obvious, the result itself is quite important.

Proposition 1.17. *The product of any family of (continuously) diagonalizable spaces is a continuously diagonalizable space.*

Similar assertion holds for crosslike spaces. In conclusion of this section, we mention a curious corollary of Theorem 1.10.

Theorem 1.18. *Suppose X is a space such that $X \times Y$ is homeomorphic to a (continuously) diagonalizable space, for some space Y. Then X is also (continuously) diagonalizable.*

Proof. Indeed, X is a retract of $X \times Y$. It remains to apply Theorem 1.10. □

2. Some necessary conditions for diagonalizablity

A space X is called *Moscow* at a point $e \in X$ if, for every open set U the closure of which contains e, there exists a G_δ-subset P of X such that $e \in P \subset \overline{U}$ (see [1, 5]). If X is Moscow at every point, we call X *a Moscow space.*

A space X is called *weakly Klebanov* at a point $e \in X$ if for every family γ of G_δ-subsets of X such that the closure of $\cup\gamma$ contains e there exists a G_δ-subset P of X such that $e \in P \subset \overline{\cup\gamma}$. We say that X is *weakly Klebanov* if X is weakly Klebanov at every point of X. Clearly, every weakly Klebanov space is Moscow, and every space of countable pseudocharacter is weakly Klebanov.

The importance of the notion of Moscow space comes from the role it plays in connection with C-embeddings; see about that [6] and [4]. Besides, a non-trivial result on Moscow spaces is the theorem that every Dugundji compactum is Moscow (see [15]); it follows that every compact (actually, every pseudocompact) topological group is a Moscow space.

The simplest example of a non-Moscow space is the one-point (Alexandroff) compactification of an uncountable discrete space. Note that the tightness of this space is countable. On the other hand, it was shown in [5] that every topological (and even semitopological) group of countable tightness is a Moscow space. This again underlines the significance of the concept of a Moscow space for topological algebra.

One of our main results is the next theorem:

Theorem 2.1. *Suppose X is a space of countable tightness diagonalizable (at $e \in X$). Then X is weakly Klebanov (at e).*

Proof. Let A be a subset of X which is the union of a family of G_δ-subsets of X, and e any point in $\overline{A}$. We have to show that there exists a G_δ-subset P such that $e \in P \subset \overline{A}$.

Since the tightness of X is countable, there exists a countable subset B of A such that $e \in \overline{B}$. For each $b \in B$ we fix a G_δ-set P_b such that $b \in P_b \subset A$.

Let us also fix a diagonalizing mapping ϕ of $X \times X$ into X at e. For each $b \in B$ consider the mapping ϕ_b of X into X given by the formula: $\phi_b(x) = \phi(x, b)$, for every $x \in X$.

Since ϕ is a diagonalizing mapping at e, ϕ_b is continuous at e. Therefore, $\phi^{-1}(P_b)$ contains a G_δ-set M_b such that $e \in M_b$, since $\phi_b(e) = \phi(e, b) = b$. Then the set $F = \cap\{M_b : b \in B\}$ is also a G_δ-set in X, and $e \in F$.

Take any point $a \in F$. We have $\phi_b(a) \in P_b \subset A$, for each $b \in B$, since $F \subset \phi_b^{-1}(P_b)$. Thus, $\phi(a, b) = \phi_b(a) \in A$, for each $b \in B$. However, $e \in \overline{B}$, and the function $\phi(a, x)$ is continuous with respect to the second argument at $x = e$. It follows that $\phi(a, e) \in \overline{A}$. Since ϕ is a diagonalizing mapping at e, we have $\phi(a, e) = a$. Therefore, $a \in \overline{A}$, that is, $F \subset \overline{A}$. The proof is complete. □

The assumption that the tightness of X is countable can be considerably weakened but can not be completely removed. The *κ-tightness* of a space X at a point $e \in X$ is said to be countable [5] if for each open subset U such that e is in the closure of U there exists a countable subset B of U such that $e \in \overline{B}$ (notation: $t_\kappa(e, X) \leq \omega$). If the κ-tightness of X is countable at every point $e \in X$, we say that the κ-tightness of X is countable, and write $t_\kappa(X) \leq \omega$.

Introducing a mild, obvious, change in the proof of Theorem 2.1, we obtain a proof of the next statement:

Theorem 2.2. *Every diagonalizable (at a point e) space X of countable κ-tightness is Moscow (at e).*

It is worth noting that for every dyadic compactum X the κ-tightness of X is countable, while if the tightness of a dyadic compactum X is countable, then X is metrizable (see [7]). In particular, the κ-tightness of every Tychonoff cube is countable. This shows that the countability of κ-tightness is much, much weaker restriction than the countability of tightness. However, the next example shows that we can not completely drop it.

Example 2.3. The space of ordinals $\omega_1 + 1$ is continuously diagonalizable, by Theorem 1.7. Nevertheless, this space is easily seen to be not Moscow [4].

Of course, this happens because the κ-tightness of $\omega_1 + 1$ is not countable (precisely at the point ω_1). Observe that the space $\omega_1 + 1$ does not admit a separately continuous antimixer, since it is compact but not dyadic. Observe also that, by Theorem 1.17, the space $(\omega_1 + 1)^\tau$ is continuously diagonalizable, for every cardinal number τ.

Example 2.4. Let τ be an uncountable cardinal number and A_τ the one-point (Alexandroff) compactification of a discrete space of cardinality τ. Then A_τ is not diagonalizable (at the unique non-isolated point of A_τ). Obviously, A_τ is a Fréchet-Urysohn space; hence, the tightness of A_τ is countable. Assume now that A_τ is diagonalizable. Then, by Theorem 2.1, A_τ is Moscow, a contradiction. It follows that A_τ is not diagonalizable.

Note that the usual convergent sequence is continuously diagonalizable by Theorem 1.7 or by Theorem 1.14. Note also, that the space A_τ is compact, zero-dimensional, Hausdorff, and satisfies the first axiom of countability at all points except one, the non-isolated point. Thus, Theorem 1.14 can not be much improved.

It is well known that a compact topological group of countable tightness is metrizable (see [7]). For diagonalizable compact spaces we have a parallel statement with a weaker conclusion. Recall that a compact space X is said to be *ω-monolithic* if, for every countable subset A of X, the closure of A in X is a space with a countable base.

Theorem 2.5. *Every diagonalizable ω-monolithic compact Hausdorff space X of countable tightness is first countable.*

Proof. Take any point $x \in X$. Since X is compact Hausdorff, it is enough to show that x is a G_δ-point in X. The space X is Fréchet-Urysohn, and X

is first countable at a dense set Y of points (since every ω-monolithic compact Hausdorff space of countable tightness has these properties [3]). Therefore, there exists a sequence $\{y_n : n \in \omega\}$ of points of Y converging to x. On the other hand, X is weakly Klebanov, by Theorem 2.1.

It remains to apply the following obvious lemma:

Lemma 2.6. *Suppose X is a weakly Klebanov space and $\{y_n : n \in \omega\}$ is a sequence of G_δ-points in X converging to x. Then x is also a G_δ-point in X.*

Note that the space A_τ in Example 2.4 is an ω-monolithic compact Hausdorff space of countable tightness. Theorem 2.5 clarifies, why it is not diagonalizable: it is because it is not first countable.

Corollary 2.7. *Every diagonalizable Corson compactum is first countable.*

Proof. Indeed, every Corson compact space is monolithic and Fréchet-Urysohn (see [3]). It remains to apply Theorem 2.5. □

In the next result we assume the Continuum Hypothesis (CH). It is not clear whether the statement remains true without this assumption.

Theorem 2.8. (CH) *Every diagonalizable sequential compact Hausdorff space X is first countable.*

Proof. Let Y be the set of all points of X at which X satisfies the first axiom of countability. Then Y is G_δ-dense in X, by a theorem in [2] (here we use (CH)). On the other hand, from Theorem 2.1 it follows that X is weakly Klebanov.

Assume now that $X \neq Y$. Then Y is not closed in X, since Y is dense in X. Therefore, since X is sequential, there exists a point $x \in X \setminus Y$ and a sequence $\{y_n : n \in \omega\}$ of points of Y converging to x. It follows from Lemma 2.6 that x is a G_δ-point in X. Since X is compact Hausdorff, we conclude that X is first countable at x. This is a contradiction with $x \notin Y$ and definition of Y. □

3. Diagonalizablity and C-embeddings

In this section, we combine our results on diagonalizablity and a result of M.G. Tkachenko to obtain several new results on C-embeddings and Stone-Čech compactifications. For more results on C-embeddings in the context of topological groups see [11]. Here is Uspenskij's 's modification [15] of Tkachenko's result from [14]:

Theorem 3.1 (Tkachenko)**.** *If X is a Moscow Tychonoff space, then every G_δ-dense subspace Y of X is C-embedded in X.*

Theorem 3.2. *Let X be a compact diagonalizable space of countable κ-tightness. Then X is the Stone-Čech compactification of any G_δ-dense subspace Y of X.*

Proof. Indeed, X is a Moscow space, by Theorem 2.1. Therefore, by Theorem 3.1, Y is C-embedded in X. It follows that Y is pseudocompact and $X = \beta Y$. □

The next statement is a typical application of Theorem 2.2.

Theorem 3.3. *If a Tychonoff space X is diagonalizable at $e \in X$, and the κ-tightness of X at e is countable, then either e is a G_δ-point in X, or the subspace $Y = X \setminus \{e\}$ is C-embedded in X.*

Proof. Assume that e is not a G_δ-point in X. Then Y is G_δ-dense in X. By Theorem 2.2, X is Moscow at e. Since Y is G_δ-dense in X, it follows, by an obvious modification of Theorem 3.1 (see [4]), that Y is C-embedded in X. □

Corollary 3.4. *Assume that a Tychonoff space X is diagonalizable at $e \in X$, the κ-tightness of X at e is countable, and the subspace $Y = X \setminus \{e\}$ is Hewitt-Nachbin complete. Then e is a G_δ-point in X.*

Proof. Assume that e is not isolated in X. Then, since $Y = X \setminus \{e\}$ is Hewitt-Nachbin complete, Y is not C-embedded in X. Now it follows from Theorem 3.3 that e is a G_δ-point in X. □

Corollary 3.5. *Assume that X is a pseudocompact Tychonoff space diagonalizable at a point $e \in X$ such that the κ-tightness of X at e is countable. Then either X is first countable at e, or the subspace $X \setminus \{e\}$ is pseudocompact.*

Proof. Since ever pseudocompact Tychonoff space is first countable at every G_δ-point, it follows from Theorem 3.3 that the subspace $Y = X \setminus \{e\}$ is C-embedded in X. Therefore, since X is pseudocompact, the space Y must be pseudocompact as well. □

Here is a result in the same direction, in which the assumption on X does not contain explicitly a restriction on the tightness of X.

Theorem 3.6. *Assume that X is a pseudocompact Tychonoff space diagonalizable at a point $e \in X$. Assume also that the next condition is satisfied: (α) For each open subset U of X such that $e \in \overline{U} \setminus U$, the subspace $\overline{U} \setminus \{e\}$ is not pseudocompact. Then X is first countable at e.*

Proof. Clearly, we can assume that the point e is not isolated in X. Then condition α implies that the subspace $X \setminus \{e\}$ is not pseudocompact. It follows from Corollary 3.5 that, to complete the proof, it remains to show that the κ-tightness of X at e is countable.

Take any open set U such that $e \in \overline{U} \setminus U$. By ($\alpha$), the subspace $Z = \overline{U} \setminus \{e\}$ is not pseudocompact. Therefore, there exists a discrete family $\xi = \{V_n : n \in \omega\}$ of non-empty open subsets in Z. However, the subspace $\overline{U}$ is pseudocompact, since X is pseudocompact and $\overline{U}$ is a canonical closed subset of X (see [9]). It follows that the sequence $(V_n : n \in \omega)$ converges to e. Clearly, $V_n \cap U$ is non-empty, for each $n \in \omega$. Choosing a point $x_n \in V_n \cap U$ for each $n \in \omega$, we obtain a sequence of points of the set U converging to e. Hence, the κ-tightness of X at e is countable. □

The condition (α) in Theorem 3.6 may look a little artificial. However, there are several natural corollaries of Theorem 3.6. Recall that a subset A of a space X is called *locally closed* if $A = B \cap C$, where B is a closed subset of X and C is an open subset of X. The next three statements follow directly from Theorem 3.6.

Corollary 3.7. *Assume that X is a pseudocompact Tychonoff space diagonalizable at a point $e \in X$ such that every locally closed pseudocompact subspace of X is closed in X. Then X is first countable at e.*

Corollary 3.8. *Assume that X is a pseudocompact Tychonoff space diagonalizable at a point $e \in X$ and such that the subspace $X \setminus \{e\}$ is Dieudonné complete. Then X is first countable at e.*

Corollary 3.9. *Assume that X is a pseudocompact Tychonoff space diagonalizable at a point $e \in X$ and such that the subspace $X \setminus \{e\}$ is metacompact. Then X is first countable at e.*

The list of corollaries to Theorem 3.6 can be easily expanded.

4. Diagonalizable separable spaces

The results obtained in the preceding sections are, in particular, applicable to separable spaces. We present several such applications below.

Theorem 4.1. *If a separable space X is diagonalizable at $e \in X$, then X is Moscow at e.*

Proof. This statement is a direct corollary of Theorem 2.1 and the obvious fact that the κ-tightness of every separable space is countable. □

A space X is called *a G_δ-extension* of a space Y if Y is a G_δ-dense subspace of X. A space X may have many different G_δ-extensions. For example, every compactification of a pseudocompact Tychonoff space X is a G_δ-extension of X, and usually there are many such compactifications.

However, it turns out that few of these extensions should be expected to be diagonalizable. This is demonstrated by the next "uniqueness" result.

Theorem 4.2. *If a Tychonoff space X is a G_δ-extension of a separable space Y, and X is diagonalizable and Hewitt-Nachbin complete, then X is the Hewitt-Nachbin completion νY of Y.*

Proof. The space X is also separable. Therefore, by Theorem 3.1, X is Moscow. Since Y is G_δ-dense in X, it follows Theorem 3.1 that Y is C-embedded in X. Therefore, since X is Hewitt-Nachbin complete, X is the Hewitt-Nachbin completion of X. □

With the help of Theorem 4.2, we could easily construct many further examples of non-diagonalizable separable spaces.

The notion of diagonalizablity can be also applied to show that G_δ-extensions of spaces, in general, should not be expected to be homogeneous. This is based on the following key lemma from [6]:

Lemma 4.3. *If a Tychonoff space X is a homogeneous G_δ-extension of a Moscow space Y, then X is also a Moscow space and Y is C-embedded in X.*

Theorem 4.4. *If a Tychonoff space X is a homogeneous G_δ-extension of a separable diagonalizable space Y, and X is Hewitt-Nachbin complete, then X is the Hewitt-Nachbin completion νY of Y.*

Proof. By Theorem 4.1, the space Y is Moscow. Since X is homogeneous and Y is G_δ-dense in X, it follows from Lemma 4.3 that X is also Moscow and Y is C-embedded in X. Since X is Hewitt-Nachbin complete, we can conclude that X is the Hewitt-Nachbin completion νY of Y. □

Corollary 4.5. *If X is a compact Hausdorff homogeneous extension of a separable pseudocompact diagonalizable space Y, then X is the Stone-Čech compactification of Y.*

Proof. Indeed, Y is G_δ-dense in X, since Y is pseudocompact, and X is Hewitt-Nachbin complete, since X is compact Hausdorff. It remains to apply Theorem 4.2. □

The next statement is proved by a similar argument.

Corollary 4.6. *If X is a Hausdorff compactification of a separable pseudocompact space Y, and X is diagonalizable, then X is the Stone-Čech compactification of Y.*

We know that every zero-dimensional Hausdorff space of countable pseudocharacter is diagonalizable. We also established several conditions under which diagonalizable spaces are Moscow or even have countable pseudocharacter. Since the class of Moscow spaces is an extension of the class of spaces of countable pseudocharacter, it is natural to ask if every zero-dimensional Moscow space is diagonalizable. Theorem 4.1 is instrumental in finding a compact counterexample.

Example 4.7. Let $\beta\omega$ be the Stone-Čech compactification of the discrete space ω, and $e \in \beta\omega \setminus \omega$. Let us show that $\beta\omega$ is not diagonalizable at e.

Assume the contrary. Then the space $Z = \beta\omega \times \beta\omega$ is, obviously, diagonalizable at the point (e, e). Since the space $\beta\omega$ is separable, the space Z is also separable. Now it follows from Theorem 4.1 that Z is Moscow at the point (e, e). However, this is not the case, as it was shown in [5]. Thus, not every compact Moscow space of countable κ-tightness is diagonalizable.

The next two results are related in an obvious way to the classical theorems in [10] and [12] (see also [6] and [4]).

Theorem 4.8. *Assume that Y_α is a separable Tychonoff space with a diagonalizable Hewitt-Nachbin complete G_δ-extension X_α, for each $\alpha \in A$, where $|A| \leq 2^\omega$. Then the next formula holds for the Hewitt-Nachbin extensions νY_α:*

$$\Pi\{\nu Y_\alpha : \alpha \in A\} = \nu\Pi\{Y_\alpha : \alpha \in A\}.$$

Proof. By Theorem 4.2, $\nu Y_\alpha = X_\alpha$, for each $\alpha \in A$. Therefore, $\Pi\{\nu Y_\alpha : \alpha \in A\}$ is a G_δ-extension of the space $\Pi\{Y_\alpha : \alpha \in A\}$. Obviously, $\Pi\{\nu Y_\alpha : \alpha \in A\}$ is Hewitt-Nachbin complete. Applying again Theorem 4.2 and Proposition 1.17, we conclude that $\Pi\{\nu Y_\alpha : \alpha \in A\} = \nu\Pi\{Y_\alpha : \alpha \in A\}$. □

Corollary 4.9. *Assume that Y_α is a separable pseudocompact space with a diagonalizable Hausdorff compactification bY_α, for each $\alpha \in A$, where $|A| \leq 2^\omega$. Then the next formula holds for the Stone-Čech compactifications βY_α:*

$$\Pi\{\beta Y_\alpha : \alpha \in A\} = \beta\Pi\{Y_\alpha : \alpha \in A\}.$$

Proof. To deduce this statement from Theorem 4.8, it is enough to observe that every pseudocompact space is G_δ-dense in each Hausdorff compactification of it, and that the Hewitt-Nachbin completion of any pseudocompact space coincides with the Stone-Čech compactification of Y. □

We conclude this section with the next obvious corollary of Theorem 4.1.

Corollary 4.10. *If a separable Tychonoff space X is diagonalizable at a point $e \in X$, and $X \setminus \{e\}$ is Hewitt-Nachbin complete, then e is a G_δ-point in X.*

5. Continuously diagonalizable spaces

Following M.G. Tkachenko [14], we say that the *o-tightness* of a space X at a point $e \in X$ is countable (and write $ot(e, X) \leq \omega$) if, for each family γ of open subsets of X such that $e \in \overline{\cup\gamma}$, there exists a countable subfamily ξ of γ such that $e \in \overline{\cup\xi}$. If this is true for every point e in X, we say that the *o-tightness* of X is countable.

Theorem 5.1. *If a space X is continuously diagonalizable at a point $e \in X$, and the o-tightness of X at e is countable, then X is Moscow at e.*

Proof. Let U be any open subset of X such that e is in the closure of U. Obviously, we may assume that e is not in U.

Let η be the family of all open subsets W of X such that, for some open neighborhood O_W of e (which we now fix), $xy \in U$ for each $x \in O_W$ and each $y \in W$ (that is, $\phi(O_W \times W) \subset U$). Then η is a base of the space U, since the operation ϕ is jointly continuous at (e, x), for each $x \in X$.

Therefore, $e \in \overline{\cup\eta}$. Since the o-tightness of X is countable, it follows that there exists a countable subfamily ξ of η such that e is in the closure of $\cup\xi$.

Put $G = \cup\xi$ and $P = \cap\{O_W : W \in \xi\}$. Then P is a G_δ-set in X, since ξ is countable, and $e \in P$, $e \in \overline{G}$.

Take any $a \in P$. We want to show that $a \in \overline{U}$. We may assume that a is not e, since $e \in \overline{U}$. Then, for each $W \in \xi$, $a \in O_W$ which implies that $aW \subset U$. Therefore, $aG \subset U$. Since ax depends continuously on the second argument x at $x = e$, and $e \in \overline{G}$, it follows that $ae \in \overline{aG} \subset \overline{U}$. Finally, since $ae = a$, we obtain: $a \in \overline{U}$, that is, $e \in P \subset \overline{U}$, and X is a Moscow space. □

Corollary 5.2. *If a space X is continuously diagonalizable at a point $e \in X$, and the Souslin number of X is countable, then X is a Moscow space.*

Proof. It is enough to observe that if the Souslin number of X is countable, then the o-tightness of X is also countable [14]. □

Corollary 5.3. *If X is a continuously diagonalizable Tychonoff space with the countable Souslin number, then every G_δ-dense subspace Y of X is C-embedded in X.*

Proof. The space X is Moscow, by Theorem 5.1. It follows from Theorem 3.1 that every G_δ-dense subspace Y of X is C-embedded in X. The next statement is a typical application of Theorem 2.2. □

Corollary 5.4. *If a Tychonoff space X is continuously diagonalizable at $e \in X$, and the o-tightness of X at e is countable, then either e is a G_δ-point in X, or the subspace $Y = X \setminus \{e\}$ is C-embedded in X.*

Proof. Assume that e is not a G_δ-point in X. Then Y is G_δ-dense in X. By Theorem 5.1, X is Moscow at e. Since Y is G_δ-dense in X, it follows, by Theorem 3.1, that Y is C-embedded in X. □

Corollary 5.5. *Suppose a Tychonoff space X is continuously diagonalizable at $e \in X$, the o-tightness of X at e is countable, and the subspace $Y = X \setminus \{e\}$ is Hewitt-Nachbin complete. Then e is a G_δ-point in X.*

Proof. Assume that e is not isolated in X. Then, since $Y = X \setminus \{e\}$ is Hewitt-Nachbin complete, Y is not C-embedded in X. Now it follows from 5.4 that e is a G_δ-point in X. □

The next result should be compared to 3.5

Corollary 5.6. *Assume that X is a pseudocompact Tychonoff space continuously diagonalizable at a point $e \in X$ such that the o-tightness of X at e is countable. Then either X is first countable at e, or the subspace $X \setminus \{e\}$ is pseudocompact.*

Proof. Since every pseudocompact Tychonoff space is first countable at every G_δ-point, from Corollary 5.4 it follows that the subspace $Y = X \setminus \{e\}$ is C-embedded in X. Therefore, since X is pseudocompact, the space Y must be pseudocompact as well. □

Many results, proved in the previous section for separable diagonalizable spaces, have their counterparts for continuously diagonalizable spaces with the countable Souslin number. Their proofs do not differ much, so we just formulate a few such results below, omitting the proofs.

Theorem 5.7. *Assume that a Tychonoff space X is a G_δ-extension of a space Y such that the Souslin number of Y is countable, and assume also that X is continuously diagonalizable and Hewitt-Nachbin complete. Then X is the Hewitt-Nachbin completion νY of Y.*

Theorem 5.8. *If a Tychonoff space X is a homogeneous G_δ-extension of a continuously diagonalizable space Y with the countable Souslin number, and X is Hewitt-Nachbin complete, then X is the Hewitt-Nachbin completion νY of Y.*

Corollary 5.9. *If X is a compact Hausdorff homogeneous extension of a pseudocompact continuously diagonalizable space Y with the countable Souslin number, then X is the Stone-Čech compactification of Y.*

Corollary 5.10. *Assume that Y_α is a separable pseudocompact space with a continuously diagonalizable Hausdorff compactification bY_α, for each $\alpha \in A$. Then the next formula holds for the Stone-Čech compactifications βY_α:*

$$\Pi\{\beta Y_\alpha : \alpha \in A\} = \beta\Pi\{Y_\alpha : \alpha \in A\}.$$

In connection with Corollaries 5.9 and 5.10, see [8] and [10].

References

[1] Arhangel′skiĭ A.V., *Functional tightness, Q-spaces, and τ-embeddings*, Comment. Math. Univ. Carol. 24:1 (1983), 105–120.
[2] Arhangel′skiĭ A.V., *On bicompacta hereditarily satisfying Souslin's condition. Tightness and free sequences.* Soviet Math. Dokl. 12 (1971), 1253–1257.
[3] Arhangel′skiĭ A.V., *Topological Function Spaces.* Kluwer Academic Publishers, 1992.
[4] Arhangel′skiĭ A.V., *Topological groups and C-embeddings.* Submitted, 1999.
[5] Arhangel′skiĭ A.V., *On a Theorem of W.W. Comfort and K.A. Ross*, Comment. Math. Univ. Carolinae 40:1 (1999), 133–151.
[6] Arhangel′skiĭ A.V., *Moscow spaces, Pestov–Tkachenko Problem, and C-embeddings.* To appear in CMUC.
[7] Arhangel′skiĭ A.V. and V.I. Ponomarev, *Fundamentals of General Topology in Problems and Exercises.* (D. Reidel Publ. Co., Dordrecht-Boston, Mass., 1984).
[8] van Douwen E., *Homogeneity of βG if G is a topological group.* Colloquium Mathematicum 41 (1979), 193–199.
[9] Engelking R., *General Topology.* PWN, Warszawa, 1977.
[10] Glicksberg I., *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. 90 (1959), 369–382.
[11] Hernandez S., Sanchis M., and M.G. Tkachenko, *Bounded sets in spaces and topological groups.* Topology and Appl. 101:1 (2000), 21–44.
[12] Hušek M., *The Hewitt realcompactification of a product.* Comment. Math. Univ. Carol. 11 (1970), 393–395.
[13] Reznichenko E.A., V.V. Uspenskij, *Pseudocompact Mal′tsev spaces.* Topology and Appl. 86 (1998), 83–104.
[14] Tkachenko M.G., *The notion of o-tightness and C-embedded subspaces of products*, Topology and Appl. 15 (1983), 93–98.
[15] Uspenskij V.V., *Topological groups and Dugundji spaces*, Matem. Sb. 180:8 (1989), 1092–1118.

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