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On fixed-point free selections and multivalued maps on $\mathbb{R}$

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Abstract

We study fixed-point free multivalued selfmaps on $\mathbb{R}$ as well as continuous fixed-point free selection of multivalued selfmaps on $\mathbb{R}$. We justify necessary conditions in some earlier results and motivate some new interesting questions.

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1. Introduction

One of the classical theorems of de Bruijn-Erdös [2] states that if f is a fixed-point free multivalued selfmap on a set X and the sizes of $f(x)$'s are bounded by a fixed natural number N, then X can be covered by finitely many sets each of which misses its image. A more precise statement is as follows: *Let $\mathcal{P}(X)$ be the set of all non-empty subsets of X and let $f : X \to \mathcal{P}(X)$ be a map with the property that $x \notin f(x)$. If there exists a natural number k such that $|f(x)| \leq k$ for all $x \in X$, then there exists a finite cover $\mathcal{F}$ of X such that F misses $\bigcup\{f(x) : x \in F\}$ for each $F \in \mathcal{F}$.*

The sets Fs in the above statement are called colors of f and the family $\mathcal{F}$ a coloring of f. In the context of a map (not necessarily continuous) on a topological space, a color of the map is traditionally a closed subset of the space. To be more precise, given a multivalued selfmap f on a topological space X, a closed $F \subset X$ is a *color* of f if $f(x)$ misses F for each $x \in F$. A finite cover of X by colors of f is called a *coloring* of f. If a coloring of f exists, f is called

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colorable. The condition of closed-ness of a color in the topological context is justified by the fact that for a space X from a sufficiently wide class, colorability of a continuous selfmap f on X by closed colors is equivalent to the existence of a continuous fixed-point free extension of f to some compactifcation of X. The notion of colorings for continuous single-valued selfmaps was introduced in [1]. A nice review of results about this topic for single-valued maps is given in [6].

It is proved in [3] that a fixed-point free continuous selfmap on $\mathbb{R}$ is colorable if the size of $f(x)$ is at most N, where N is a fixed positive number determined by f. This result is a natural topologization of the mentioned de Bruijn-Erdös theorem. In the first part of this paper we will consider a few routes to extend this result to certain fixed-point free multivalued selfmaps on $\mathbb{R}$ with compact images. For this mission, it is natural to consider the following restrictions on f:

(1) There exists $M > 0$ such that $\sup\{d(x,y) : y \in f(x)\} < M$ for all x; and/or
(2) There exists $m > 0$ such that $d(x, f(x)) > m$ for all x.

We will show that (1) alone or (2) alone leads to an example. At the same time if a multivalued selfmap on $\mathbb{R}$ (not necessarily continuous) satisfies both (1) and (2), then it is colorable (Theorem 2.1).

It is clear that if f has a fixed point then it is not colorable. If, however, f is multivalued, it may have a fixed-point free continuous selection. It is shown in [4] that a continuous fixed-point free map on $\mathbb{R}^n$ is always colorable. This observation inspired the second part of the paper, in which we discuss possible conditions that guarantee the existence of continuous fixed-point free selections of multivalued selfmaps on $\mathbb{R}$.

By $\exp\mathbb{R}$ we denote the space of all non-empty closed subsets of $\mathbb{R}$ endowed with the Vietoris topology. By $\exp_K(\mathbb{R})$ we denote the space of all non-empty compact subsets of $\mathbb{R}$ with the topology inherited from $\exp\mathbb{R}$. A standard neighborhood in $\exp\mathbb{R}$ is in form

$$\langle U_1, ..., U_m\rangle = \{A \in \exp\mathbb{R} : A \subset U_1 \cup ... \cup U_m \text{ and } U_i \cap A \neq \varnothing \text{ for all } i = 1, ..., m\},$$

where $U_1, ..., U_m$ are open sets of $\mathbb{R}$. Throughout the paper when we say that f is a multivalued selfmap on $\mathbb{R}$, we mean that f is a map from $\mathbb{R}$ to $\exp\mathbb{R}$. We reserve the letter "d" to measure distances in $\mathbb{R}$.

In notations and terminology, we will follow [5].

2. Continuous Fixed-Point Free Multivalued Selfmaps on $\mathbb{R}$

In this section, we will consider mulitivalued fixed-point free selfmaps on $\mathbb{R}$ with compact images. Our goal is to find a large subclass of such maps whose members are colorable. It is natural to assume that a more "volatile" graph is less likely to be colorable while a fixed-point fee map with a "more orderly conduct" should naturally be colorable. To continue our discussion, it is appropriate at this point to define how exactly we measure the volatility of a map f. For this purpose, we introduce the following function-determined constants:

$$SQ(f) = \inf\{d(x,y) : x \in \mathbb{R}, y \in f(x)\}, ST(f) = \sup\{d(x,y) : x \in \mathbb{R}, y \in f(x)\}.$$

One can view $SQ(f)$ as the squeeze measure of f, which is, simply the greatest lower bound of the distances between points and their images.

Similarly, the constant $ST(f)$ can be viewed as a stretch measure. The authors think of a fixed-point free map f as being volatile if its squeeze measure $SQ(f)$ is 0 or its stretch measure $ST(f)$ is ∞. Thus, we may think of f is not volatile if both $SQ(f)$ and $ST(f)$ are positive real numbers. In this section we will show that any non-volatile fixed-point free (not necessarily continuous) multivalued map on $\mathbb{R}$ is colorable. We will also show that for either of the two measures of volatility, there exists an example of a non-colorable continuous fixed-point free multivalued selfmap on $\mathbb{R}$ that obeys the given measure and violates the other. We start with our affirmative result.

Theorem 2.1. *Let $f : \mathbb{R} \to \exp(\mathbb{R})$ be a map and m, M positive numbers. Suppose that $m \le d(x,y) \le M$ for every $x \in \mathbb{R}$ and $y \in f(x)$. Then f is colorable.*

Proof. Put $\mathcal{I} = \{[\frac{m}{2} \cdot n, \frac{m}{2} \cdot (n+1)] : n \textit{ is an integer}\}$. Clearly $\mathcal{I}$ is a cover of $\mathbb{R}$. Let K be an integer such that $K \cdot \frac{m}{2} > M$. We define our colors $\{A_i : i = 0, ..., K\}$ as follows.

A_i is the union of all elements of $\mathcal{I}$ that can be obtained by shifting $[\frac{m}{2} \cdot i, \frac{m}{2} \cdot i + \frac{m}{2}]$ a multiple of $K+1$ units to the right or to the left. That is, A_i is the union of the largest subfamily of $\mathcal{I}$ that contains $[\frac{m}{2} \cdot i, \frac{m}{2} \cdot i + \frac{m}{2}]$ and has neighbors at distance exactly $\frac{m}{2} \cdot K$ from each other.

It is clear that each A_i is closed and $\{A_i : i = 0, ..., K\}$ is a cover of $\mathbb{R}$. Let us show that A_i is a color. Note that A_i is the union of a disjoint collection of segments of length $m/2$. Since $d(x, f(x)) \ge m$ for each x, we conclude that the image of each segment I of this collection misses I. Also, since the distance between any two distinct segments I and J of this collection is $\frac{m}{2} \cdot K > M$, the image of I misses J. Therefore, the image of A_i misses A_i. Hence, A_i is a color. $\square$

Recall that given a colorable multivalued map f on a topological space, the chromatic number of f is defined as the smallest number of colors necessary to color f. Observe that the argument of Theorem 2.1 implies that the chromatic numbers does not exceed $\lceil \frac{2M}{m} \rceil + 1$.

We next show that either part (alone) of the inequality in the hypothesis of Theorem 2.1 is not sufficient for the desired conclusion. We first describe an example that shows that positiveness of SQ alone does not guarantee colorability.

Example 2.2. There exists a continuous fixed-point free non-colorable map from $\mathbb{R}$ to $\exp_K(\mathbb{R})$ such that $SQ(f) > 0$, $ST(f) = \infty$, and the image of each point is a segment.

Construction. For each $x \leq -1$, put $f(x) = [x + 1/2, 0]$. For each $x \geq -1$, put $f(x) = [x + 1/2, x + 1]$. This map is clearly continuous, fixed-point free, $SQ(f) = 1/2$, and $ST(f) = \infty$. Let us show that f is not colorable. Since $n \in f(m)$ for every pair of negative integers n, m with $n > m$, we conclude that n and m have different colors. Therefore, $-1, -2, -3, ...$ must have mutually distinct colors. □

Next, we will show that finiteness of ST alone is not sufficient for colorability. Lemma 2.3 and Example 2.4 can serve as a manual for constructing examples with the desired properties. We will then demonstrate the manual at work. We start with the following technical statement.

Lemma 2.3. *Let a map $f : \mathbb{R} \to \exp(\mathbb{R})$, $a \in \mathbb{R}$, and $N \in \omega \setminus \{0, 1\}$ satisfy the following requirement:*

$$f(a + \frac{m}{N}) = [a + \frac{m+1}{N}, a + 1 + \frac{m+1}{N}] \text{ for each } m = 0, ..., N - 1$$

Then f cannot be colored by fewer than N colors.

Proof. If f is not colorable, then we are done. Otherwise, fix a coloring of f. If $m > k$, then $a + \frac{m}{N} \geq a + \frac{k+1}{N}$. If $m \leq N - 1$ then $a + \frac{m}{N} \leq a + 1 + \frac{1}{N}$. Therefore,

(P) $a + \frac{m}{N} \in f(a + \frac{k}{N})$ for $k < m \leq N - 1$

Property (P) implies that $a, a + \frac{1}{N}, ..., a + \frac{(N-1)+1}{N}$ must have mutually distinct colors. □

We are now ready to construct an example showing that finiteness of SQ alone is not sufficient for colorability of a fixed-point free continuous map from $\mathbb{R}$ to $\exp_K(\mathbb{R})$.

Example 2.4. There exists a fixed-point free non-colorable continuous map from $\mathbb{R}$ into $\exp_K(\mathbb{R})$ such that $ST(f)$ is finite, $SQ(f) = 0$, and the image of each point is a unit segment.

Construction. We state the next claim for further reference and it is a direct consequence of continuity of a composition of continuous functions.

Claim. Let $h : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \exp(\mathbb{R})$ be maps such that $f(x) = [h(x), h(x) + 1]$ for each $x \in \mathbb{R}$. If h is continuous, then f is continuous too.

Fix a sequence of real numbers $a_2, ..., a_n, ...$ such that

(A) $a_{n+1} - a_n > 5$, for each n

For each $n = 2, 3...$ put

(B) $g(a_n) = a_n + 1/n, g(a_n + \frac{1}{n}) = a_n + \frac{2}{n}, ..., \ g(a_n + \frac{n-1}{n}) = a_n + 1$

Observe that $x < g(x)$ for all x in $\{a_n + m/n : m = 0, ..., n - 1\}$, where $n \in \omega \setminus \{0, 1\}$. Moreover,

(C) $g(a_n), g(a_n + \frac{1}{n}), ..., g(a_n + \frac{(n-1)}{n})$ fall in $[a_n, a_n + 5) \subset [a_n, a_{n+1})$.

Now applying (A), we conclude that $x < g(x)$ for all x in $S = \{a_n + \frac{m}{n} : n = 2, 3... \textit{ and } m = 0, 1, ..., n-1\}$. Since S is closed, there exists a continuous extension $h : \mathbb{R} \to \mathbb{R}$ of g such that $x < h(x)$ for all $x \in \mathbb{R}$.

Define $f : \mathbb{R} \to \exp_K(\mathbb{R})$ by letting $f(x) = [h(x), h(x) + 1]$. By Claim, f is continuous. By (B) and Lemma 2.3, f is not colorable. Since $x < h(x)$ for all x in the domain, f is fixed-point free. □

One can use the argument of the example and the accompanying lemma to create well-defined maps with the desired properties. In particular, the referee of this paper suggested the map $f : \mathbb{R} \to \exp_K(\mathbb{R})$ defined as follows:

$$f(x) = [x + \frac{1}{x}, x + \frac{1}{x} + 1] \textit{ if } x \geq 1 \textit{ and } f(x) = [x + 1, x + 2] \textit{ otherwise.}$$

Following the argument of our example and the lemma, one can see that this map has the desired properties.

We have now shown that there are examples of non-colorable continuous maps from $\mathbb{R}$ to $\exp_K \mathbb{R}$ that have positive squeeze component and ones that have finite stretch component.

3. Continuous Fixed-Point Free Selections

We start this section with a simple observation that motivates this section's discussion.

Proposition 3.1. *A continuous map $f : \mathbb{R} \to \exp_K(\mathbb{R})$ has a fixed-point free continuous selection if and only if $\{x \in \mathbb{R} : x = \max f(x)\}$ or $\{x \in \mathbb{R} : x = \min f(x)\}$ is empty.*

Proof. To prove necessity, let s_f be a continuous fixed-point free selection. Then the graph of s_f is completely above the diagonal or completely below the diagonal in $\mathbb{R}^2$. This means that $s_f(x) > x$ for all x or $s_f(x) < x$ for all x. Assume that $s_f(x) > x$ for all x. Then $x < \max f(x)$ for all x. Hence $\{x \in \mathbb{R} : x = \max f(x)\} = \varnothing$. Similarly, $\{x \in \mathbb{R} : x = \min f(x)\} = \varnothing$ if $s_f(x) < x$ for all x.

To prove sufficiency, assume that $\{x \in \mathbb{R} : x = \max f(x)\} = \varnothing$. The map s_f defined by letting $s_f(x) = \max f(x)$ is then fixed-point free and is known to be continuous. □

An argument similar to the one in Proposition 3.1 implies the following statement.

Proposition 3.2. *Let X be a closed subset of $\mathbb{R}$ and $f : X \to \exp_K(X)$ be continuous. Then f has a fixed-point free continuous selection iff the sets $\{x \in \mathbb{R} : x = \max f(x)\}$ and $\{x \in \mathbb{R} : x = \min f(x)\}$ are separated by clopen neighborhoods.*

Note that the statement of Proposition 3.1 is no longer true if we replace $\exp_K(\mathbb{R})$ by $\exp(\mathbb{R})$. This is shown in the next example.

Example 3.3. There exists a continuous map $f : \mathbb{R} \to \exp(\mathbb{R})$ such that the set $\{x \in \mathbb{R} : x = \max f(x)\}$ is empty but every continuous selection of f has a fixed point.

Construction. For $n \in \mathbb{N}$, let $F_n = ((-\infty, n]) \times \{n\}) \cup \{(x, 2n - x) : n \leq x \leq 2n\} \cup [2n, \infty) \times \{0\}$ and $E = \bigcup_{n\in\mathbb{N}} F_n$. Define $f : \mathbb{R} \to \exp(\mathbb{R})$ by $f(x) = \{y : (x, y) \in E\} = E_x$ (vertical slice at x.) The graph of any continuous selection would have to "follow" some F_n and hence would have n as a fixed point. □

Our observation prompts the following natural questions.

Question 3.4. *Does Proposition 3.1 hold of we replace* $\exp_K(\mathbb{R})$ *by* $\{A \in \exp(\mathbb{R}) : A$ *is connected*$\}$ *?*

In general, it would be interesting to indentify conditions under which a continuous multivalued map on $\mathbb{R}^n$ with compact images has a fixed-point free continuous selection. It would be also interesting to consider the n-dim versions of Proposition 3.2 and Question 3.4. In addition to continuity we can require our selection to have some other natural properties.

Question 3.5. *Suppose that f is a continuous multivalued map on $\mathbb{R}$ such that $f(x)$ is a closed non-trivial segment and f has a continuous fixed-point free selection. Does f have a differentiable fixed-point free selection? Does f have a continuous fixed-point free selection differentiable at at least one point? What if we replace $\mathbb{R}$ with $\mathbb{R}^n$ and "closed segment" with "closed ball"?*

Note that "non-triviality" of images is important since any continuous non-differentiable (or nowhere differentiable) map with graph strictly above the line $y = x$ would give a trivial example.

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