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Graph topologies on closed multifunctions

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Dedicated by the first two authors to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. In this paper we study function space topologies on closed multifunctions, i.e. closed relations on $X \times Y$ using various hypertopologies. The hypertopologies are in essence, **graph topologies** i.e topologies on functions considered as graphs which are subsets of $X \times Y$. We also study several topologies, including one that is derived from the Attouch-Wets filter on the range. We state embedding theorems which enable us to generalize and prove some recent results in the literature with the use of known results in the hyperspace of the range space and in the function space topologies of ordinary functions.

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1. Introduction.

Recently McCoy [24] studied relations among four hyperspace topologies (viz. Fell topology, Fell uniform topology, Vietoris topology and Hausdorff-Bourbaki topology) and the corresponding topologies on set valued maps. In this paper we plan to study the subject comprehensively in more general situations. We recall that for a topological space Z the hyperspace, 2^Z, of closed subsets of Z has a number of natural topologies on it obtained from the topology on Z. In our setting (X, τ_1) and (Y, τ_2) denote Hausdorff topological spaces and Z the product space $X \times Y$ equipped with the product topology $\tau = \tau_1 \times \tau_2$. If δ_1 and δ_2 are compatible proximities on X and Y respectively, then on Z is assigned the product proximity $\delta = \delta_1 \times \delta_2$. The hyperspace $2^Z = 2^{X \times Y}$ can be considered as the space $\mathbf{F}$ of all set valued maps on X to 2^Y taking points of X to (possibly empty) closed subsets of Y. We do not distinguish between a function $f \in \mathbf{F}$ and its graph $\{(x, f(x)) : x \in X\} \subset Z = X \times Y$. Thus our study includes topologies on the spaces of partial maps studied first in 1936 and which are being studied intensively in recent times ([1], [2], [3], [13], [19], [20], [23], [27], [33], [34]).

Given a Hausdorff topological space Z, for each subset E of Z, $cl_Z E$, $int E$ and E^c stand for the closure, interior and complement of E in Z. Moreover

$E^- = \{A \in 2^Z : A \cap E \neq \varnothing\}$;
$E^+ = \{A \in 2^Z : A \subset E\}$.

Futhermore, if δ is a compatible proximity on Z (for details see [30]), we set

$E_\delta^{++} = \{A \in 2^Z : A \ll_\delta E\}$. (Note: $A \ll_\delta E$ iff $A \not\delta E^c$ where $\not\delta$ denotes the negation of δ).

We omit δ if it is clear from the context and write E_δ^{++} simply as E^{++}.

We recall that the set of all compatible proximities on Z is partially ordered as follows: $\delta_1 \leq \delta_2$ iff whenever A, $B \subset Z$ and $A \not\delta_1 B$, then $A \not\delta_2 B$ (see [30]).

Some special cases of δ are:

δ_0 the **fine** LO**-proxmity** on Z given by
$A\delta_0 B$ iff $cl_Z A \cap cl_Z B \neq \varnothing$.
δ_0 is called the *Wallman proximity*.

It is well known that δ_0 is, by far, the most important compatible LO-proximity on Z, and that δ_0 is EF iff Z is normal (Urysohn's Lemma).

If Z is Tychonoff and $\mathcal{V}$ is a compatible uniformity on Z, then

$\delta(\mathcal{V})$ denotes the EF-proximity on Z given by
$A\delta(\mathcal{V})B$ iff $V(A) \cap B \neq \varnothing$ for each $V \in \mathcal{V}$.
$\delta(\mathcal{V})$ is called the *uniform proximity* (induced by $\mathcal{V}$).

If Z is a metrizable space with metric d, then

$\delta(d)$ is the EF-proximity on Z given by
$A\delta(d)B$ iff $D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} = 0$.
$\delta(d)$ is called the *metric proximity* (induced by d).

For $Z = X \times Y$, we use the symbol Δ (resp. Δ_1, Δ_2) to denote a subfamily of $CL(Z) = 2^Z \setminus \{\varnothing\}$ (resp. of $CL(X) = 2^X \setminus \{\varnothing\}$, $CL(Y) = 2^Y \setminus \{\varnothing\}$) which is a **cobase**, i.e.
(a) is **closed under finite unions**; and
(b) contains **the singletons**.
A **cover** is a cobase which is also **closed hereditary**.
Moreover, we assume
(c) $\Delta_1 \times \Delta_2 \subset \Delta$.
In some cases, in addition to the above condition, we suppose

(d) $p_1(\Delta) \subset \Delta_1$ and $p_2(\Delta) \subset \Delta_2$, where p_1 and p_2 are projections from Z to X and Y respectively.

A typical and important example of a cover is $\Delta = K(Z)$**, the family of all nonempty compact subsets of** Z**,** $\Delta_1 = K(X)$**,** $\Delta_2 = K(Y)$**. Moreover, in this case** $(c) - (d)$ **also hold.**

In what follows, unless explicitly stated, we assume always that $\Delta \subset CL(Z)$ (resp. $\Delta_1 \subset CL(X)$, $\Delta_2 \subset CL(Y)$) is a cobase.

We now describe some hypertopologies on 2^Z (for details see [4]). Suppose δ is a compatible LO-proximity on Z.

The *lower Vietoris topology* τ_V^- on 2^Z has a subbase $\{W^- : W \in \tau\}$.
The *upper* Δ*-topology* $\tau(\Delta)^+$ on 2^Z has a base $\{W^+ : W^c \in \Delta\}$.
The *upper proximal* Δ*-topology* $\sigma(\Delta, \delta)^+$ on 2^Z has a base $\{W^{++} : W^c \in \Delta\}$.
The Δ *topology* $\tau(\Delta)$ on 2^Z equals $\tau_V^- \vee \tau(\Delta)^+$.
The *proximal* Δ *topology* $\sigma(\Delta, \delta)$ on 2^Z equals $\tau_V^- \vee \sigma(\Delta, \delta)^+$.
The *upper* ΔU*-topology* $\tau(\Delta U)^+$ on 2^Z has a base
$\{W^+ : W^c \in \Delta \text{ or } clW \in \Delta\}$.
The *upper proximal* ΔU *topology* $\sigma(\Delta U, \delta)^+$ on 2^Z has a base
$\{W^{++} : W^c \in \Delta \text{ or } clW \in \Delta\}$.
The ΔU*-topology* $\tau(\Delta U)$ on 2^Z equals $\tau_V^- \vee \tau(\Delta U)^+$ ([8] and [16]).
The *proximal* ΔU*-topology* $\sigma(\Delta U, \delta)$ on 2^Z equals $\tau_V^- \vee \sigma(\Delta U, \delta)^+$.

Special cases:

(1) **<u>Vietoris and proximal topologies:</u>** when $\Delta = CL(Z)$,

the *upper Vietoris topology* $\tau(V)^+ = \tau(CL(Z))^+$;
the *Vietoris topology* $\tau(V) = \tau(CL(Z))$;
the *upper proximal topology* $\sigma(\delta)^+ = \sigma(CL(Z), \delta)^+$ and
the *proximal topology* $\sigma(\delta) = \sigma(CL(Z), \delta) = \sigma$, if δ is understood.

The paper [7] **deals with only metric proximities, and** [18] **remains unpublished. It is not widely known that proximal hypertopologies can be studied in more general situations and not merely in metric spaces.** (However, see the recent papers [11], [12] and [16]).

We note that the Vietoris topology is itself a proximal topology, i.e. $\tau_V = \sigma(\delta_0)$.

(2) **<u>Fell topology:</u>** when $\Delta = K(Z)$,

the *upper Fell topology* (also called the *co-compact topology*) $\tau(F)^+ = \tau(K(Z))^+$;
the *Fell topology* $\tau(F) = \tau(K(Z))$;

the *U-topology* $\tau(U) = \tau(K(Z)U)$ (see [8]).

When $\Delta = K(Z)$ and δ is EF, we have
$\tau(F) = \tau(K(Z)) = \sigma(K(Z), \delta) = \sigma(F, \delta)$ and
$\tau(U) = \tau(K(Z)U) = \sigma(K(Z)U, \delta) = \sigma(U, \delta)$.

In this case the Fell topology equals the proximal Fell topology and this explains the reason for several beautiful results. In generalizing results concerning Fell topology to Δ-topologies, we find that some are true in $\tau(\Delta)$ while others are true in $\sigma(\Delta)$! (Also see below about weak topologies).

(3) **Ball and proximal ball topologies:**

When Z is a metric space, $\Delta = \mathcal{B}$ is the cobase generated by all finite unions of all closed balls of nonnegative radii and δ is the metric proximity, we have

the *ball topology* $\tau(\mathcal{B}) = \tau(\Delta)$ ([4]);
the *proximal ball topology* $\sigma(\mathcal{B}) = \sigma(\Delta, \delta)$ ([17]).

The proximal ball topology is very close to the Wjisman topology. In fact, the two are equal in metric spaces satisfying some simple conditions that are present in a normed linear space ([17]).

For $Z = X \times Y$, when we wish to refer to hypertopologies on 2^Y, we use the suffix 2 e.g.
$\tau_2(V)$ denotes the Vietoris topology on 2^Y;
$\tau_2(F)$ denotes the Fell topology on 2^Y;
$\sigma_2(\delta_2) = \sigma_2$ denotes the proximal topology w.r.t. δ_2 on 2^Y etc..

(4) **Weak topologies:**

If $Z = X \times Y$, then for each of the topologies involving Δ described above, we also have an associated **weak topology** wherein Δ is replaced by $\Delta_1 \times \Delta_2$ (see [31]) and we attach the letter "w". Thus $\tau(w\Delta) = \tau(\Delta_1 \times \Delta_2)$ and important special examples are:

the **weak Vietoris topology** $\tau(wV) = \tau(CL(X) \times CL(Y)) \subset \tau(V) = \tau(CL(Z))$;
the **weak Fell topology** $\tau(wF) = \tau(K(X) \times K(Y)) \subset \tau(F) = \tau(K(Z))$.

For single-valued functions with closed graphs, it was shown in [21] that $\tau(wF) = \tau(F)$. The proof also works for $\mathbf{F}$ and combining this result with the fact that when the proximity δ is EF, $\tau(F) = \sigma(F, \delta)$ we have
$\tau(wF) = \tau(F) = \sigma(wF, \delta) = \sigma(F, \delta)$.

In generalizing McCoy's results involving Fell topology we find that our generalizations hold if we replace an appropriate member from the above four.

(5) **Hausdorff-Bourbaki and Attouch-Wets topologies:**

Definition 1.1. Let Y be a Tychonoff space, $\mathcal{V}$ a compatible uniformity and $\Delta_2 \subset CL(Y)$.

(i) For each $V \in \mathcal{V}$ set:

$V_H = \{(A, B) \in 2^Y \times 2^Y : A \subset V(B) \text{ and } B \subset V(A)\}$.

The family $\{V_H : V \in \mathcal{V}\}$ is a base for a uniformity $\mathcal{V}^H$ on 2^Y called the *Hausdorff-Bourbaki uniformity* (cf. [4]) (or the *HB-uniformity* for short).

(ii) Whereas for each $D \in \Delta_2$ and $V \in \mathcal{V}$ set:

$[D, V] = \{(A, B) \in 2^Y \times 2^Y : A \cap D \subset V(B) \text{ and } B \cap D \subset V(A)\}$.

The family $\{[D, V] : D \in \Delta_2 \text{ and } V \in \mathcal{V}\}$ is a base for a filter $\mathcal{V}^{\Delta_2}$ on 2^Y called the Δ_2-*Attouch-Wets filter* (cf. [5], [6] and [16]) (or the Δ_2-*AW filter* for short).

Remark 1.2. Let Y be a locally compact space, $\mathcal{V}$ a compatible uniformity and $\Delta_2 = K(Y)$. Then the corresponding Δ_2-AW filter $\mathcal{V}^{\Delta_2}$ on 2^Y is a uniformity (see [4] or [5]) and it will be denoted with $\mathcal{U}(F)$. Moreover, if 2^Y is equipped with the Fell topology $\tau_2(F)$, it is known that $\mathcal{U}(F)$ is compatible with $\tau_2(F)$ (see [4] and [10]).

Observe that in this case $(2^Y, \tau_2(F))$ is a compact Hausdorff space and thus $\mathcal{U}(F)$ is the unique uniformity on 2^Y corresponding to the Fell topology and it is generated by all $\tau_2(F) \times \tau_2(F)$ open neighbourhoods of the diagonal in $2^Y \times 2^Y$.

Thus, if Y is a Tychonoff space with a compatible uniformity $\mathcal{V}$, $\Delta_2 \subset CL(Y)$ and $\mathcal{V}^H$ and $\mathcal{V}^{\Delta_2}$ the associated HB-uniformity and Δ_2-AW filter on 2^Y respectively, then on the space $\mathbf{F}$:

(a) A typical basic open set in the **HB-uniform convergence topology** $\tau(UC\Delta_1, \mathcal{V}^H)$ **on** Δ_1 is of the form

$< f, A, V_H >= \{g \in \mathbf{F} : \text{for all } x \in A,\ (f(x), g(x)) \in V_H\}$, where $f \in \mathbf{F}$, $A \in \Delta_1$ and $V_H \in \mathcal{V}^H$.

If $\Delta_1 = K(X)$, we get one of the most important topologies, namely the **HB-uniform convergence topology on compacta** $\tau(UCC, \mathcal{V}_H)$.

If we replace A by X, we get another important topology: the **HB-uniform convergence topology** $\tau(UC, \mathcal{V}^H)$.

If Δ_1 is the family of all finite subsets of X, then we have the **pointwise HB-convergence topology** $\tau_p(\mathcal{V}^H)$.

When $\mathcal{V}^H$ is understood, we may omit it and just write $\tau(UCC)$ and $\tau(UC)$.

(b) The topology on $\mathbf{F}$ generated by

$\{< f, A, M >: f \in \mathbf{F}, A \in \Delta_1 \text{ and } M \in \mathcal{V}^{\Delta_2}\}$
is called the **Δ_2-AW convergence topology** $\tau(UC\Delta_1, \mathcal{V}^{\Delta_2})$ **on** Δ_1.

If $A = X$, we have the **Δ_2-AW convergence topology** $\tau(UC, \mathcal{V}^{\Delta_2})$. As before, we replace Δ_1 by C for "compacta".

If $\Delta_1 = K(X)$, we obtain the **Δ_2-AW convergence topology on compacta** $\tau(UCC, \mathcal{V}^{\Delta_2})$.

If Δ_1 is the family of all finite subsets of X, then we have the **pointwise Δ_2-AW convergence topology** $\tau_p(\mathcal{V}^{\Delta_2})$.

By Remark 1.2 it follows that whenever Y is a locally compact space and 2^Y is equipped with the Fell topology $\tau_2(F)$, then the corresponding Δ_2-AW filter $\mathcal{U}(F)$ is a uniformity which is independent of the uniformity $\mathcal{V}$ chosen on Y.

Note that the topology $\tau(UCC, \mathcal{U}(F))$ on $\mathbf{F}$ is just what McCoy calls "**Fell uniform topology (on compact sets)**" (see [24]).

(6) **Pseudo uniform topologies:**

In [22] and [25] function space topologies akin to uniform topologies were studied. The range space was not necessarily uniformizable. Here we introduce a similar concept. Let W be a *symmetric* neighbourhood of the diagonal in $(2^Y \times 2^Y, \tau_2 \times \tau_2)$.

For each $f \in \mathbf{F}$ and $A \in \Delta_1$ we set
$W^\star(f, A) = \{g \in \mathbf{F} : \text{for all } x \in A,\ (f(x), g(x)) \in W\}$.

The topology on $\mathbf{F}$ generated by $\{W^\star(f, A) : f \in \mathbf{F}, A \in \Delta_1$ and W a symmetric $\tau_2 \times \tau_2$ neighbourhood of the diagonal in $2^Y \times 2^Y\}$ is the **τ_2-pseudo uniform topology on** Δ_1: $ps(\tau(UC\Delta_1, \tau_2))$.

If $A = X$, we have the **pseudo τ_2-uniform topology** $ps(\tau(UC, \tau_2))$.

As before, if $\Delta_1 = K(X)$, we replace Δ_1 by C for "compacta" and we have the **pseudo τ_2-uniform topology on compacta** $ps(\tau(UCC, \tau_2))$.

In case $(2^Y, \tau_2)$ is uniformizable and we restrict W's to symmetric entourages, we do get a uniform topology. This is true as in (5) above or in (6) when Y is a locally compact space and 2^Y is equipped with the Fell topology $\tau_2(F)$ on 2^Y and in this case $ps(\tau(UC\Delta_1, \tau_2(F)) = \tau(UC\Delta_1, \mathcal{U}(F))$ (cf. above Remark 1.2).

Although McCoy got his results in *uniform* setting, we find that some of his results do not need uniformity at all!

Finally, if τ_2 is a given hypertopology on 2^Y, then $\tau_p(\tau_2)$ is the corresponding **τ_2-pointwise convergence topology** on $\mathbf{F}$ which agrees with the pseudo τ_2 uniform topology on Δ_1 $ps(\tau(UC\Delta_1, \tau_2))$ when Δ_1 is the family of all finite subsets of X.

Those interested in more details are referred to [4] for hypertopologies, [30] for proximities, [26] and [28] for function space topologies, [9], [10] and [24] for uniform topologies and convergences on spaces of multifunctions.

2. Basic results.

One of the most valuable result in function space topologies is the embedding of the range space in the function space (cf. Theorem 2.1.1, page 15 in [26]). In this section we prove similar results for multifunctions which are of fundamental importance in our work. We need to introduce "upper" hypertopologies that are specially meant for the family $\mathcal{C} = \{X \times E : E \in CL(Y)\}$ of **constant multifunctions**. These topologies depend on Δ_2 alone, unlike other hypertopologies which depend on either Δ or $\Delta_1 \times \Delta_2$. We use the suffix r (for *range*) for such topologies.

On $CL(Z)$, we have the **upper r-Δ_2-topology** $\tau(r\Delta_2)^+$ which is generated by the basis $\{(X \times V)^+ : V^c \in \Delta_2\} \cup \{CL(Z)\}$.

Similarly, we have the **upper r-Δ_2U-topology** $\tau(r\Delta_2 U)^+$ which is generated by the basis $\{(X \times V)^+ : V^c \in \Delta_2$ or $clV \in \Delta_2\} \cup \{CL(Z)\}$.

If δ_2 is a LO-proximity on Y, then we define an associated "proximity" rδ_2 on $\mathcal{C}$ by $(X \times E) \ll (X \times V)$ w.r.t. rδ_2 iff $E \ll V$ w.r.t. δ_2.

If δ_2 is an EF-proximity on Y, then it is easy to see that rδ_2 on $\mathcal{C}$ is also EF.

Naturally we also have the proximal versions:

the **upper proximal r-Δ_2-topology** $\sigma(r\Delta_2, r\delta_2)^+$;
the **upper proximal r-Δ_2U-topology** $\sigma(r\Delta_2 U, r\delta_2)^+$.

We have on $CL(Z)$

the **r-Δ_2-topology** $\tau(r\Delta_2) = \tau(r\Delta_2)^+ \vee \tau_V^-$.

Similarly the analogues:

the **r-Δ_2U-topology** $\tau(r\Delta_2 U) = \tau(r\Delta_2 U)^+ \vee \tau_V^-$;
the **proximal r-Δ_2-topology** $\sigma(r\Delta_2, r\delta_2) = \sigma(r\Delta_2, r\delta_2)^+ \vee \tau_V^-$ and
the **proximal r-Δ_2U-topology** $\sigma(r\Delta_2 U, r\delta_2) = \sigma(r\Delta_2 U, r\delta_2)^+ \vee \tau_V^-$.

Let $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ denote the set of all subsets of Y and Z, respectively. Consider the map $j : \mathcal{P}(Y) \hookrightarrow \mathcal{P}(Z)$ defined by $j(E) = (X \times E) \in \mathcal{P}(Z)$. Obviously,

(a) $j\colon CL(Y) \hookrightarrow \mathcal{C}$ is a bijection;

(b) $j(V^+) \subset [j(V)]^+$;
(c) $j(V^{++}_{\delta_2}) \subset [j(V)]^{++}_{r\delta_2}$;
(d) $j(V^-) \subset [j(V)]^-$;
(e) $j(E) \in W^-$ and $W \in \tau$ together imply $E \in [p_2(W)]^-$, where $p_2\colon Z \to Y$ is the projection.
(f) Let $\Delta_1 \subset CL(X)$, $D \in \Delta_1$, $\mathcal{U}$ a filter on 2^Y, M a symmetric member of $\mathcal{U}$, $A \in CL(Y)$ and $f = j(A) = X \times A$. Then:
$< f, x, M > \cap \mathcal{C} =< f, D, M > \cap \mathcal{C} =< f, X, M > \cap \mathcal{C} = j(M(A))$
for each $x \in X$.

So, we have the following results.

Theorem 2.1. *Let X and Y be Hausdorff spaces with compatible LO-proximities. The following are embeddings:*

(a) $j\colon (CL(Y), \tau_2(\Delta_2)^+) \hookrightarrow (CL(Z), \tau(r\Delta_2)^+)$;
(b) $j\colon (CL(Y), \tau_2(\Delta_2 U)^+) \hookrightarrow (CL(Z), \tau(r\Delta_2 U)^+)$;
(c) $j\colon (CL(Y), \sigma_2(\Delta_2, \delta_2)^+) \hookrightarrow (CL(Z), \sigma(r\Delta_2, r\delta_2)^+)$;
(d) $j\colon (CL(Y), \sigma_2(\Delta_2 U, \delta_2)^+) \hookrightarrow (CL(Z), \sigma(r\Delta_2 U, r\delta_2)^+)$;
(e) $j\colon (CL(Y), \tau_2(\Delta_2)) \hookrightarrow (CL(Z), \tau(r\Delta_2))$;
(f) $j\colon (CL(Y), \tau_2(\Delta_2 U)) \hookrightarrow (CL(Z), \tau(r\Delta_2 U))$;
(g) $j\colon (CL(Y), \sigma_2(\Delta_2, \delta_2)) \hookrightarrow (CL(Z), \sigma(r\Delta_2, r\delta_2))$;
(h) $j\colon (CL(Y), \sigma_2(\Delta_2 U, \delta_2)) \hookrightarrow (CL(Z), \sigma(r\Delta_2 U, r\delta_2))$.

Remark 2.2. Let Y be a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y, $\Delta_1 \subset CL(X)$ and $\Delta_2 \subset CL(Y)$. If $\mathcal{V}^H$ and $\mathcal{V}^{\Delta_2}$ are respectively the associated HB-uniformity and Δ_2-AW filter on 2^Y, then on $\mathcal{C}$:

(1) $\tau(UC, \mathcal{V}^H) = \tau(UC\Delta_1, \mathcal{V}^H) = \tau_p(\mathcal{V}^H)$;
(2) $\tau(UC, \mathcal{V}^{\Delta_2}) = \tau(UC\Delta_1, \mathcal{V}^{\Delta_2}) = \tau_p(\mathcal{V}^{\Delta_2})$.

Thus the following are embeddings:

(1a) $j\colon (CL(Y), \tau_2(\mathcal{V}^H)) \hookrightarrow (CL(Z), \tau(UC, \mathcal{V}^H))$;
(1b) $j\colon (CL(Y), \tau_2(\mathcal{V}^H)) \hookrightarrow (CL(Z), \tau(UC\Delta_1, \mathcal{V}^H))$.
(2a) $j\colon (CL(Y), \mathcal{V}^{\Delta_2}) \hookrightarrow (CL(Z), \tau(UC, \mathcal{V}^{\Delta_2}))$;
(2b) $j\colon (CL(Y), \mathcal{V}^{\Delta_2}) \hookrightarrow (CL(Z), \tau(UC\Delta_1, \mathcal{V}^{\Delta_2}))$.

Similarly, if Y is a Hausdorff space, τ_2 a given hypertopology on 2^Y and $\Delta_1 \subset CL(X)$, then on $\mathcal{C}$:

(3) $ps(\tau(UC, \tau_2)) = ps(\tau(UC\Delta_1, \tau_2)) = \tau_p(\tau_2)$.

Thus the following are embeddings:

(3a) $j\colon (CL(Y), \tau_2) \hookrightarrow (CL(Z), ps(\tau(UC, \tau_2)))$;
(3b) $j\colon (CL(Y), \tau_2) \hookrightarrow (CL(Z), ps(\tau(UC\Delta_1, \tau_2)))$.

Lemma 2.3. *Let X and Y be Hausdorff spaces. Then, on the family $\mathcal{C}$ of constant multifunctions:*

(a) $\tau(w\Delta)^+ \leq \tau(r\Delta_2)^+ \leq \tau(r\Delta_2 U)^+ \leq \tau(V)^+$;
and if $p_2(\Delta) \subset \Delta_2$, then
$\tau(w\Delta)^+ \leq \tau(\Delta)^+ \leq \tau(r\Delta_2)^+ \leq \tau(r\Delta_2 U)^+ \leq \tau(V)^+$.

(b) $\tau(w\Delta)^+ \leq \tau(\Delta)^+ \leq \tau(\Delta U)^+ \leq \tau(V)^+$;
and if $p_2(\Delta) \subset \Delta$, *then*
$\tau(w\Delta)^+ \leq \tau(\Delta)^+ \leq \tau(\Delta U)^+ \leq \tau(r\Delta_2 U)^+ \leq \tau(V)^+$.

(c) $\tau(w\Delta) \leq \tau(r\Delta_2) \leq \tau(r\Delta_2 U) \leq \tau(V)$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\tau(w\Delta) \leq \tau(\Delta) \leq \tau(r\Delta_2) \leq \tau(r\Delta_2 U) \leq \tau(V)$.

(d) $\tau(w\Delta) \leq \tau(\Delta) \leq \tau(\Delta U) \leq \tau(V)$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\tau(w\Delta) \leq \tau(\Delta) \leq \tau(\Delta U) \leq \tau(r\Delta_2 U) \leq \tau(V)$.

Lemma 2.4. *Let X and Y be Hausdorff spaces with compatible LO-proximities. Then, on the family $\mathcal{C}$ of constant multifunctions:*

(a) $\sigma(w\Delta)^+ \leq \sigma(r\Delta_2)^+ \leq \sigma(r\Delta_2 U)^+ \leq \sigma^+$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\sigma(w\Delta)^+ \leq \sigma(\Delta)^+ \leq \sigma(r\Delta_2)^+ \leq \sigma(r\Delta_2 U)^+ \leq \sigma^+$.

(b) $\sigma(w\Delta)^+ \leq \sigma(\Delta)^+ \leq \sigma(\Delta U)^+ \leq \sigma^+$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\sigma(w\Delta)^+ \leq \sigma(\Delta)^+ \leq \sigma(\Delta U)^+ \leq \sigma(r\Delta_2 U)^+ \leq \sigma^+$.

(c) $\sigma(w\Delta) \leq \sigma(r\Delta_2) \leq \sigma(r\Delta_2 U) \leq \sigma$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\sigma(w\Delta) \leq \sigma(\Delta) \leq \sigma(r\Delta_2) \leq \sigma(r\Delta_2 U) \leq \sigma$.

(d) $\sigma(w\Delta) \leq \sigma(\Delta) \leq \sigma(\Delta U) \leq \sigma$;
and if $p_2(\Delta) \subset \Delta_2$, *then*
$\sigma(w\Delta) \leq \sigma(\Delta) \leq \sigma(\Delta U) \leq \sigma(r\Delta_2 U) \leq \sigma$.

We say that Z is **locally** Δ iff for each $z \in Z$ with $z \in V \in \tau$, there is $D \in \Delta$ with $z \in intD \subset D \subset V$. (*Note that this is a generalization of local compactness in which case* $\Delta = K(Z)$).

Lemma 2.5. *Let X and Y be Hausdorff spaces with compatible LO-proximities, $Z = X \times Y$ and $\Delta \subset CL(Z)$ a cover. If Z is locally Δ, then:*

(a) $\tau(\Delta)^+ = \tau(\Delta U)^+$ *if and only if* $Z \in \Delta$ *i.e.* $\Delta = CL(Z)$.

(b) $\sigma(\Delta)^+ = \sigma(\Delta U)^+$ *if and only if* $Z \in \Delta$ *i.e.* $\Delta = CL(Z)$.

Proof. We prove only (a). It suffices to show that $\tau(\Delta U)^+ \leq \tau(\Delta)^+$ implies $Z \in \Delta$. Suppose U is a nonempty subset of Z with $A \subset U$ and $clU \in \Delta$ (note that such U exists since Z is locally Δ). Then there is an open subset V in Z with $V^c \in \Delta$ such that $A \subset V \subset U$. Clearly $Z = clU \cup V^c \in \Delta$. □

Corollary 2.6. *Let X and Y be Hausdorff spaces with compatible LO-proximities, $Z = X \times Y$ and $\Delta \subset CL(Z)$ a cover. If Z is locally Δ, then:*

(a) $\tau(\Delta) = \tau(\Delta U)$ *if and only if* $Z \in \Delta$ *i.e.* $\Delta = CL(Z)$ (cf. [14], Theorem 3.2).

(b) $\sigma(\Delta) = \sigma(\Delta U)$ *if and only if* $Z \in \Delta$ *i.e.* $\Delta = CL(Z)$.

(c) *When* $\Delta = K(Z)$, *we have* $\tau(F) = \tau(U)$ *if and only if X is compact.*

Remark 2.7. In the following relations, vertical lines show embeddings:

(a)
$$\begin{array}{lccccccccc}
CL(Y): & & & \tau_2(\Delta_2) & \leq & \tau_2(\Delta_2U) & \leq & \tau_2(V) & & \\
\downarrow \mathrm{j} & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}\subset\mathbf{F}: & \tau(w\Delta) & \leq & \tau(r\Delta_2) & \leq & \tau(r\Delta_2U) & \leq & \tau(rCL(Y)) & \leq & \tau(V).
\end{array}$$

(b)
$$\begin{array}{lccccccccccc}
CL(Y): & & & & & \tau_2(\Delta_2) & \leq & \tau_2(\Delta_2U) & \leq & \tau_2(V) & & \\
\downarrow \mathrm{j} & & & & & & & \downarrow & & \downarrow & & \\
\mathcal{C}\subset\mathbf{F}: & \tau(w\Delta) & \leq & \tau(\Delta) & \leq & \tau(\Delta U) & \leq & \tau(r\Delta_2U) & \leq & \tau(rCL(Y)) & \leq & \tau(V).
\end{array}$$

(c)
$$\begin{array}{lccccccccccc}
CL(Y): & & & & & \sigma_2(\Delta_2) & \leq & \sigma_2(\Delta_2U) & \leq & \sigma_2 & \leq & \tau_2(\mathcal{V}^H) \\
\downarrow \mathrm{j} & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}\subset\mathbf{F}: & \sigma(w\Delta) & \leq & \sigma(\Delta) & \leq & \sigma(r\Delta_2) & \leq & \sigma(r\Delta_2U) & \leq & \sigma(rCL(Y)) & \leq & \tau(\mathcal{V}^H).
\end{array}$$

(d)
$$\begin{array}{lccccccccccc}
CL(Y): & & & & & \sigma_2(\Delta_2) & \leq & \sigma_2(\Delta_2U) & \leq & \sigma_2 & \leq & \tau_2(\mathcal{V}^H) \\
\downarrow \mathrm{j} & & & & & & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{C}\subset\mathbf{F}: & \sigma(w\Delta) & \leq & \sigma(\Delta) & \leq & \sigma(\Delta U) & \leq & \sigma(r\Delta_2U) & \leq & \sigma(rCL(Y)) & \leq & \tau(\mathcal{V}^H).
\end{array}$$

(e) If the proximities involved are EF, then

$$\begin{array}{lccccccccccccc}
CL(Y): & & & & & \sigma_2(\Delta_2) & \leq & \sigma_2(\Delta_2U) & \leq & \sigma_2 & \leq & \tau_2(V) & & \\
\downarrow \mathrm{j} & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}\subset\mathbf{F}: & \sigma(w\Delta) & \leq & \sigma(\Delta) & \leq & \sigma(r\Delta_2) & \leq & \sigma(r\Delta_2U) & \leq & \sigma(rCL(Y)) & \leq & \tau(rCL(Y)) & \leq & \tau(V).
\end{array}$$

(f)
$$\begin{array}{lccccccccccccc}
CL(Y): & & & & & \sigma_2(\Delta_2) & \leq & \sigma_2(\Delta_2U) & \leq & \sigma_2 & \leq & \tau_2(V) & & \\
\downarrow \mathrm{j} & & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}\subset\mathbf{F}: & \sigma(w\Delta) & \leq & \sigma(\Delta) & \leq & \sigma(r\Delta_2) & \leq & \sigma(r\Delta_2U) & \leq & \sigma(rCL(Y)) & \leq & \tau(rCL(Y)) & \leq & \tau(V).
\end{array}$$

(g)
$$\begin{array}{lccccc}
CL(Y): & \tau_2(\Delta_2) & \leq & \tau_2(V) & & \\
\downarrow \mathrm{j} & \downarrow & & \downarrow & & \\
\mathcal{C}\subset\mathbf{F}: & ps(\tau(UC\Delta_1,\tau_2(\Delta_2))) & \leq & \tau(rCL(Y)) & \leq & \tau(V).
\end{array}$$

3. Generalization of McCoy's Results.

In this section we begin comparing some of the topologies defined in the previous section and find conditions for their pairwise equivalence. We study some simple ones which are analogues of those in McCoy's paper and state McCoy's results just below the analogues. Again, we recall that (X,τ_1) and (Y,τ_2) are Hausdorff spaces. We set $Z = X\times Y$ and assign the product topology $\tau = \tau_1 \times \tau_2$. If δ_1 and δ_2 are compatible proximities on X and Y respectively, then on Z is assigned the product proximity $\delta = \delta_1 \times \delta_2$. The family 2^Z of closed subsets of Z can be identified with the space $\mathbf{F}$ of all set valued maps on X to 2^Y taking points of X to closed (possibly empty) subsets of Y. Other assumptions will be stated at the places where they are needed.

McCoy assumed that both X and Y are locally compact spaces and Y is a non-trivial complete metric space. He then studied four topologies: $\tau(F)$, $\tau(V)$, $\tau(UCC,\mathcal{U}(F))$ and $\tau(\mathcal{V}^H)$. In this section we pursue $\tau(\Delta)$, $\tau(w\Delta)$, $\sigma(\Delta)$, $\sigma(w\Delta)$, $\tau(V)$, $\tau(\mathcal{V}^H)$, $\tau(UC\Delta_1,\mathcal{V}^H)$ and $\tau(UC,\mathcal{V}^H)$ as well as $\tau(UC\Delta_1,\mathcal{V}^{\Delta_2})$ and $\tau(UC,\mathcal{V}^{\Delta_2})$. Moreover, we also consider $ps(\tau(UC,\tau_2))$ and $ps(\tau(UC\Delta_1,\tau_2))$ for an arbitrary topology τ_2 on 2^Y.

Theorem 3.1. *Let X and Y be Hausdorff spaces with compatible LO-proximities δ_1 and δ_2, respectively. Then on $\mathbf{F}$:*

(a) $\tau(w\Delta)^+ = \tau(\Delta_1\times\Delta_2)^+ \le \tau(\Delta)^+ \le \tau(\Delta U)^+ \le \tau(V)^+$.
(b) $\tau(w\Delta) \le \tau(\Delta) \le \tau(\Delta U) \le \tau(V)$.
Moreover, if $\delta = \delta_1\times\delta_2$ is EF, then:
(c) $\sigma(w\Delta,\delta)^+ \le \sigma(\Delta,\delta)^+ \le \sigma(\Delta U,\delta)^+ \le \sigma(\delta)^+ \le \tau(V)^+$.
(d) $\sigma(w\Delta,\delta) \le \sigma(\Delta,\delta) \le \sigma(\Delta U,\delta) \le \sigma(\delta) \le \tau(V)$.
(Cf. [24] Prop. 4.1) *If X and Y are Hausdorff spaces, then $\tau(F) \le \tau(U) \le \tau(V)$.*

The following results and Lemmas play a key role.

Lemma 3.2. *Let X be a Hausdorff space, Y a Tychonoff space, $\mathcal{U}$ and $\mathcal{V}$ compatible uniformities on Y with $\mathcal{U}\subset\mathcal{V}$ and $\mathcal{U}^H$ and $\mathcal{V}^H$ the corresponding HB-uniformities on 2^Y associated to $\mathcal{U}$ and $\mathcal{V}$, respectively. Then on $\mathbf{F}$:*

(a) $\tau(UC\Delta_1,\mathcal{U}^H) \le \tau(UC\Delta_1,\mathcal{V}^H)$;
(b) $\tau(UC,\mathcal{U}^H) \le \tau(UC,\mathcal{V}^H)$.

Furthermore, if $\Delta_2\subset CL(Y)$ and $\mathcal{U}^{\Delta_2}$ and $\mathcal{V}^{\Delta_2}$ are the corresponding Δ_2-AW filters on 2^Y associated to $\mathcal{U}$ and $\mathcal{V}$, respectively, then:

(c) $\tau(UC\Delta_1,\mathcal{U}^{\Delta_2}) \le \tau(UC\Delta_1,\mathcal{V}^{\Delta_2})$;
(d) $\tau(UC,\mathcal{U}^{\Delta_2}) \le \tau(UC,\mathcal{V}^{\Delta_2})$.

Proof. (a) and (b) (resp. (c) and (d)) follow from the fact that if $\mathcal{U}\subset\mathcal{V}$ on Y, then $\mathcal{U}^H\subset\mathcal{V}^H$ (resp. $\mathcal{U}^{\Delta_2}\subset\mathcal{V}^{\Delta_2}$) on 2^Y. □

Lemma 3.3. *Let Y be a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y, $\Delta_2\subset CL(Y)$, $\mathcal{V}^H$ and $\mathcal{V}^{\Delta_2}$ the corresponding HB-uniformity and Δ_2-AW filter on 2^Y, respectively. Then, on 2^Y, $\mathcal{V}^{\Delta_2}\subset\mathcal{V}^H$.*

Proof. Let $[D,V]$ be basic element of $\mathcal{V}^{\Delta_2}$ where $D\in\Delta$ and $V\in\mathcal{V}$. We claim that the corresponding element $V_H\in\mathcal{V}^H$ is such that $V_H\subset[D,V]$. Assume not. Then there exists $(A,B)\in 2^Y\times 2^Y$ such that $(A,B)\in V_H$ but $(A,B)\notin[D,V]$. Thus either (i) $A\cap D\not\subset V(B)$ or (ii) $B\cap D\not\subset V(A)$. If (i) occurs, then there exists $y\in(A\cap D)\setminus V(B)$; a contradiction because $A\subset V(B)$. Similarly, if (ii) occurs. □

Corollary 3.4. *Let X be a Hausdorff space, Y a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y, $\Delta_1 \subset CL(X)$, $\Delta_2 \subset CL(Y)$, $\mathcal{V}^H$ and $\mathcal{V}^{\Delta_2}$ the corresponding HB-uniformity and Δ_2-AW filter on 2^Y. Then on* $\mathbf{F}$*:*

(a) $\tau(UC\Delta_1, \mathcal{V}^{\Delta_2}) \leq \tau(UC\Delta_1, \mathcal{V}^H)$;
(b) $\tau(UC, \mathcal{V}^{\Delta_2}) \leq \tau(UC, \mathcal{V}^H)$.

Proof. (a) and (b) follow from above Lemma 3.3. □

We recall that if (Z, τ) is a Tychonoff space with a compatible EF-proximity δ, then a uniformity $\mathcal{U}$ on Z is called *compatible w.r.t.* δ iff the uniform proximity $\delta(\mathcal{U})$ induced by $\mathcal{U}$ equals δ (see Section 1 and [30]). δ admits a unique compatible totally bounded uniformity $\mathcal{U}_w$ ([30]).

Theorem 3.5. (Cf. [18]) *Let Y be a Tychonoff space and δ_2 a compatible EF-proximity on Y. The corresponding proximal topology $\sigma_2(\delta_2)$ on 2^Y is always uniformizable. In fact, it is the topology induced on 2^Y by the HB-uniformity $\mathcal{U}_w^H$ which is derived from the unique totally bounded uniformity $\mathcal{U}_w$ on Y compatible w.r.t. δ_2.*

Lemma 3.6. (Cf. Theorem 2.1 in [16]) *Let Y be a Tychonoff space, δ_2 a compatible EF-proximity on Y and $\Delta_2 \subset CL(Y)$ a cover. If the proximal Δ_2 topology $\sigma_2(\Delta_2, \delta_2)$ is uniformizable, then the proximal Δ_2-topology $\sigma_2(\Delta_2, \delta_2)$ equals the topology $\tau(\mathcal{U}_w^{\Delta})$ induced by the Δ-AW filter $\mathcal{U}_w^{\Delta_2}$, where $\mathcal{U}_w$ is the unique totally bounded uniformity on Y compatible w.r.t. δ_2.*

Proof. Let $\mathcal{U}_w$ be the unique totally bounded uniformity on Y compatible with δ_2. Without loss of generality we assume that **all entourages** $W \in \mathcal{U}_w$ **are open and symmetric**.

First, let $\{A_\lambda : \lambda \in \Lambda\} \subset 2^Y$ be a net $\tau(\mathcal{U}_w^{\Delta_2})$-converging to $A \in 2^Y$. We claim that the net $\{A_\lambda : \lambda \in \Lambda\}$ $\sigma_2(\Delta_2, \delta_2)$-converges to A.

(i) If $A \in V^-$ where $V \in \tau$, then there exist $a \in A \cap V$ and a $W \in \mathcal{U}_w$ such that $W(a) \subset V$. Since $A \in [\{a\}, W](A) \subset V^-$, eventually
$A_\lambda \in [\{a\}, W](A) \subset V^-$.

(ii) If $A \in (D^c)^{++}_{\delta_2}$ where $D \in \Delta_2$, then $D \ll_{\delta_2} A^c$. Since $\mathcal{U}_w$ is compatible w.r.t. δ_2 and by assumption $\sigma_2(\Delta_2, \delta_2)$ is uniformizable there are $S \in \Delta_2$ and $W \in \mathcal{U}_w$ such that $D \subset W(D) \subset S \subset W(S) \subset A^c$ (see Theorem 4.4.5, Lemma 4.4.3 and Definition 4.4.2 in [4]). Since $A \in [S, W](A)$, $W(A) \cap S = \varnothing$ and eventually $A_\lambda \in [S, W](A)$, then eventually $A_\lambda \in (D^c)^{++}_{\delta_2}$.

Thus $\sigma_2(\Delta_2, \delta_2) \leq \tau(\mathcal{U}_w^{\Delta_2})$.

On the other hand, let $\{A_\lambda : \lambda \in \Lambda\} \subset 2^Y$ be a net $\sigma_2(\Delta_2, \delta_2)$-converging to $A \in 2^Y$. We claim that the net $\{A_\lambda : \lambda \in \Lambda\}$ $\tau(\mathcal{U}_w^{\Delta_2})$-converges to A. So, let $[W, D](A)$ a $\tau(\mathcal{U}_w^{\Delta})$-neigbourhood at A where $D \in \Delta_2$ and $W \in \mathcal{U}_w$. Let $V \in \mathcal{U}_w$ be such that $V^2 \subset W$. We have two cases:

(i) $A \in (D^c)^{++}_{\delta_2}$. Then eventually $A_\lambda \in (D^c)^{++}_{\delta_2}$ and obviously, $\varnothing = A_\lambda \cap D \subset W(A)$ and $\varnothing = A \cap D \subset W(A_\lambda)$, i.e eventually $A_\lambda \in [W, D][A]$.

(ii) $A \notin (D^c)^{++}_{\delta_2}$. Then $V(A) \cap D \neq \varnothing$. Since V is totally bounded, there are $x_j \in A$, $1 \leq j \leq n$ such that $A \subset \bigcup_{j=1}^{n} V(x_j) \subset V^2(A)$. Since $A \cap V(x_j) \neq \varnothing$ for each j, eventually $A_\lambda \cap V(x_j) \neq \varnothing$ and so $x_j \in V(A_\lambda)$. Hence, eventually $A \cap D \subset \bigcup_{j=1}^{n} V(x_j) \subset V^2(A_\lambda) \subset W(A_\lambda)$. Note that $(D \cap V(A)^c) \in \Delta_2$ and $A \in (D^c \cup V(A))^{++}_{\delta_2} \in \sigma_2(\Delta, \delta_2)$. So eventually, $A_\lambda \in (D^c \cup V(A))^{++}_{\delta_2}$. Thus eventually $A_\lambda \cap D = [A_\lambda \cap (D \cap V(A))] \subset W(A)$, i.e. eventually $A_\lambda \in [W, D](A)$. Thus $\tau(\mathcal{U}_w^\Delta) \leq \sigma_2(\Delta_2, \delta_2)$. Combining the earlier part we get $\tau(\mathcal{U}_w^{\Delta_2}) = \sigma_2(\Delta_2, \delta_2)$. $\square$

Remark 3.7.

(a) In [15] it is shown that if $\tau_2(\Delta_2)$ is uniformizable, then there is a compatible EF-proximity δ_2 on Y such that $\tau_2(\Delta_2) = \sigma_2(\Delta_2, \delta_2)$ (see Lemma 2.2 in [15]).

(b) Above Lemma and Remark 3.7 (a) show that the appropriate extension of $\mathcal{U}(F)$ (see Remark 1.2) for uniformizable Δ_2- and proximal Δ_2-topologies are the Δ_2-AW filters induced by compatible totally bounded uniformities $\mathcal{U}_w$ on Y. Thus, as in the definition of $\mathcal{U}(F)$ (see Remark 1.2), we reserve the symbol $\mathcal{U}(\tau_2)$ to denote the corresponding compatible AW filter associated with the totally bounded uniformity $\mathcal{U}_w$ on Y whenever τ_2 is a uniformizable (proximal) Δ_2-topology on 2^Y.

Theorem 3.8. *Let X be a Hausdorff space, Y a Tychonoff space with a compatible EF-proximity δ_2 and $\mathcal{U}_w$ the unique totally bounded uniformity associated to δ_2, $\Delta_1 \subset CL(X)$, $\Delta_2 \subset CL(Y)$ a cover and 2^Y equipped with the proximal Δ_2-topology $\sigma_2(\Delta_2)$ induced by δ_2. If $\sigma_2(\Delta_2)$ is uniformizable and $\mathcal{U}(\sigma_2(\Delta_2))$ and $\mathcal{U}^H$ are respectively the corresponding Δ_2-AW filter compatible w.r.t. $\sigma(\Delta_2)$ and HB-uniformity on 2^Y associated to $\mathcal{U}_w$, then on* $\mathbf{F}$*:*

(a) $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2))) \leq \tau(UC\Delta_1, \mathcal{U}^H)$;
(b) $\tau(UC, \mathcal{U}(\sigma_2(\Delta_2))) \leq \tau(UC, \mathcal{U}^H)$.

Proof. (a) and (b) follow by Corollary 3.4 and Lemma 3.6. $\square$

Next Theorem generalizes Propositions 4.2 and 4.5 and shows that the assumption of local compactness on the base space X it is not nedded (see also Proposition 4.1 in [10]).

Theorem 3.9. *Let X be a Hausdorff space, Y a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y and $\mathcal{V}^H$ the corresponding HB-uniformity on 2^Y. Suppose δ_2 is an EF-proximity on Y with $\delta_2 \leq \delta_2(\mathcal{V})$, 2^Y equipped with the proximal topology σ_2 induced by δ_2. If $\mathcal{U}(\sigma_2)$ is the corresponding compatible HB-uniformity associated with $\mathcal{U}_w$, the unique totally bounded uniformity compatible w.r.t. δ_2, then on* $\mathbf{F}$*:*

(a) $\tau(UC\Delta_1, \mathcal{U}(\sigma_2)) \leq \tau(UC\Delta_1, \mathcal{V}^H)$.

(b) $\tau(UC,\mathcal{U}(\sigma_2)) \leq \tau(UC,\mathcal{V}^H)$.
(c) (Cf. Prop. 4.2 in [24] and Prop. 4.1 in [10]) *If X is a Hausdorff space, Y a locally compact space and $\tau_2(F)$ the Fell topology on 2^Y, then $\tau(UCC,\mathcal{U}(F)) \leq \tau(UCC,\mathcal{V}^H)$.*
(d) *Furthermore, there is equality either in (a) or in (b) if and only if $\mathcal{V}$ is totally bounded.*
(e) (Cf. Prop. 4.5 in [24]) *If X is a Hausdorff space, Y a locally compact and completely metrizable space with metric d, $\mathcal{V}$ the metric uniformity associated with d and $\tau_2(F)$ the Fell topology on 2^Y, then $\tau(UCC,\mathcal{U}(F)) = \tau(UCC,\mathcal{V}^H)$ if and only if Y is compact.*

Proof. Let $\mathcal{U}_w$ and $\mathcal{U}'_w$ be the totally bounded uniformities on Y compatible with δ_2 and $\delta_2(\mathcal{V})$, respectively. From Theorems 12.7 and 12.14 in [30] $\mathcal{U}_w \subset \mathcal{U}'_w$ and $\mathcal{U}'_w \subset \mathcal{V}$ and hence $\mathcal{U}_w \subset \mathcal{V}$.

Thus (a) and (b) follow from Theorem 3.5 and (a) and (b) in Lemma 3.2.

(d) It follows from the fact that equality either in (a) or in (b) is equivalent to $\sigma_2(\delta_2) = \tau_2(\mathcal{V}^H)$, which in turn, is equivalent to the total boundedness of $\mathcal{V}$. Whereas (c) and (e) follow from above (a) and (d) respectively when $\Delta_1 = K(X)$, Remark 1.2 and the well-known relation $\tau_2(F) = \sigma_2(F,\delta_2) \leq \sigma_2(\delta_2)$. □

Next Theorem shows that in Propositions 4.6 and 4.8 in [24] the assumption of local compactness on the base space X can be dropped. First we give the following Remark.

Remark 3.10. It is well known (see [32]) that on fuction spaces the lower Vietoris topology is coarser than the topology of pointwise convergence. Thus, whenever τ_2 is a given topology on 2^Y, we have:

($\star$) $\tau(V^-) \leq \tau_p(\tau_2)$.

Theorem 3.11. *Let X be a Hausdorff space with a compatible LO- proximity δ_1, Y a Tychonoff space with a compatible EF-proximity δ_2, $\Delta_1 \subset CL(X)$, $\Delta_2 \subset CL(Y)$ a cover, $Z = X \times Y$ equipped with the product proximity $\delta = \delta_1 \times \delta_2$ and 2^Y equipped with the proximal Δ_2-topology $\sigma_2(\Delta_2)$ induced by δ_2. If $\sigma_2(\Delta_2)$ is uniformizable and $\mathcal{U}(\sigma_2(\Delta_2))$ is the corresponding compatible Δ_2-AW filter associated to $\mathcal{U}_w$, the unique totally bounded uniformity compatible w.r.t. δ_2, then on* $\mathbf{F}$*:*

(a) $\sigma(w\Delta,\delta) \leq \tau(UC\Delta_1,\mathcal{U}(\sigma_2(\Delta_2))) \leq \tau(UC,\mathcal{U}(\sigma_2(\Delta_2)))$.
(b) *Furthermore, under the conditions of Theorem (3.9) we have*
$\sigma(w\Delta,\delta) \leq \tau(UC\Delta_1,\mathcal{U}(\sigma_2(\Delta_2)) \leq \tau(UC\Delta_1,\mathcal{V}^H)$.
(Cf. [24] Prop. 4.3 and Prop. 4.4) *If X is a Hausdorff space, Y a locally compact space and $\mathcal{V}$ a compatible uniformity on Y, then*
$\tau(F) \leq \tau(UCC,\mathcal{U}(F)) \leq \tau(UCC,\mathcal{V}^H)$.

Proof. First we show (a). So, let $M = U \times V$, $U \in \tau_1$, $V \in \tau_2$ and $f \in M^-$. Hence, there is a point $(x,y) \in f \cap M$. Then by ($\star$) in the above Remark there exists a $\tau_p(\sigma_2(\Delta_2))$-neighbourhood $\mathcal{H}$ of f such that $\mathcal{H} \subset M^-$ and clearly $\mathcal{H}$ is also a $\tau(UC\Delta_1,\mathcal{U}(\sigma_2(\Delta_2))$-neighbourhood of f.

Next, suppose $D = A \times B$ where $A \in \Delta_1$, $B \in \Delta_2$ and $f \ll_\delta D^c$. Since $\delta = \delta_1 \times \delta_2$ and δ_2 is EF, there is an open set V in Y with $B \ll_{\delta_2} V$ and $f \ll_\delta (A \times V^c)$.

Let $\mathcal{U}_w$ be the unique totally bounded uniformity on Y compatible with δ_2. Thus, there is a $W \in \mathcal{U}_w$ such that $B \subset W(B) \subset W^2(B) \subset V$ (see [30]).

Clearly, $< f, A; [B, W] >$ is $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2))$-neighbourhood of f. We claim $< f, A; [B, W] > \subset (D^c)^{++}_\delta$. In fact, if $g \in < f, A; [B, W] >$, then $g(x) \in [B, W]$ for each $x \in A$. Now, $W^2(B) \cap V^c = \varnothing$ together with $g(x) \in [B, W]$ for each $x \in A$ imply $g \ll_\delta A \times V^c$. Hence $g \in (D^c)^{++}_\delta$. So the first inclusion follows. The second one is trivial.

(b) It follows from (a) above and Corollary 3.4. $\square$

Remark 3.12. By (a) in Remark 3.7 we give statements and proofs only for the $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2)))$ topology. Similar ones for the $\tau(UC\Delta_1, \mathcal{U}(\tau_2(\Delta_2)))$ topology, when $\tau_2(\Delta_2)$ is uniformizable and Δ_2 is a cover, are left to the reader.

Theorem 3.13. *Let X be a Hausdorff space with a compatible LO-proximity δ_1, Y a Tychonoff space with a compatible EF-proximity δ_2, $Z = X \times Y$ equipped with the product proximity $\delta = \delta_1 \times \delta_2$, $\Delta_2 \subset CL(Y)$ a cover and 2^Y equipped with the proximal Δ_2-topology $\sigma_2(\Delta_2)$ induced by δ_2. Let $\sigma_2(\Delta_2)$ be uniformizable and $\mathcal{U}(\sigma_2(\Delta_2))$ the corresponding compatible Δ_2-AW filter. Then on* **F***:*

(a) *If Δ_1 is the family of all finite subsets of X, then* $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2))) \leq \sigma(w\Delta)$.

(b) *If $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2))) \leq \sigma(w\Delta)$, then X is discrete.*

(Cf. [24] Prop. 4.6 and Prop. 4.8) *If X is a Hausdorff space and Y a locally compact space, then the following are equivalent:*

(α) X is discrete;

(β) $\tau(F) = \tau(UCC, \mathcal{U}(F))$;

(γ) $\tau(UCC, \mathcal{U}(F)) \leq \tau(F) \leq \tau(UCC, \mathcal{V}^H)$.

(c) *Let Y be a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y and $\mathcal{V}^H$ the corresponding HB-uniformity on 2^Y. Then the HB- uniform topology on Δ_1 $\tau(UC\Delta_1, \mathcal{V}^H)$ equals the weak proximal Δ topology $\sigma(w\Delta)$ if and only if each member of Δ_1 is finite and $\mathcal{V}$ is totally bounded.*

([24] Prop. 4.7) *If Y is a completely metrizable space with metric d and $\mathcal{V}$ is the d-metric uniformity, then $\tau(UCC, \mathcal{V}^H) = \tau(F)$ if and only if X is discrete and Y is compact.*

Proof. (a) It suffices to observe that if Δ_1 is the family of all finite subsets of X, then $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2))) = \tau_p(\sigma_2(\Delta_2))$ and clearly $\tau_p(\sigma_2(\Delta_2)) \leq \sigma(w\Delta)$.

(b) Suppose X is not discrete. Then there exists a point x_0 in X which is not isolated. Denote by $\mathcal{N}(x_0)$ the family of all open neighbourhoods of x_0 and let y_0, y_1 be two distinct points in Y. For $U \in \mathcal{N}(x_0)$ define $f_U(x) = \{y_0, y_1\}$ for $x \notin U$ and $f_U(x) = \{y_0\}$ for $x \in U$. It is easy to verify that $f_U \in \mathbf{F}$ and that the net $\{f_U : U \in \mathcal{N}(x_0)\}$ $\sigma(w\Delta)$-converges to a multifunction f defined by $f(x) = \{y_0, y_1\}$ for $x \in X$. Since $\{f_U(x_0) : U \in \mathcal{N}(x_0)\}$ does not $\tau_2(V^-)$ converge to

$f(x_0)$, it follows that $\{f_U : U \in \mathcal{N}(x_0)\}$ cannot $\tau(UC\Delta_1, \mathcal{U}(\sigma_2(\Delta_2)))$ converge to f.

(c) It follows from the above and the fact that the equality is equivalent to $\sigma_2(\Delta_2) = \tau_2(\mathcal{V}^H)$, which in turn, is equivalent to the total boundedness of $\mathcal{V}$. □

Theorem 3.14. *Let X be a Hausdorff space, Y a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y and $\mathcal{V}^H$ the corresponding HB-uniformity on 2^Y. Then on $\mathcal{C}$:*

(a) *If $\mathcal{V}$ is totally bounded, then $\tau(UC, \mathcal{V}^H) \leq \tau(rCL(Y)) \leq \tau(V)$.*
(b) *If $\mathcal{V}$ is not totally bounded, then on $\mathcal{C}$, $\tau(UC, \mathcal{V}^H) \not\leq \tau(V)$.*
Thus, if on $\mathbf{F}$ $\tau(UC, \mathcal{V}^H) \leq \tau(V)$, then $\mathcal{V}$ is totally bounded.

([24] Prop. 4.9) *If X is a locally compact space, Y a locally compact completely metrizable space with metric d and $\mathcal{V}$ the d-metric uniformity, then on $\mathbf{F}$ $\tau(UCC, \mathcal{V}^H) \leq \tau(V)$ if and only if X is discrete and Y is compact.*

Proof. (a) Let $U \in \mathcal{V}$ be open and symmetric and $f = (X \times E) \in \mathcal{C}$. Total boundedness of $\mathcal{V}$ implies there is a finite set $\{y_k : 1 \leq k \leq n\}) \subset Y$ such that $E \subset \bigcup_{k=1}^{n} U(y_k)$. Consider a typical $\tau(UC, \mathcal{V}^H)$- neighbourhood of f, i.e. $< f, X, U^2 >$. It is easy to see that $[X \times U(E)]^+ \cap \bigcap_{k=1}^{n} U(y_k)^-$ is a $\tau(wV)$-neighbourhood of f which is contained in $< f, X, U^2 >$.

(b) If $\mathcal{V}$ is not totally bounded there is an open $U \in \mathcal{V}$ and a sequence $\{y_k : k \in IN\} \subset Y$ such that $Y \not\subset \bigcup_{k=1}^{n} U(y_k)$ for each $n \in IN$. Then it is clear that $\mathcal{F}_\mathcal{C} = \{f = X \times E : E \subset Y$ is finite$\}$ is a subset of $\mathcal{C}$ which is not dense in $(\mathcal{C}, \tau(UC, \mathcal{V}_H))$ but which is dense in $(\mathcal{C}, \tau(V))$. □

Proposition 3.15. *Let X be a Hausdorff space, Y a Tychonoff space, $\mathcal{V}$ a compatible uniformity on Y and $\mathcal{V}^H$ the corresponding HB- uniformity on 2^Y. If on $\mathbf{F}$ $\tau(V) \leq \tau(UC\Delta_1, \mathcal{V}^H)$, then $X \in \Delta_1$ and Y is Atsuji (i.e. $\Delta_1 = CL(X)$ and every real-valued continuous function on Y is uniformly continuous).*

Proof. First we show $X \in \Delta_1$. Assume not and let y_1, y_2 be distinct points of Y. Define $f = X \times \{y_1\}$ and $D = X \times \{y_2\}$. Then $f \in (D^c)^+$ but for any $A \in \Delta_1$ and any $V \in \mathcal{V}$, $< f, A; V_H >$ is not contained in $(D^c)^+$. In fact, choose $x' \in (X \setminus A)$ (which exists since we are assumming $X \notin \Delta_1$) and set $g = f \cup \{(x', y_2)\}$ then $g \in < f, A; V_H >$, but $g \notin (D^c)^+$ because $(x', y_2) \in g(x') \cap D$; a contradiction. Then, the result follows from the fact that on $\mathcal{C}$ $\tau(V) \leq \tau(UC, \mathcal{V}^H)$ if and only if $\tau_2(V) \leq \tau_2(\mathcal{V}^H)$ on $CL(Y)$ which in turn is equivalent to Y being Atsuji. □

Next Example suggested by Ľubica Holá shows that the converse is not in general true.

Example 3.16. Let $X = [1, +\infty)$ and $Y = [0, 1]$ subspaces of the real line and $\mathcal{V}^H$ the Hausdorff metric uniformity on 2^Y. Set $f = X \times \{0\}$ and for each natural number n define $f_n = X \times \{\frac{1}{n}\}$. Then the sequence $\{f_n : n \in IN\}$ $\tau(UC, \mathcal{V}^H)$-converges to f but it fails to $\tau(V)$-converge to f. In fact, take $G = \{(x, y) \in X \times Y : y < \frac{1}{x}\}$. Then $f \in G$ but $f_n \notin G$, for each $n \in IN$.

However, if $\Delta_1 = K(X)$ we have the next result.

Proposition 3.17. *Let X be a Hausdorff space, Y a Tychonoff and locally Δ_2 space, $\mathcal{V}$ a compatible uniformity on Y, then on $\mathbf{F}$ $\tau(V) \leq \tau(UCC, \mathcal{V}^H)$ if and only if X is compact and Y is Atsuji.*

> (Cf. [24] Prop. 4.10) *If X is a Hausdorff space, Y a locally compact and completely metrizable space with metric d and $\mathcal{V}$ the d-metric uniformity, then on $\mathbf{F}$ $\tau(V) \leq \tau(UCC, \mathcal{V}^H)$ if and only if X is compact and Y is a topological sum of a compact space and a discrete space.*

Proof. It is known from [29] that on $C(X, Y)$, the family of all continuous functions on X to Y, the graph topology equals the Vietoris topology. Moreover, observe that $\tau(UCC, \mathcal{V}^H)$ on $C(X, Y)$ equals the compact open topology τ_k. Thus, from a result analogous to 2(d) page 14 of [26] it follows that on $C(X, Y)$, $\tau(V) \leq \tau_k$ if and only if $X \in K(X)$. The statement then follows from the known result that Y is Atsuji if and only if $\tau_2(V) \leq \tau_2(\mathcal{V}^H)$. □

The proof of the next Proposition is left to the reader.

Proposition 3.18. *Let X and Y be Hausdorff spaces. Then on* $\mathbf{F}$
$\tau(\Delta) = \tau(V)$ *if and only if* $Z \in \Delta$ *i.e.* $\Delta = CL(Z)$.

> ([24] Prop. 4.11) *$\tau(F) = \tau(V)$ if and only if Z is compact i.e. X and Y are both compact.*

To study comparisons between the pseudo uniform topologies with some other topologies we give the following Lemma.

Lemma 3.19. *Let X be a Hausdorff space, Y a Tychonoff and locally Δ_2 space, δ_1 a compatible LO-proximity on X, δ_2 a compatible EF-proximity on Y, $Z = X \times Y$ equipped with the product proximity $\delta = \delta_1 \times \delta_2$, $\tau_2(V^-)$ and $\sigma_2(\Delta_2)$ respectively the lower Vietoris topology and the proximal Δ_2-topology on 2^Y. Then on $\mathbf{F}$:*

(a) $\tau_p(\tau_2(V^-)) \leq ps(\tau(UC\Delta_1, \sigma_2(\Delta_2))) \leq ps(\tau(UC, \sigma_2(\Delta_2)))$.

If Y is a Hausdorff and locally Δ_2 space and 2^Y is equipped with the Δ_2-topology $\tau_2(\Delta_2)$, then:

(b) $\tau_p(\tau_2(V^-)) \leq ps(\tau(UC\Delta_1, \tau_2(\Delta_2))) \leq ps(\tau(UC, \tau_2(\Delta_2)))$.

Proof. We prove only (a). To show (b) few changes are needed.

Suppose $< f, \{x\}, V^- >$ is a $\tau_p(\tau_2(V^-))$ neighbourhood of f, where $V \in \tau_2$ and $f \in \mathbf{F}$. So there is a point $y \in f(x) \cap V$. Since Y is locally Δ_2, there is a $D \in \Delta_2$ such that $y \in intD \subset D \subset V$. Since the proximity δ_2 is EF we also have $y \in intD \subset D \ll V$. Then $W = (V^- \times V^-) \cup [(D^c)^{++} \times (D^c)^{++}]$ is a symmetric neighbourhood of the diagonal in $(2^Y \times 2^Y, \sigma_2 \times \sigma_2)$. Clearly, $f \in W^\star(f, \{x\}) \subset V^-$. □

Theorem 3.20. *Let X be a Hausdorff space with a compatible LO-proximity δ_1, Y a Tychonoff space with a compatible EF-proximity δ_2, $Z = X \times Y$ equipped with the product proximity $\delta = \delta_1 \times \delta_2$ and 2^Y equipped with the proximal Δ_2-topology $\sigma_2(\Delta_2)$ induced by δ_2. If Y is locally Δ_2, then on $\mathbf{F}$:*

(a) $\sigma(w\Delta, \delta) \leq ps(\tau(UC\Delta_1, \sigma_2(\Delta_2))) \leq ps(\tau(UC, \sigma_2(\Delta_2)))$.

If Y is a Hausdorff and locally Δ_2 space and 2^Y is equipped with the Δ_2 topology $\tau_2(\Delta_2)$, then:

(b) $\tau(w\Delta) \leq ps(\tau(UC\Delta_1, \tau_2(\Delta_2))) \leq ps(\tau(UC, \tau_2(\Delta_2)))$.

Proof. Again it suffices to show (a). By above Lemma and Remark 3.10 it suffices to show that on $\mathbf{F}$

$$\sigma(w\Delta, \delta)^+ \leq ps(\tau(UC\Delta_1, \sigma_2(\Delta_2))) \leq ps(\tau(UC, \sigma_2(\Delta_2))).$$

Thus, suppose $D = A \times B$ where $A \in \Delta_1$, $B \in \Delta_2$ and $f \ll D^c$. There is an open set V in Y with $B \ll_{\delta_2} V$ and such that $f \ll A \times V$ (because δ_2 is EF). Set $S = (V^- \times V^-) \cup [(B^c)^{++} \times (B^c)^{++}]$ a symmetric neighbourhood of the diagonal in $(2^Y \times 2^Y, \sigma_2 \times \sigma_2)$. Then $f \in S^\star(f, A) \subset (D^c)^{++}$. So the first inclusion follows. The second one is trivial.

Clearly (b) follows from above with obvious changes. □

Theorem 3.21. *Let X and Y be Hausdorff spaces with Y locally Δ_2, δ_1 a compatible LO-proximity on X, δ_2 a compatible LO-proximity on Y and $\delta = \delta_1 \times \delta_2$ the product proximity on $Z = X \times Y$. Let σ_2 and τ_2 denote the proximal Δ_2-topology and the Δ_2-topology on 2^Y, respectively. Then on $\mathbf{F}$:*

(a) *If Δ_1 is the family of all finite subsets of X, then $ps(\tau(UC\Delta_1, \sigma_2)) \leq \sigma(w\Delta)$ and $ps(\tau(UC\Delta_1, \tau_2)) \leq \tau(w\Delta)$.*

(b) *If $ps(\tau(UC\Delta_1, \sigma_2)) \leq \sigma(w\Delta)$ or $ps(\tau(UC\Delta_1, \sigma_2)) \leq \tau(w\Delta)$, then X is discrete.*

Proof. To check (a) observe that if Δ_1 is the family of all finite subsets of X, then $ps(\tau(UC\Delta_1, \sigma_2(\Delta_2))) = \tau_p(\sigma_2(\Delta_2))$ as well as $ps(\tau(UC\Delta_1, \tau_2(\Delta_2))) = \tau_p(\tau_2(\Delta_2))$ and clearly $\tau_p(\sigma_2(\Delta_2)) \leq \sigma(w\Delta)$ as well as $\tau_p(\tau_2(\Delta_2)) \leq \tau(w\Delta)$.

(b) We prove only the second part, i.e if $ps(UC\Delta_1, \tau_2(\Delta_2)) \leq \tau(w\Delta)$, then X is discrete. Assume not. Then there exists a point x_0 in X wich is not isolated. Denote by $\mathcal{N}(x_0)$ the family of all open neighbourhoods of x_0 and let y_0, y_1 be two different points in Y. For $U \in \mathcal{N}(x_0)$ define $f_U(x) = \{y_0, y_1\}$ for $x \notin U$ and $f_U(x) = \{y_0\}$ for $x \in U$. It is easy to verify that $f_U \in \mathbf{F}$ and that the net $\{f_U : U \in \mathcal{N}(x_0)\}$ $\tau(w\Delta)$-converges to f defined by $f(x) = \{y_0, y_1\}$

for $x \in X$. Since $\{f_U(x_0) : U \in \mathcal{N}(x_0)\}$ does not $\tau_2(V^-)$ converge to $f(x_0)$, it follows that $\{f_U : U \in \mathcal{N}(x_0)\}$ cannot $ps(\tau(UC\Delta_1, \tau_2(\Delta_2)))$ converge to f. $\square$

Theorem 3.22. *Let X and Y be Hausdorff spaces. If Y is locally Δ_2, then on $\mathbf{F}$ $\tau(wV) \leq ps(\tau(UC\Delta_1, \tau_2(\Delta_2)))$ if and only if $X \in \Delta_1$ and $Y \in \Delta_2$ (i.e. $\Delta_1 = CL(X)$ and $\Delta_2 = CL(Y)$).*

> (Cf. [24] Prop. 4.12) *If X is a Hausdorff space and Y is a locally compact space, then $\tau(V) \leq \tau(UCC, \mathcal{U}(F))$ if and only if Z is compact i.e. X and Y are both compact.*

Proof. First observe that from Remark 2.8 (g) it follows that on $\mathcal{C}$ $\tau(wV) \leq ps(\tau(UC\Delta_1, \tau_2(\Delta_2))) \Leftrightarrow \tau_2(V) \leq \tau_2(\Delta_2) \Leftrightarrow Y \in \Delta_2$ i.e. $\Delta_2 = CL(Y)$. Next, by (b) in Theorem 3.20 to show that $\tau(wV) \leq ps(\tau(UC\Delta_1, \tau(V)))$ is equivalent to $X \in \Delta_1$ it suffices to prove that the inequality $\tau(wV) \leq ps(\tau(UC\Delta_1, \tau_2(V))$ implies $X \in \Delta_1$. Assume not. Let y_1, y_2 be distinct points of Y. Set $f = X \times \{y_1\}$ and $D = X \times \{y_2\}$. Clearly, $D \in CL(X) \times CL(Y)$ and $f \in (D^c)^+$. We claim that for each $ps(\tau(UC\Delta_1, \tau_2(V)))$ neighbourhood $W^\star(f, A)$ of f there exists $g \in W^\star(f, A)$ such that $g \notin (D^c)^+$. In fact, since $A \neq X$ there exists some $x' \in (X \setminus A)$. Set $g = f \cup \{(x', y_2)\}$. Then $g \in W^\star(f, A)$ but $g \notin (D^c)^+$ showing thereby that X must be in Δ_1. $\square$

The following Example, due to Ľubica Holá, shows that the uniform Hausdorff convergence topology $\tau(UC, \mathcal{V}^H)$ is in general not finer than the pseudo proximal uniform topology $ps(\tau(UC, \sigma_2))$.

Example 3.23. In the real line with the usual metric d, set $X = \bigcup_{n \in IN} (\frac{1}{n+1}, \frac{1}{n})$. Let $Y = X$, $\mathcal{V}$ the metric uniformity on Y associated to d, $V_n = (\frac{1}{n+1}, \frac{1}{n})$, $\eta_n = \frac{1}{n} - \frac{1}{n+1}$ and $y_n \in V_n$ be fixed for each $n \in IN$. Let $f\colon X \to Y$ defined by $f(V_n) = y_n$. Let $W = \bigcup_{n \in IN} (V_n^- \times V_n^-) \cup \{\varnothing, \varnothing\}$. Then W is a symmetric $\sigma_2 \times \sigma_2$ open neigbourhood of the diagonal of $2^Y \times 2^Y$ such that the corresponding $W^\star(f, X) = \{g \in \mathbf{F} : \forall x \in X (f(x), g(x)) \in W\} \notin \tau(UC, \mathcal{V}^H)$.

Assume not, i.e. $W^\star(f, X) \in \tau(UC, \mathcal{V}^H)$. Then there exists a positive real ε such that $< f, X; \varepsilon > \subset W^\star(f, X)$, where $< f, X; \varepsilon > = \{g \in \mathbf{F} : H_d(f(x), g(x)) < \varepsilon \quad \forall x \in X\}$ (here H_d denotes the Hausdorff distance associated to d). Let $\eta < \varepsilon$. For each $x \in X$, set $g(x) = f(x) + \eta$ and let $n \in IN$ be such that $\eta_n < \eta$. Of course $g \in < f, X; \varepsilon >$, but $g \notin W^\star(f, X)$. In fact choose $x_n = y_n \in V_n$. Then $f(x_n) \in V_n$, but $g(x_n) \notin V_n$.

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