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Orderability and continuous selections for Wijsman and Vietoris hyperspaces

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. Bertacchi and Costantini obtained some conditions equivalent to the existence of continuous selections for the Wijsman hyperspace of ultrametric Polish spaces. We introduce a new class of hypertopologies, the macro-topologies. Both the Wijsman topology and the Vietoris topology belong to this class. We show that subject to natural conditions, the base space admits a closed order such that the minimum map is a continuous selection for every macro-topology. In the setting of Polish spaces, these conditions are substantially weaker than the ones given by Bertacchi and Costantini. In particular, we conclude that Polish spaces satisfying these conditions can be endowed with a compatible order and that the minimum function is a continuous selection for the Wijsman topology, just as it is for $[0, 1]$. This also solves a problem implicitely raised in Bertacchi and Costantini's paper.

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1. Introduction.

We define a "macro-topology" to be an admissible hyperspace topology, finer than the lower Vietoris topology. This new class of hypertopologies contains the Wijsman topology, the Vietoris topology and a rich subclass of Δ-topologies. With the help of a new object called "lexor" and using the "extra-dense" sets (new objects as well), we construct a suitable order on a generic topological space. We give conditions under which this order is closed, showing its "sub-compatibility" under some further, but natural hypotheses. The properties of this order make it possible to prove the continuity of the minimum function $A \to min(A)$ from a "macro-hyperspace" to the base space.

When the base space is metrizable and complete, in particular, we find a sufficient condition for the existence of such an order, and consequently, for the minimum map to be a continuous selection for the Wijsman hyperspace, just as happens in the setting of compact orderable metric spaces, such as $[0, 1]$.

The result about Wijsman hyperspaces leads to a deeper study of the conditions that in [3] the authors assign on a metric space (X, d) in order to obtain a continuous selection for the associated Wijsman hyperspace. As a final result, they prove that every separable complete "ultrametric" space has a Wijsman continuous selection if and only if a further condition, called "condition $(\sharp)$", holds at each point. We focus our attention on a particular subfamily of $\mathbb{R}^{P(X)\setminus\{\varnothing\}}$, the collection of all the real-valued functions defined on the set of all nonempty subsets of X, whose elements are determined by a finite number of points and real numbers. We call "n-coordinated-functions" the elements of this family determined by n points and n real numbers. Starting on them, we introduce the notions of "n-coordinated-set" and "star-set" (when $n = 1$), showing the existence of a natural relationship between both of them and the Wijsman basic and subbasic open subsets, respectively. Having a base of star-sets is proved to be both a weaker condition than condition $(\sharp)$ given in [3], and a stronger condition than the one developed to prove the existence of the order.

2. Preliminaries.

Let X be a nonempty set. We denote with (X, τ) a topological space and with (X, d) a metric space, which is understood to be endowed with the topology induced by the metric d. Given a topology on X, let $CL(X)$ be the collection of all nonempty closed subsets of X and $C(X, \mathbb{R})$ the set of all the real-valued continuous functions on X.

For every $E \subseteq X$, $\overline{E}$ and E^c stand for the closure and the complement of E in X, respectively. We also set:

$$E^- = \{A \in CL(X) : A \cap E \neq \varnothing\},$$

$$E^+ = \{A \in CL(X) : A \subseteq E\} = \{A \in CL(X) : A \cap E^c = \varnothing\}.$$

For every $V \subseteq X$, $CL(X)\setminus(V^c)^+ = V^-$.

Let (X, τ) be a topological space and Δ be a nonempty subfamily of $CL(X)$. The Δ-*topology* τ_Δ on $CL(X)$ has as a subbase all sets of the form U^-, where U is an open set, plus all the sets of the form $(B^c)^+$, where $B \in \Delta$ (see [13]).

When $\Delta = CL(X)$, the corresponding Δ-topology is the well-known Vietoris topology.

Let:

$\mathcal{V}^- = \{U^- : U \text{ } open \text{ } in \text{ } X\}$,

$\mathcal{V}^+ = \{U^+ : U \text{ } open \text{ } in \text{ } X\}$.

The family $\mathcal{V}^-$ forms a subbase for the *lower Vietoris topology* τ_V^- on $CL(X)$; while the family $\mathcal{V}^+$ determines a subbase for the *upper Vietoris topology* τ_V^+ on $CL(X)$. The supremum of these two hypertopologies is the *Vietoris topology*: $\tau_V = \tau_V^+ \bigvee \tau_V^-$.

In general, given $\Delta \subseteq CL(X)$, $\tau_\Delta = \tau_\Delta^+ \bigvee \tau_V^-$, where τ_Δ^+ denotes the *upper* Δ*-topology* on $CL(X)$.

Let (X, d) be a metric space. The open ball with center $x \in X$ and radius $\epsilon > 0$ is given by $S_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. The diameter of a nonempty subset A of X and the distance from $x \in X$ to A are expressed by the familiar formulas:

$diam(A) = sup\{d(a, b) : a, b \in A\}$ and

$d(x, A) = inf\{d(x, a) : a \in A\}$.

For every $x \in X$, $d(x, -)$ denotes the distance functional from $CL(X)$ to $\mathbb{R}$ defined by $d(x, -)(A) = d(x, A)$ for every $A \in CL(X)$. The *Wijsman Topology* τ_{W_d} on $CL(X)$ is the weak topology determined by the family of distance functionals $\{d(x, -) : x \in X\}$. Equivalently, the Wijsman Topology on $CL(X)$ can be defined by having as a subbase all the sets of the form:

$\mathcal{A}^-(x, \alpha) = \{A \in CL(X) : d(x, A) < \alpha\}$ and

$\mathcal{A}^+(x, \alpha) = \{A \in CL(X) : d(x, A) > \alpha\}$,

where $x \in X$ and $\alpha > 0$.

A net of closed subsets of X, $\{A_\lambda\}_{\lambda \in \Lambda}$, τ_{W_d}-converges to $A \in CL(X)$ if for every $x \in X$, $\lim_\lambda d(x, A_\lambda) = d(x, A)$, i.e. $(CL(X), \tau_{W_d})$ can be embedded in $C(X, \mathbb{R})$, equipped with the topology of the pointwise convergence, under the identification map $A \to d(-, A)$ (cf. Sections 1.2 and 2.1 in [1]).

We can also present the Wijsman topology as split in two halves (Section 4.2 in [1]; see also [5], [8], [11]): $\tau_{W_d} = \tau_{W_d}^+ \bigvee \tau_{W_d}^-$. A subbase for $\tau_{W_d}^+$ consists of all the sets of the form $\mathcal{A}^+(x, \alpha)$ ($x \in X$ and $\alpha > 0$), whereas $\tau_{W_d}^-$, coinciding with the lower Vietoris topology, has as a subbase all the sets of the form U^- (U open in X).

A map $f : CL(X) \to X$ is a *selection* for $CL(X)$ if $f(C) \in C$ for every $C \in CL(X)$. By *continuous selection* we mean a selection $f : CL(X) \to X$ also continuous with respect to the hypertopology on $CL(X)$.

If X is a set linearly ordered by a relation $<$, we denote by $\tau_<$ the order topology induced by $<$ on X, i.e. the topology having as a subbase all the rays $(\leftarrow, x)$ and $(x, \rightarrow)$, where $x \in X$. In particular, all the intervals of the form

(a,b), where $a<b$, are open with respect to $\tau_<$. Given a topology τ on X, the order relation $<$ on X is called *compatible w.r.t.* τ if $\tau=\tau_<$ and *closed* if the set $(X\times X)\setminus\leq$ is open in the product topology on $X\times X$ ($\leq$ is the partial order induced by $<$). Every compatible order is a closed order. We denote by $\langle x,y\rangle$ a point in $X\times X$. We will also use the following terminology: *$<$-interval* for any of the possible intervals (a,b), $[a,b)$, $(a,b]$ or $[a,b]$, where $a\leq b$; *right $<$-ray* for any ray of the form $(a,\rightarrow)$ or $[a,\rightarrow)$; *left $<$-ray* for any ray of the form $(\leftarrow,a)$ or $(\leftarrow,a]$.

3. Admissibility and macro-topologies.

Let (X,τ) be a topological space. A hyperspace topology on $CL(X)$ is called *admissible* if the relative topology induced on X by the identification map $x\rightarrow\{x\}$, coincides with the initial topology on X ([12]).

It is understood that X must satisfy some separation properties in order to get the admissibility of the corresponding hyperspace. If (X,τ) is a T_1-space, then τ_Δ is admissible (Remark 5.1 in [7]): in particular, the Vietoris topology is admissible, whenever the base space is T_1. If (X,d) is a metric space, then the Wijsman topology on $CL(X)$ is admissible (Lemma 2.1.4 in [1]).

Definition 3.1. Let (X,τ) be a topological space. A topology Ω on $CL(X)$ is called a *macro-topology* if it is admissible and $\tau_V^-<\Omega$.

If (X,τ) is a T_1-space, then every Δ-topology on $CL(X)$ is a macro-topology: in particular, the Vietoris topology is a macro-topology. It is also clear that, given a metric space (X,d), the relative Wijsman topology is a macro-topology.

The following will turn out to be a useful result.

Lemma 3.2. *Let (X,τ) be a T_1-space and H any hypertopology on $CL(X)$ such that the identification map $x\rightarrow\{x\}$ is continuous. Let $B\subseteq X$. If B^- is a H-closed subset of $CL(X)$, then B is a closed subset of X.*

Proof. Suppose B is not closed in X, i.e. there exists $x\in\overline{B}\setminus B$. Then there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ of points of B converging to x. Since the identification map is continuous, $\{\{x_\alpha\}\}_{\alpha\in\Lambda}$ converges to $\{x\}$ with respect to H. For every $\alpha\in\Lambda$, $\{x_\alpha\}\in B^-$, hence $\{x\}$ must be in $cl_H(B^-)=B^-$, so that $x\in B$. $\square$

Corollary 3.3. *Let (X,τ) be a topological space, Ω a macro-topology on $CL(X)$ and $B\subseteq X$. If B^- is a Ω-closed subset of $CL(X)$, then B is a closed subset of X.*

Corollary 3.4. *Let (X,τ) be a T_1-space and $B\subseteq X$. If B^- is a τ_Δ-closed subset of $CL(X)$, then B is a closed subset of X.*

Corollary 3.5. *Let (X,d) be a metric space and $B\subseteq X$. If B^- is a τ_{W_d}-closed subset of $CL(X)$, then B is a closed subset of X.*

Remark 3.6. Let (X,τ) be a topological space and Ω a macro-topology on $CL(X)$. If $A\in CL(X)$ and $U=A^c$, then the following two chains of implications are equivalent:

- A^- is Ω-closed $\Rightarrow A$ is closed in $X \Rightarrow A^+$ is Ω-closed;
- U^+ is Ω-open $\Rightarrow U$ is open in $X \Rightarrow U^-$ is Ω-open.

The implication U *is open in* $X \Rightarrow U^-$ *is* Ω*-open* follows from $\tau_V^- < \Omega$; while A^- *is* Ω*-closed* $\Rightarrow A$ *is closed in* X has just been shown (Corollary 3.3).

4. When the minimum map is a continuous selection for macro-hyperspaces.

Proposition 4.1. *Let (X, τ) be a topological space, H be any hypertopology on $CL(X)$ and $<$ be a linear order on X such that τ is generated by a family $\mathcal{R}$ of right $<$-rays. If:*

(i) *for every $A \in CL(X)$, $min(A)$ exists;*
(ii) *for every $R \in \mathcal{R}$, R^+ is a H-open subset of $CL(X)$.*

Then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), H)$ to (X, τ).

Proof. Only the continuity of the mapping $A \to min(A)$ needs to be proved. Let $A \in CL(X)$ and R be a basic open right $<$-ray containing $min(A)$. Then R^+ is H-open neighbourhood of A such that for every $C \in R^+$, $min(C) \in R$. $\square$

The dual of Proposition 4.1 is also true: we leave the proof to the reader.

Proposition 4.2. *Let (X, τ) be a topological space, H be any hypertopology on $CL(X)$ and $<$ be a linear order on X such that τ is generated by a family $\mathcal{L}$ of left $<$-rays. If:*

(i) *for every $A \in CL(X)$, $min(A)$ exists;*
(ii) *for every $L \in \mathcal{L}$, L^- is a H-open subset of $CL(X)$.*

Then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), H)$ to (X, τ).

Definition 4.3. Let (X, τ) be a topological space. A linear order $<$ on X is called *sub-compatible w.r.t.* τ if $\tau_< \leq \tau$ and τ has a subbase consisting of right and left $<$-rays.

A topological space (X, τ) is sub-orderable (see 14.B.13 and 15.A.14 in [4]) if there exists a linear order $<$ sub-compatible w.r.t. τ. It is well-known that the class of sub-orderable space concides with the one of subspaces of orderable spaces (17.A.22 in [4]). Obviously, every compatible order is sub-compatible, just as every ordarable space is also sub-orderable.

A combination of Proposition 4.1 and Proposition 4.2 leads to the following:

Proposition 4.4. *Let (X, τ) be a topological space, H be any hypertopology on $CL(X)$ and $<$ be sub-compatible w.r.t. τ. If:*

(i) *for every $A \in CL(X)$, $min(A)$ exists;*
(ii) *for every subbasic open right $<$-ray R, R^+ is a H-open subset of $CL(X)$;*
(iii) *for every subbasic open left $<$-ray L, L^- is a H-open subset of $CL(X)$.*

Then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), H)$ to (X, τ).

Given a sub-orderable space, conditions (ii) and (iii) of the previous proposition indeed quite often can be verified: for instance if H is any Δ-topology, where Δ contains all the left rays which are complement of subbasic right rays (see Lemma 3.8 in [6]); or, more concretely, if H is the Vietoris topology. After Section 6 and the main result, it will be clear that (ii) holds true for the Wijsman topology if X is a complete metric space having a countable base with a peculiar property (see Corollary 7.2).

In case the base space is orderable, the subbasic open rays involved are of the form $(\leftarrow, x)$ and $(x, \rightarrow)$, where $x \in X$. Moreover, if H is finer than the lower Vietoris topology, condition (iii) is always satisfied. Therefore, the following are immediate consequences of Proposition 4.4.

Corollary 4.5. *Let (X, τ) be a topological space, Ω be a macro-topology on $CL(X)$ and $<$ be a compatible linear order on X. Suppose that:*

(i) *for every $A \in CL(X)$, $min(A)$ exists;*

(ii) *for every $x \in X$, $(x, \rightarrow)^+$ is a Ω-open subset of $CL(X)$.*

Then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \Omega)$ to (X, τ).

Corollary 4.6. *Let (X, d) be a metric space whose topology admits a compatible linear order $<$ with respect to which each $A \in CL(X)$ has a smallest element. If for every $x \in X$, $(x, \rightarrow)^+$ is a τ_{W_d}-open subset of $CL(X)$, then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

Another consequence of Proposition 4.4 is a classical and well-known result about the existence of continuous selections for the Vietoris hyperspace of orderable spaces ([9], [10]).

Corollary 4.7. *Let (X, τ) be a topological space whose topology admits a compatible order $<$ with respect to which each closed subset has a smallest element. Then the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \tau_V)$ to (X, τ).*

The following includes an important result of Beer, Lechicki, Levi and Naimpally (Corollary 5.6 in [2]): we give here a more direct and easier proof.

Proposition 4.8. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is compact;*

(2) *$\tau_{W_d} = \tau_V$ on $CL(X)$;*

(3) *$\tau_{W_d}^+ = \tau_V^+$ on $CL(X)$.*

Proof. (2) $\Leftrightarrow$ (3). It follows from the fact that $\tau_{W_d}^- = \tau_V^-$ and $\tau_{W_d}^+ \leq \tau_V^+$ always happens.

(1) $\Rightarrow$ (3). Let U be open in X and $A \in U^+$. U^c is closed in X, and hence compact. Also $A \cap U^c = \varnothing$. Since A and U^c are closed, $d(x, A) > 0$, whenever $x \in U^c$. For every $x \in U^c$, choose r_x such that $0 < r_x < d(x, A)$. The family

$\{S_d(x, r_x) : x \in U^c\}$ is an open cover for the compact U^c. Then there exist $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n > 0$ ($n \in \omega$) such that $U^c \subseteq \bigcup_{i=1}^n S_d(x_i, r_i)$ and $A \cap S_d(x_i, r_i) = \varnothing$ for every $i = 1, \cdots, n$. Then, $A \in \bigcap_{i=1}^n \mathcal{A}^+(x_i, r_i) \subseteq U^+$.

(3) $\Rightarrow$ (1). If $\tau^+_{W_d} = \tau^+_V$, X is separable (see following Remark) and $(CL(X), \tau_{W_d}) = (CL(X), \tau_V)$ is metrizable (see Theorem 2.1.5 in [1]). By Theorem 4.6 in [12], X is compact. □

Remark 4.9. Let (X, d) be a metric space and $\tau_{\delta(d)}$ be the topology on $CL(X)$ generated by all the sets of the form V^- and $V^{++} = \{F \in CL(X) : d(F, V^c) > 0\}$, where V runs over the open subsets of X. We can rappresent this topology as splitted in two parts: $\tau_{\delta(d)} = \tau^+_{\delta(d)} \bigvee \tau^-_V$: it is called *d-proximal topology* (see [1], [2], [8] among the others). It is known, but it is also easy to verify, that $\tau^+_{W_d} \leq \tau^+_{\delta(d)} \leq \tau^+_V$. So, if $\tau^+_{W_d} = \tau^+_V$, then $\tau^+_{W_d} = \tau^+_{\delta(d)}$. By Lemma 5.4 in [2] (*If (X, d) is a metric space which is not second countable, then $\tau_{W_d} \neq \tau_{\delta(d)}$.*), X is separable.

Corollary 4.10. *Let (X, d) be a compact orderable metric space. The following are equivalent:*

(*a*) *for every $x \in X$, $(x, \rightarrow)^+$ is a τ_V-open subset of $CL(X)$;*
(*b*) *for every $x \in X$, $(x, \rightarrow)^+$ is a τ_{W_d}-open subset of $CL(X)$.*

Remark 4.11. It follows immediately from Corollary 4.10 that Corollary 4.6 and Corollary 4.7 are equivalent formulations of the same result for compact orderable metric spaces.

Corollary 4.6 (or Corollary 4.7) and Proposition 4.8 yield:

Corollary 4.12. *Let (X, d) be a compact orderable metric space. Then the mapping $A \rightarrow min(A)$ is a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

Corollary 4.13. *The mapping $A \rightarrow min(A)$ is a continuous selection from $(CL([0, 1]), \tau_{W_d})$ to $([0, 1], d)$, where d denotes the restriction to $[0, 1]$ of the Euclidean metric on $\mathbb{R}$.*

5. Introducing a closed linear order on X: lexors and extra-dense sets.

We have just shown (in Section 4) how the existence of a sub-compatible order on a topological space (X, τ) with well specified properties ((*i*), (*ii*) and (*iii*) of Proposition 4.4), is a sufficient condition for the minimum map from $CL(X)$ to X to be a continuous selection when $CL(X)$ is endowed with any hypertopology.

But, **when does this order exist?**

Answering this question begins our main goal. This is why the present section, which focuses on the construction of such an order, can actually be considered as the heart of the paper.

We start with some useful definitions.

Definition 5.1. Let X be a set. If $\{\mathcal{U}_n\}_{n\in\omega}$ is a family of (arbitrary) covers of X satisfying the property:

($\mathcal{I}$) if $\{A_n\}_{n\in\omega}$ is a family of subsets of X such that $A_n \in \mathcal{U}_n$ for every $n \in \omega$, then $|\bigcap_{n\in\omega} A_n| \leq 1$,

and for every $n \in \omega$, $<_n$ is a well-order (of any type) on $\mathcal{U}_n$, then we call $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ a *lexor* of X.

Definition 5.2. Let X be a set and $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a lexor of X. For every $A \subseteq X$ and every $n \in \omega$, let $I_A(n) = U$ if U is the $<_n$-minimal element in $\mathcal{U}_n$ such that $(A \cap \bigcap_{m<n} I_A(m)) \cap U \neq \varnothing$. If $A = \{x\}$, then we write $I_x(n)$ instead of $I_A(n)$: for every $x \in X$, $I_x(n) = U$ if U is the $<_n$-minimal element in $\mathcal{U}_n$ such that $x \in U$. The sequence $\{I_A(n)\}_{n\in\omega}$ is called the *path of* A, and denoted by I_A (I_x if $A = \{x\}$).

Remark 5.3. Given a set X, a lexor $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ of X and $A \subseteq X$, the path I_A is well-defined, since each $<_n$ is a well-order.

Definition 5.4. Let X be a set and $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a lexor of X. Given $x, y \in X$, let $\Delta(x, y) = min\{n : I_x(n) \neq I_y(n)\}$, and $<$ be the relation on X defined by:

($\mathcal{LO}$) $x < y$ if $I_x(\Delta(x, y)) <_{\Delta(x,y)} I_y(\Delta(x, y))$.

$<$ is called the *order generated by the lexor* $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$, and denoted by $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$.

The terminology used in the definition above is justified by the following proposition.

Proposition 5.5. *Let X be a set and $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a lexor of X. The relation $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$ is a linear order on X.*

Proof. $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$ *is antisymmetric*: Let $x, y \in X$, $x \neq y$, and suppose that $I_x(n) = I_y(n)$ for every n; then $x, y \in \bigcap_{n\in\omega} I_x(n)$; a contradiction, since $|\bigcap_{n\in\omega} I_x(n)| \leq 1$. Hence, $x < y$ or $y < x$.

$<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$ *is transitive*: Let $x, y, z \in X$ be such that $x < y$ and $y < z$. If $\Delta(x, y) \leq \Delta(y, z)$, then $\Delta(x, z) = \Delta(x, y)$ and $I_x(\Delta(x, z)) = I_x(\Delta(x, y)) <_{\Delta(x,y)} I_y(\Delta(x, y)) \leq_{\Delta(x,y)} I_z(\Delta(x, y)) = I_z(\Delta(x, z))$, so that $x < z$. If $\Delta(y, z) < \Delta(x, y)$, then $\Delta(x, z) = \Delta(y, z)$ and $I_x(\Delta(x, z)) = I_x(\Delta(y, z)) = I_y(\Delta(y, z)) <_{\Delta(y,z)} I_z(\Delta(y, z)) = I_z(\Delta(x, z))$, so that $x < z$. □

Remark 5.6. We have not yet assigned a topology on X: there is nothing really topological in the discussion above. Moreover, the condition ($\mathcal{I}$) on the family of covers $\{\mathcal{U}_n\}_{n\in\omega}$ allows us to get the antisymmetry of $<$. Without this condition $<$ would still be a linear preorder.

Definition 5.7. Let (X, τ) be a topological space. A lexor $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ of X is said to be *complete* if the following property holds:

($\mathcal{II}$) If $A \in CL(X)$, $U_n \in \mathcal{U}_n$ for every $n \in \omega$ and $\{U_n \cap A\}_{n\in\omega}$ has the finite intersection property, then $|\bigcap_{n\in\omega} U_n \cap A| \neq 0$.

Proposition 5.8. *Let (X, τ) be a topological space and $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a complete lexor of X. Order X by $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$. Then for every $A \in CL(X)$, $min(A)$ exists.*

Proof. Fix $A \in CL(X)$ and consider the path at A, I_A. Then the sequence $\{A \cap I_A(n)\}_{n\in\omega}$ has the finite intersection property, and by $(\mathcal{I})$ and $(\mathcal{II})$, $|\bigcap_{n\in\omega}(A \cap I_A(n))| = 1$. Let $\{x_A\} = \bigcap_{n\in\omega}(A \cap I_A(n))$. Suppose that there exists $x \in A$ such that $x < x_A$. Then $(A \cap \bigcap_{m<\Delta(x,x_A)} I_A(m)) \cap I_x(\Delta(x, x_A)) \neq \varnothing$, contradicting the minimality of $I_A(\Delta(x, x_A)) = I_{x_A}(\Delta(x, x_A))$. Therefore, $x_A = min(A)$. □

We also introduce the notion of "extra-dense" for subsets of an ordered space.

Definition 5.9. Let $(X, <)$ be an ordered space. A subset D of X is called *extra-dense* if for every $x, y \in X$, $x < y$, $D \cap (x, y] \neq \varnothing$.

Proposition 5.10. *Let (X, τ) be a topological space and $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a complete lexor of X such that each initial segment of each $(\mathcal{U}_n, <_n)$ is clopen in X. Then $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$ is a closed order on X, with respect to which there exists an extra-dense subset D with the following property:*

($\mathcal{D}$) *for every $d \in D$, there exists $k \in \omega$ and a finite sequence $(U_1, \cdots, U_k) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_k$ such that $(\leftarrow, d) = \bigcup_{n\leq k} \bigcup_{V<_n U_n} V$.*

If, moreover, $\bigcup_{n\in\omega}\{U \setminus \bigcup_{V<_n U} V : U \in \mathcal{U}_n\}$ is a (clopen) base for τ, then τ has as a subbase all left $(<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}})$-rays of the form $(\leftarrow, x)$, where $x \in X$, and all right $(<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}})$-rays of the form $[d, \rightarrow)$, where $d \in D$; in particular $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$ is sub-compatible w.r.t. τ.

Proof. For $x \in X$ and $k \in \omega$, let $L(x, k) = \bigcap_{n\leq k}(I_x(n) \setminus \bigcup_{V<_n I_x(n)} V)$. Since the initial segments of each $(\mathcal{U}_n, <_n)$ are clopen, $L(x, k)$ is clopen: given $n \in \omega$, an initial segment of the well-order $(\mathcal{U}_n, <_n)$ is a subset of the form $\bigcup_{V<_n U} V$ for some $U \in \mathcal{U}_n$; since $L(x, k) = \bigcap_{n\leq k}((\bigcup_{V<_n \mathcal{S}(I_x(n))} V) \setminus (\bigcup_{V<_n I_x(n)} V))$, where $\mathcal{S}(I_x(n))$ is the $<_n$-successor of $I_x(n)$, and by assumption, $\bigcup_{V<_n I_x(n)} V$ and $\bigcup_{V<_n \mathcal{S}(I_x(n))} V$ are clopen, $L(x, k)$ is the intersection of finitely many clopen subsets, and hence clopen. By Proposition 5.8, $L(x, k)$ has a minimum. Moreover, given $x, y \in X$, the following is true:

(!) if $x < y$, then $x < min(L(y, \Delta(x, y))) \leq y$.

To see this, let $x < y$ and $L = L(y, \Delta(x, y))$. By definition, $I_x(\Delta(x, y)) <_{\Delta(x,y)} I_y(\Delta(x, y))$. Obviuosly, $y \notin I_x(\Delta(x, y))$. Since $y \in L$, $min(L) \leq y$. Moreover, for every n, $min(L)$ must be in the minimal element of $\mathcal{U}_n$ to intersect $L \cap \bigcap_{m<n} I_L(m)$. Consequently, $I_y(n) = I_{min(L)}(n)$ for every $n \leq \Delta(x, y)$, so that $\Delta(x, min(L)) = \Delta(x, y)$ and $x < min(L)$.

Let $D = \{min(L(x, k)) : x \in X, k \in \omega\}$. By (!), D is an extra-dense subset of X. To show $(\mathcal{D})$, let $d \in D$. Then there exists $x \in X$ and $k \in \omega$ such that $d = min(L(x, k))$. Using (!), it is possible to check that:

$$(\leftarrow, d) = \bigcup_{n \leq k} \bigcup_{V <_n I_x(n)} V.$$

In fact, let $F = \bigcup_{n \leq k} \bigcup_{V <_n I_x(n)} V$. If $a \in F$, then by definition, $a < x$ and, by (!), $a < min(L(x, \Delta(a, x))) \leq x$, where $\Delta(a, x) \leq k$. Since $L(x, k) \subseteq L(x, \Delta(a, x))$, $min(L(x, \Delta(a, x))) \leq min(L(x, k))$, so that $a < min(L(x, k))$. Viceversa, if $a \in (\leftarrow, d)$, i.e. $a < d$, then $I_a(\Delta(a, d)) <_{\Delta(a,d)} I_d(\Delta(a, d))$. Notice that $a \notin L(x, k)$, otherwise $min(L(x, k)) = d \leq a$. Hence, $\Delta(a, d) \leq k$. We show that $I_x(\Delta(a, d)) = I_d(\Delta(a, d))$, so that $a \in F$. It is simple to check that for every $n \leq k$, $I_{L(x,k)}(n) = I_x(n)$. On the other hand, since $d \in I_{L(x,k)}(n)$ for every n, $I_{L(x,k)}(n) = I_d(n)$ whenever $n \leq k$ (otherwise, there would exist $n \leq k$ such that $I_d(n) <_n I_{L(x,k)}(n) = I_x(n)$ and $I_d(n) \cap L(x, k) \neq \varnothing$, while for every $n \leq k$ and every $V <_n I_x(n)$, $V \cap L(x, k) = \varnothing$). Hence, $I_x(n) = I_d(n)$ for every $n \leq k$. In particular, $I_x(\Delta(a, d)) = I_d(\Delta(a, d))$.

Let $x, y \in X$ such that $y < x$ (i.e. $\langle x, y\rangle \in (X \times X) \setminus \leq$). To show the closeness of $<$ we need to find an open neighbourhood of $\langle x, y\rangle$ in the product topology on $X \times X$ entirely contained in the complement of $\leq$. By (!), $y < d \leq x$, where $d = min(L(x, \Delta(x, y))) \in D$, and $L(x, \Delta(x, y))$ is open subset of X containing x. By $(\mathcal{D})$, $(\leftarrow, d)$ is the union of finitely many initial segments, so it is clopen in X. Hence the set $L(x, \Delta(x, y)) \times (\leftarrow, d)$ is the required neighbourhood of $\langle x, y\rangle$.

Finally, suppose that $\bigcup_{n\in\omega}\{U \setminus \bigcup_{V <_n U} V : U \in \mathcal{U}_n\}$ is a clopen base for the topology on X. From the closeness of $<$ it follows that $\tau_< \leq \tau$. In particular, all sets of the form $(a, \rightarrow)$ and $(\leftarrow, a)$, where $a \in X$, are open in X. Moreover, by $(\mathcal{D})$, for every $d \in D$, $[d, \rightarrow)$ is clopen in X.

Now, fix $x \in X$ and let $U \setminus \bigcup_{V <_j U} V$, where $j \in \omega$ and $U \in \mathcal{U}_j$, be a basic open neighbourhood of x. Note that $x \in U \setminus \bigcup_{V <_j U} V$ if and only if $U = I_x(j)$. Let $d = min(L(x, j))$, $T = (L(x, j) \cup \bigcup_{n \leq j} \bigcup_{V <_n I_x(n)} V)^c$ and $t = min(T)$. We claim that $x \in [d, t) \subseteq L(x, j)$. Since $L(x, j) \subseteq I_x(j) \setminus \bigcup_{V <_j I_x(j)} V$, this will prove that all $<$-intervals $[d, x)$, with $d \in D$ and $x \in X$, form a base for τ.

Since $x \in L(x, j)$, $d \leq x$. On the other hand, by definition of T, $\Delta(x, t) \leq j$ and $I_x(\Delta(x, t)) <_{\Delta(x,t)} I_t(\Delta(x, t))$, so that $x < t$.

Now suppose that there exists $y \in [d, t) \setminus L(x, j)$. Since $d \leq y$, by $(\mathcal{D})$, $y \in [d \rightarrow) = (\bigcup_{n \leq j} \bigcup_{V <_n I_x(n)} V)^c$. But $y \notin L(x, j)$, hence $y \in T$; a contradiction to the fact that $y < t$ (if $y \in T$, then $t = min(T) \leq y$). $\square$

6. The key result.

Proposition 6.1. *Let (X, τ) be a T_1 space and H be any hypertopology on $CL(X)$ such that the map $x \rightarrow \{x\}$ is continuous. Let $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ be a countable family of well-orders of type $\leq \omega$ each being an open cover for X. Suppose that:*

(*i*) *if* $\{A_n\}_{n\in\omega}$ *is a family of subsets of* X *with the finite intersection propety, such that for every* $n \in \omega$ *there exists* U_n *for which* $A_n \subseteq U_n \in \mathcal{U}_n$*, then* $|\bigcap_{n\in\omega} \overline{A_n}| = 1$*;*

(*ii*) *for every* $n \in \omega$ *and for every* $U \in \mathcal{U}_n$ *,* U^- *is a* H*-closed subset of* $CL(X)$*.*

Then there exists a closed linear order $<$ *on* X *such that:*

(1) *for every* $A \in CL(X)$*,* $min(A)$ *exists;*

(2) X *has an extra-dense subset* D *such that* $(\leftarrow, d)^-$ *is a* H*-closed subset of* $CL(X)$*, whenever* $d \in D$*.*

If, moreover, $\bigcup_{n\in\omega}\{U\setminus\bigcup_{V<_nU} V : U \in \mathcal{U}_n\}$ *is a (clopen) base for* τ*, then* $<$ *is sub-compatible w.r.t.* τ*.*

Proof. By (ii) and Lemma 3.2, each cover $\mathcal{U}_n$ consists of clopen subsets. Hence, by (i), both $(\mathcal{I})$ and $(\mathcal{II})$ hold for the family $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$, which is hence a complete lexor of X. Let $<$ be $<_{\{(\mathcal{U}_n,<_n)\}_{n\in\omega}}$. The initial segments of each $(\mathcal{U}_n, <_n)$ are unions of finitely many clopen subsets ($\mathcal{U}_n$ consists of clopen and $<_n$ is of type $\leq \omega$). Via Proposition 5.8 and Proposition 5.10, we only need to show (2).

Let $d \in D$. By $(\mathcal{D})$, there exists $k \in \omega$ and a finite sequence $(U_1, \cdots, U_k) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_k$ such that $(\leftarrow, d) = \bigcup_{n\leq k}\bigcup_{V<_nU_n} V$. Consequently, $(\leftarrow, d)^- = \bigcup_{n\leq k}\bigcup_{V<_nU_n} V^-$, which is a H-closed subset of $CL(X)$ by (ii). □

7. The main result.

Combining Proposition 6.1 and Proposition 4.4, we can formulate our main result. As a consequence of it, we derive an important result related to the problem of finding conditions, as "natural" as possible, on the base space in order to prove the existence of continuous selections for the Wijsman hyperspace. This will move our interest (see Section 8) to a comparison with a result of Bertacchi and Costantini in the setting of ultrametric Polish spaces.

Theorem 7.1. *Let* (X, τ) *be a topological space,* Ω *be a macro-topology on* $CL(X)$ *and* $\{(\mathcal{U}_n, <_n)\}_{n\in\omega}$ *be a countable family of well-orders of type* $\leq \omega$ *such that* $\bigcup_{n\in\omega}\{U\setminus\bigcup_{V<_nU} V : U \in \mathcal{U}_n\}$ *is a base for* τ *and each* $\mathcal{U}_n$ *is an open cover for* X*. Suppose that:*

(*i*) *if* $\{A_n\}_{n\in\omega}$ *is a family of subsets of* X *with the finite intersection propety, such that for every* $n \in \omega$ *there exists* U_n *for which* $A_n \subseteq U_n \in \mathcal{U}_n$*, then* $|\bigcap_{n\in\omega} \overline{A_n}| = 1$*;*

(*ii*) *for every* $n \in \omega$ *and for every* $U \in \mathcal{U}_n$ *,* U^- *is a* Ω*-closed subset of* $CL(X)$*.*

Then there exists a sub-compatible linear order $<$ *on* X *such that the mapping* $A \to min(A)$ *is a continuous selection from* $(CL(X), \Omega)$ *to* (X, τ)*.*

Corollary 7.2. *Let* (X, τ) *be a complete metrizable space and* Ω *be a macro-topology on* $CL(X)$*. If* X *has a countable base* $\mathcal{B}$ *such that* B^- *is* Ω*-clopen, whenever* $B \in \mathcal{B}$*, then there exists a sub-compatible linear order* $<$ *on* X *such* *that the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \Omega)$ to (X, τ).*

Proof. For $i \in \omega$, let $\mathcal{B}_i = \{B \in \mathcal{B} : diam(B) < \frac{1}{i}\}$ (the diameters are taken with respect to a compatible metric on X). Order each $\mathcal{B}_i$ by a well-order $<_i$ of type $\leq \omega$. Each $\mathcal{B}_i$ is a countable open cover for X, satisfying (ii) of Theorem 7.1, while $\bigcup_{i\in\omega}\{B \setminus \bigcup_{V<_i B} V : B \in \mathcal{B}_i\}$ is a base for X. Since X is complete, the family $\{\mathcal{B}_i\}_{i\in\omega}$ satisfies (i) of Theorem 7.1, as well. □

Corollary 7.3. *Let (X, d) be a complete metric space having a countable base $\mathcal{B}$ such that B^- is τ_{W_d}-clopen, whenever $B \in \mathcal{B}$. Then there exists a sub-compatible linear order $<$ on X such that the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

8. Coordinated-functions, coordinated-sets and star-sets.

Dealing with the problem of finding conditions equivalent to the existence of continuous selections for the Wijsman topology, Bertacchi and Costantini introduce the following notion (see Definition 2 in [3]), which turns out to play a fundamental role in such a research:

Definition 8.1. Let (X, d) be a metric space and $x \in X$. We say that the *condition* $(\sharp)$ holds at x if for every $\epsilon > 0$ there exist $\delta, \theta \in \mathbb{R}$, with $0 < \delta < \theta \leq \epsilon$, such that $S_d(x, \delta) = S_d(x, \theta)$.

A *Polish space* is a separable completely metrizable space. The main result of Bertacchi and Costantini (Theorem 3 in [3]) can be written as follows:

Proposition 8.2. *Let (X, d) be a Polish ultrametric space. The following are equivalent:*

- (a) *condition $(\sharp)$ holds at each $x \in X$;*
- (b) *there exists a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

In this section, we give some conditions weaker than condition $(\sharp)$: under such conditions each Polish space is proved to have a countable base satisfying the requirement of Corollary 7.2. We can then conclude, without caring about ultrametric properties, that a continuous selection not only always exists for Wijsman hyperspaces, but that there is a very natural map which is a selection, namely, the minimum map.

We introduce the following notion:

Definition 8.3. Let (X, d) be a metric space, $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n > 0$ $(n \in \omega)$. The *n-coordinated-function* determined by $x_1, \cdots, x_n$, $r_1, \cdots, r_n$ is the function $f_{x_i}^{r_i} : P(X)\setminus\{\varnothing\} \to \mathbb{R}$ defined by $f_{x_i}^{r_i}(A) = max(r_i - d(x_i, A))$, for every nonempty $A \subseteq X$. We write *coordinated-function* instead of *1-coordinated-function*.

Definition 8.4. Let (X, d) be a metric space and V be a proper subset of X. V is a *n-coordinated-set* if there exist $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n > 0$ $(n \in \omega)$ such that

(i) $f_{x_i}^{r_i}(a) \geq 0$ for every $a \in V$;
(ii) $f_{x_i}^{r_i}(V^c) < 0$.

A *coordinated-set* is a 1-*coordinated-set*.

For the next definition we need the notion of "excess". Given a metric space (X, d) and $A, B \subseteq X$, the *excess of A over B with respect to d* is defined by the formula

$$e_d(A, B) = sup\{d(a, B) : a \in A\}.$$

Excess may assume value $+\infty$ and is not symmetric (see Section 1.5 in [1] for more details and examples). In particular, $e_d(A, x)$ will denote the excess of the set A over the singleton $\{x\}$. Notice that $e_d(x, A) = d(x, A)$, while $e_d(A, x)$ is quite different.

Definition 8.5. Let (X, d) be a metric space and V be a proper subset of X. V is a *star-set* if there exists $x_V \in X$, such that:

$(*)$ $e_d(V, x_V) < d(x_V, V^c)$.

We say that V is a *star-set around x*.

Lemma 8.6. *Let (X, d) be a metric space, $x \in X$ and V be a proper subset of X. V is a coordinated-set if and only if it is a star-set.*

Proof. If V is a coordinated-set, there exist $x \in X$ and $r > 0$ such that $f_x^r(a) \geq 0$ for every $a \in V$ and $f_x^r(V^c) < 0$, i.e. $r \geq d(x, a)$ if $a \in V$ and $r < d(x, V^c)$. Hence $d(a, x) \leq r < d(x, V^c)$ whenever $a \in V$. Therefore, $e_d(V, x) = sup\{d(a, x) : a \in V\} < d(x, V^c)$.

Suppose now that V is a star-set and that $(*)$ holds for some x. Let $r = e_d(V, x)$. Then, $d(x, a) \leq e_d(V, x) = r < d(x, V^c)$ for every $a \in V$. The coordinated-function f_x^r satisfies (i) and (ii) of the definition of coordinated-set. □

Lemma 8.7. *Let (X, d) be a metric space and $x \in X$. The following are equivalent:*

(a) *condition $(\sharp)$ holds at x;*
(b) *for every $\epsilon > 0$ there exists a positive $\delta < \epsilon$ such that $S_d(x, \delta)$ is a star-set around x;*
(c) $inf\{r > 0 : S_d(x, r)$ *is a star-set around* $x\} = 0$.

Proof. $(a) \Rightarrow (b)$. Given $\epsilon > 0$, there exist $\delta, \theta \in \mathbb{R}$ such that $0 < \delta < \theta \leq \epsilon$ and $S_d(x, \delta) = S_d(x, \theta)$. It is easy to check that $e_d(S_d(x, \delta), x) < d(x, S_d(x, \delta)^c)$, so that $(*)$ holds at x.

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are easy to check. □

Corollary 8.8. *Let (X, d) be a metric space. If condition $(\sharp)$ holds at each $x \in X$, then X has a base $\mathcal{B}$ of star-sets of the form $S_d(x, \delta)$.*

Proof. Suppose condition ($\sharp$) holds at each point. By Lemma 8.7, for every $x \in X$ and $\epsilon > 0$, there exists $\delta_\epsilon < \epsilon$ such that $S_d(x, \delta_\epsilon)$ is a star-set. Since $\{S_d(x, \epsilon) : x \in X, \epsilon > 0\}$ is a base for the topology on X, so is the family $\mathcal{B} = \{S_d(x, \delta_\epsilon) : x \in X, \epsilon > 0\}$. □

Proposition 8.9. *Let (X, d) be a separable metric space. If condition ($\sharp$) holds at each $x \in X$, then X has a countable base $\mathcal{B}$ consisting of star-sets.*

Proof. By Corollary 8.8, X has a base $\mathcal{A}$ consisting of star-sets. Since X is separable, it also has a countable base. Hence, there exists a countable subfamily $\mathcal{B} \subseteq \mathcal{A}$, which is still a base for X (in general, if X is a regular second countable space and $\mathcal{B}$ is a base for X, then there exists a countable subcollection $\mathcal{B}' \subseteq \mathcal{B}$ which is again a base for X). □

Lemma 8.10. *Let (X, d) be a metric space, V an open subset of X and $n \in \omega$. The following are equivalent:*

(1) *V is an n-coordinated-set;*
(2) *$(V^c)^+ = \bigcap_{i=1}^n \mathcal{A}^+(x_i, r_i)$, where $x_i \in X$ and $r_i > 0$ for every i.*

Proof. Let $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n > 0$. The following are equivalent:

- $f_{x_i}^{r_i}(a) \geq 0$ for every $a \in V$ and $f_{x_i}^{r_i}(V^c) < 0$;
- for every $a \in V$ there exists i such that $d(x_i, a) \leq r_i$, and $d(x_i, V^c) > r_i$ for every i;
- for every $A \in CL(X)$, $A \cap V = \varnothing$ if and only if $d(x_i, A) > r_i$ for all i;
- $(V^c)^+ = \bigcap_{i=1}^n \mathcal{A}^+(x_i, r_i)$.

□

Let (X, τ) be a topological space and $A \subseteq X$. A is a *basic closed* subset of X if A^c is a basic open subset.

Corollary 8.11. *Let (X, d) be a metric space and V an open subset of X. Then:*

(i) *V is a star-set if and only if $(V^c)^+$ is a $\tau_{W_d}^+$-subbasic open subset of $CL(X)$;*
(ii) *V is a n-coordinated-set, for some $n \in \omega$, if and only if $(V^c)^+$ is a $\tau_{W_d}^+$-basic open subset of $CL(X)$;*
(iii) *V is a n-coordinated-set, for some $n \in \omega$, if and only if V^- is $\tau_{W_d}^+$-basic closed subset of $CL(X)$.*

In particular,

(iv) *if V is a n-coordinated-set, for some $n \in \omega$, then $(V^c)^+$ and V^- are τ_{W_d}-clopen subsets of $CL(X)$.*

Proof. Recall that $V^- = CL(X) \setminus (V^c)^+$ (see Preliminaries) and V^- is open in the Wijsman topology, if V is open (Remark 3.6). □

Remark 8.12. From Corollary 8.11(iv), it follows immediately that if (X, d) has a base $\mathcal{B}$ consisting of n-coordinated-sets ($n \in \omega$), then $(B^c)^+$ and B^- are τ_{W_d}-clopen subsets of $CL(X)$, for every $B \in \mathcal{B}$.

We close with the preannounced result.

Proposition 8.13. *Let (X, d) be a Polish space having a countable base of star-sets. Then there exists a sub-compatible order on X such that the mapping $A \to min(A)$ is a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

Proof. Let $\mathcal{B}$ be the given base of star-sets. By Corollary 8.11(iv), B^- is τ_{W_d}-clopen, whenever $B \in \mathcal{B}$. Apply Corollary 7.3. $\square$

The fact that condition $(\sharp)$ is sufficient for the Wijsman hyperspace to admit a continuous selection, stated by Bertacchi and Costantini in the setting of ultrametric spaces (see $(a) \Rightarrow (b)$ of Proposition 8.2), follows now as an easy consequence and without requiring d to be an ultrametric. This also solves the problem which is implicitely raised in the last two lines of [3].

Corollary 8.14. *Let (X, d) be a Polish space such that condition $(\sharp)$ holds at each point of X. Then there exists a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d).*

Proof. By Proposition 8.9, X has a countable base of star-sets. Apply Proposition 8.13. $\square$

Notice that the converse of Proposition 8.13 holds if (X, d) is an ultrametric space: use $(b) \Rightarrow (a)$ of Proposition 8.2 and Proposition 8.9.

This yields to the following result, which completes the main one of [3] and shows that the converse of Proposition 8.9 is also valid provided that d is an ultrametric.

Proposition 8.15. *Let (X, d) be a Polish ultrametric space. The following are equivalent:*

- (a) *X has a countable base of star-sets;*
- (b) *there exists a sub-compatible order on X such that the minimum map $A \to min(A)$ is a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d);*
- (c) *there exists a continuous selection from $(CL(X), \tau_{W_d})$ to (X, d);*
- (d) *condition $(\sharp)$ holds at each $x \in X$.*

Proof. $(a) \Rightarrow (b)$ follows from Propositon 8.13; $(b) \Rightarrow (c)$ is trivial; $(c) \Rightarrow (d)$ is $(b) \Rightarrow (a)$ of Proposition 8.2; $(d) \Rightarrow (a)$ follows from Proposition 8.9. $\square$

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