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A note on separation and compactness in categories of convergence spaces

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Abstract. In previous papers, various notions of compact, T_3, T_4, and Tychonoff objects in a topological category were introduced and compared. The main objective of this paper is to characterize each of these classes of objects in the categories of filter and local filter convergence spaces as well as to examine how these various generalizations are related.

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1. Introduction

The following facts are well known:

(1) A topological space X is compact if and only if the projection $\pi_2 \colon X \times Y \to Y$ is closed for each topological space Y,
(2) A topological space X is Hausdorff if and only if the diagonal, Δ, is closed in $X \times X$,
(3) For a topological space X the following are equivalent:
   (i) X is Tychonoff (completely regular T_1);
   (ii) X is homeomorphic to a subspace of a compact Hausdorff space;
   (iii) X is homeomorphic to a subspace of some T_4 space.

The facts (1) and (2) are used by several authors (see, [7, 14, 22] and [25]) to motivate a closer look at analogous situations in a more general categorical setting. Categorical notions of compactness and Hausdorffness with respect to a factorization structure were defined in the case of a general category by Manes [25] and Herrlich, Salicrup and Strecker [22]. A categorical study of these notions with respect to an appropriate notion of "closedness" based on closure operators (in the sense of [17]) was done in [18] (for the categories of various types of filter convergence spaces) and [14] (for abstract categories). Baran in [2] and [4] introduced the notion of "closedness" and "strong closedness" in set-based topological categories and used these notions in [7] to generalize each of the notions of compactness and Hausdorffness to arbitrary set-based topological categories.

By using (i) and (ii) of (3), in [7] and [14], there are various ways of generalizing the usual Tychonoff separation axiom to arbitrary set based topological categories.

We further recall from [2] and [8] that for a T_1 topological space X, the following are equivalent:

(a) X is T_3;
(b) For every non-void subset F of X, the quotient space X/F (defined in 2.1 below) is T_2 if it is T_1;
(c) For every non-void closed subset F of X, the quotient space X/F is a PreT_2 space,

where a topological space is called PreT_2 [2](or R_1 in [13]) if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods. The equivalence of (b) and (c) follows from the facts that for T_1 topological spaces, T_2 is equivalent to PreT_2, and F is closed iff X/F is T_1. We note also:

(d) A topological space X is T_4 iff X is T_1 and for every non-void subset F of X, the space X/F is T_3 if it is T_1.

In view of (b) - (d), in [2] and [8], there are various ways of generalizing each of the usual T_3 and T_4 separation axioms to arbitrary set based topological categories.

The aim of this paper is to introduce, by using (3), various generalizations of Tychonoff objects for an arbitrary set based topological category and compare them with the ones that were given in [7, 9], and [14]. Furthermore, each of the classes of T_3 and T_4-objects, compact and strongly compact objects, and Tychonoff objects in the categories of filter and local filter convergence spaces are characterized and relationships among various forms of these Tychonoff objects are investigated in these categories.

2. Preliminaries

Let **E** be a category and **Set** be the category of sets. The functor $U : \mathbf{E} \to \mathbf{Set}$ is said to be *topological*, and **E** is said to be a *topological category over* **Set**, if U is concrete (i.e., faithful and amnestic, (i.e., if $U(f) = id$ and f is an isomorphism, then $f = id$)), has small (i.e., set) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift [19, 21, 26] or [29].

Note that a topological functor $U : \mathbf{E} \to \mathbf{Set}$ is said to be *normalized* if there is only one structure on the empty set and on a point [2] or [26].

Let **E** be a topological category and $X \in \mathbf{E}$. Then F is called a *subspace* of X if the inclusion map $i : F \to X$ is an initial lift (i.e, an embedding) and we denote this by $F \subset X$.

The categorical terminology is that of [20].

Let B be a set and $p \in B$. Let $B \bigvee_p B$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is a terminal object in **Set**). More precisely, if i_1 and $i_2 : B \to B \bigvee_p B$ denote the inclusions of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is a pushout diagram. A point x in $B \bigvee_p B$ will be denoted by x_1 (x_2) if x is in the first (resp. the second) component of $B \bigvee_p B$. Note that $p_1 = p_2$. The skewed p-axis map $S_p : B \bigvee_p B \to B^2$ is given by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\nabla_p : B \bigvee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$ ([2] p. 334 or [4] p. 386).

Note that the maps S_p and ∇_p are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\nabla_p i_j = id, j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p.

The infinite wedge product $\bigvee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times \dots$ be the countable cartesian product of B. Define $A_p^\infty : \bigvee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, \dots, x, p, p, \dots)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, \dots, x, p, p, \dots)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to B$ by $\triangledown_p^\infty(x_i) = x$ for all i, [2] p. 335 or [4] p. 386.

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, p, \dots, p, id, p, \dots) : B \to B^\infty$, where the identity map, id, is in the j-th place.

Definition 2.1. (cf. [2] p. 335 or [4] p. 386). Let $U : \mathbf{E} \to \mathbf{Set}$ be topological and X an object in $\mathbf{E}$ with $UX = B$. Let F be a non-empty subset of B. We denote by X/F the final lift of the epi U-sink $q : U(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifies F with a point $*$ ([2] p. 336).

Let p be a point in B.

(1) X is T_1 at p iff the initial lift of the U-source $\{S_p : B \bigvee_p B \to U(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.
(2) p is closed iff the initial lift of the U-source $\{A_p^\infty : \bigvee_p^\infty B \to B^\infty = U(X^\infty)$ and $\triangledown_p^\infty : \bigvee_p^\infty B \to UD(B) = B\}$ is discrete.
(3) $F \subset X$ is strongly closed iff X/F is T_1 at $*$ or $F = \varnothing$.
(4) $F \subset X$ is closed iff $*$, the image of F, is closed in X/F or $F = \varnothing$.
(5) If $B = F = \varnothing$, then we define F to be both closed and strongly closed.

Remark 2.2. (1). In **Top**, the category of topological spaces, the notion of closedness coincides with the usual closedness [2], and F is strongly closed iff F is closed and for each $x \notin F$ there exists a neighbourhood of F missing x [2]. If a topological space is T_1, then the notions of closedness and strong closedness coincide [2].

(2). In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other [4]. Even if $X \in \mathbf{E}$ is T_1, where $\mathbf{E}$ is a topological category, then these notions are still independent of each other ([8] p. 64).

Let A be a set and L a function on A that assigns to each point x of A a set of filters (proper or not, where a filter δ is proper iff δ does not contain the empty set, $\varnothing$, i.e., $\delta \neq [\varnothing]$), called the "filters converging to x". L is called a *convergence structure on* A (and (A, L) a *filter convergence space*) iff it satisfies the following two conditions:

1. $[x] = [\{x\}] \in L(x)$ for each $x \in A$ (where $[F] = \{B \subset A \mid F \subset B\}$).
2. $\beta \supset \alpha \in L(x)$ implies $\beta \in L(x)$ for any filter β on A.

A map $f : (A, L) \to (B, S)$ between filter convergence spaces is called *continuous* iff $\alpha \in L(x)$ implies $f(\alpha) \in S(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) \mid D \in \alpha\}$. The category of filter convergence spaces and continuous maps is denoted by **FCO** (see [15] p.45 or [30] p.354). A filter convergence space (A, L) is said to be a *local filter convergence space* (in [29], it is called a convergence space) if $\alpha \cap [x] \in L(x)$ whenever $\alpha \in L(x)$ ([28] p.1374 or [29] p.142). These spaces are the objects of the full subcategory **LFCO** (in [29] **Conv**) of **FCO**. Note that both of these categories are (normalized) topological categories [28], or [29].

More on these categories can be found in [1, 16, 24, 28, 29], and [30].

For filters α and β we denote by $\alpha \cup \beta$ the smallest filter containing both α and β.

Remark 2.3. An epimorphism $f : (A, S) \to (B, L)$ in **FCO** (resp., **LFCO**) is final iff for each $b \in B$, $\alpha \in L(b)$ implies that $f(\beta) \subset \alpha$ for some point $a \in A$ and filter $\beta \in S(a)$ with $f(a) = b$ ([28] p.1374 or [29] p.143).

Remark 2.4. A source $\{f_i : (B, L) \to (B_i, L_i),\ i \in I\}$ in **FCO** (resp., **LFCO**) is initial iff $\alpha \in L(a)$, for $a \in B$, precisely when $f_i(\alpha) \in L_i(f_i(a))$ for all $i \in I$ ([15] p.46, [28] p.1374 or [29] p.20).

We give the following useful lemmas which will be needed later.

Lemma 2.5. (cf. [3], Lemma 3.16). *Let $\varnothing \neq F \subset B$ and let $q : B \to B/F$ be the epi map that is the identity on $B \backslash F$ and identifies F to the point $*$.*

(1) *For $a \in B$ with $a \notin F$, $q(\alpha) \subset [a]$ iff $\alpha \subset [a]$,*
(2) *$q(\alpha) \subset [*]$ iff $\alpha \cup [F]$ is proper.*

Lemma 2.6. (cf. [10], Lemma 3.2). *Let $f : A \to B$ be a map.*

(1) *If α and β are proper filters on A, then $f(\alpha) \cup f(\beta) \subset f(\alpha \cup \beta)$.*
(2) *If δ is a proper filter on B, then $\delta \subset ff^{-1}(\delta)$, where $f^{-1}(\delta)$ is the proper filter generated by $\{f^{-1}(D) \mid D \in \delta\}$.*

Lemma 2.7. (cf. [8], Lemma 1.4) *Let α and β be proper filters on B. Then $q(\alpha) \cup q(\beta)$ is proper iff either $\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\beta \cup [F]$ are proper.*

3. T_2-OBJECTS

Recall, in [2] and [6], that there are various ways of generalizing the usual T_2 separation axiom to topological categories. Moreover, the relationships among various forms of T_2-objects are established in [6].

Let B be a set and $B^2 \bigvee_\Delta B^2$ the wedge product of B^2, i.e. two disjoint copies of B^2 identified along the diagonal, Δ. A point (x, y) in $B^2 \bigvee_\Delta B^2$ will be denoted by $(x, y)_1$ (resp. $(x, y)_2$)) if (x, y) is in the first (resp., second) component of $B^2 \bigvee_\Delta B^2$ [10]. Recall that the principal axis map $A : B^2 \bigvee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \bigvee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$, $S(x, y)_2 = (x, x, y)$, and the fold map, $\bigtriangledown : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\bigtriangledown(x, y)_i = (x, y)$ for i = 1,2 [2].

Definition 3.1. Let $U : \mathbf{E} \to \mathbf{Set}$ be topological and X an object in $\mathbf{E}$ with $UX = B$.

1. X is T_0' iff the initial lift of the U-source
$$\{id : B^2 \bigvee_\Delta B^2 \to U(B^2 \bigvee_\Delta B^2)' = B^2 \bigvee_\Delta B^2 \text{ and}$$
$$\bigtriangledown : B^2 \bigvee_\Delta B^2 \to UD(B^2) = B^2\}$$
is discrete, where $(B^2 \bigvee_\Delta B^2)'$ is the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 \bigvee_\Delta B^2\}$. Here, i_1 and i_2 are the canonical injections.
2. X is T_1 iff the initial lift of the U-source $\{S : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$ and $\bigtriangledown : B^2 \bigvee_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete.
3. X is PreT_2' iff the initial lift of the U-source $\{S : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3\}$ and the final lift of the U-sink $\{i_1, i_2 : U(X^2) = B^2 \to B^2 \bigvee_\Delta B^2\}$ coincide.
4. X is ΔT_2 iff the diagonal, Δ, is closed in X^2.
5. X is ST_2 iff Δ is strongly closed in X^2.
6. X is T_2' iff X is T_0' and PreT_2'.

Remark 3.2. (1). Note that for the category **Top** of topological spaces, T_0', T_1, PreT_2', and all of the T_2's reduce to the usual T_0, T_1, PreT_2 and T_2 separation axioms, respectively [2].

(2) If $U : \mathbf{E} \to \mathbf{B}$, where $\mathbf{B}$ is a topos [23], then parts (1) - (3), (5), and (6) of Definition 3.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if $\mathbf{B}$ has infinite products and infinite wedge products, then Definition 3.1 (4) also makes sense.

Lemma 3.3. *Let (B, L) be in* **FCO** *(resp.,* **LFCO***) and $\varnothing \neq F \subset B$.*

(1) *(B, L) is T_1 iff for each distinct pair of points x and y in B, $[x] \notin L(y)$.*
(2) *All objects (B, L) in* **FCO** *(resp.,* **LFCO***) are T_0'.*
(3) *$\varnothing \neq F \subset B$ is closed iff for any $a \notin F$, if there exists $\alpha \in L(a)$ such that $\alpha \cup [F]$ is proper, then $[a] \notin L(c)$ for all $c \in F$.*

(4) $\varnothing \neq F \subset B$ *is strongly closed iff for any* $a \in B$ *with* $a \notin F$, $[a] \notin L(c)$ *for all* $c \in F$ *and* $\alpha \cup [F]$ *is improper for all* $\alpha \in L(a)$.
(5) (B, L) *is* ΔT_2 *iff for all* $x \neq y$ *in* B, $[x] \notin K(y)$ *iff* (B, L) *is* T_1.
(6) (B, L) *is* ST_2 *iff for all* $x \neq y$ *in* B, $L(x) \cap L(y) = \{[\varnothing]\}$.
(7) (B, L) *is* $PreT_2'$ (T_2') *iff* (B, L) *is discrete, i.e, for all* x *in* B, $L(x) = \{[\varnothing], [x]\}$.

Proof. (1), (2), and (7) are proved in [5]. The proof of (3)-(6) are given in [4]. □

Corollary 3.4. *Let* (B, L) *be in* **FCO** *(resp.* **LFCO***) and* $\varnothing \neq F \subset B$.
(1) *If* (B, L) *is* T_1, *then* B/F *is* T_1 *iff* F *is strongly closed.*
(2) *If* (B, L) *is* T_1, *then* F *is always closed.*
(3) *If* (B, L) *is* T_1, *then* F *is strongly closed iff* $\forall x \in B$ *if* $x \notin F$ *and* $\alpha \in L(x)$, *then* $\alpha \cup [F]$ *is improper.*
(4) *If* (B, L) *is* T_2', *then all the subsets of* B *are both closed and strongly closed.*

4. T_3-OBJECTS

We now recall, ([2] and [8]), various generalizations of the usual T_3 separation axiom to arbitrary set based topological categories and characterize each of them for the topological categories **FCO** and **LFCO**.

Definition 4.1. Let $U : \mathbf{E} \to \mathbf{Set}$ be topological and X an object in $\mathbf{E}$ with $UX = B$. Let F be a non-empty subset of B.

1. X is ST_3' iff X is T_1 and X/F is PreT_2' for all strongly closed $F \neq \varnothing$ in $U(X)$.
2. X is T_3' iff X is T_1 and X/F is PreT_2' for all closed $F \neq \varnothing$ in $U(X)$.
3. X is ΔT_3 iff X is T_1 and X/F is ΔT_2 if it is T_1, for all $F \neq \varnothing$ in $U(X)$.
4. X is ST_3 iff X is T_1 and X/F is ST_2 if it is T_1, for all $F \neq \varnothing$ in $U(X)$.

Remark 4.2. (1). For the category **Top** of topological spaces, all of the T_3's reduce to the usual T_3 separation axiom ([2] and [8]).

(2). If $U : \mathbf{E} \to \mathbf{B}$, where $\mathbf{B}$ is a topos [23], then Parts (1), (3), and (4) of Definition 4.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if $\mathbf{B}$ has infinite products and infinite wedge products, then Definition 4.1 (2), also, makes sense.

Theorem 4.3. *Let* (B, L) *be in* **FCO** *(resp.* **LFCO***).*
(1) (B, L) *is* ΔT_3 *iff* (B, L) *is* T_1.
(2) (B, L) *is* ST_3 *iff* (B, L) *is* ST_2.
(3) (B, L) *is* ST_3' *iff for all* $x \neq y$ *in* F, $[x] \notin L(y)$ *and for any* $x \in B$ *and for any proper filter* $\alpha \in L(x)$, *either* $\alpha = [x]$ *or* $F \in \alpha$ *for all non-empty strongly closed subsets* F *of* B.
(4) (B, L) *is* T_3' *iff for all* $x \neq y$ *in* F, $[x] \notin L(y)$ *for any* $x \in B$ *and for any proper filter* $\alpha \in L(x)$ *either* $\alpha = [x]$ *or* $F \in \alpha$ *for any non-empty subset* F *of* B.

Proof. (1). This follows from Definition 4.1 and Corollary 3.4.

(2). Suppose (B, L) is ST_3. Take F ={a}, a one point set. It now follows from Lemma 3.3 and Corollary 3.4 that (B, L) is ST_2. Conversely, suppose (B, L) is ST_2. By Corollary 3.4, (B, L) is T_1. Suppose B/F is T_1, then by Corollary 3.4, F is a strongly closed subset of B. We show that B/F is ST_2. Let $x \neq y$ in B and $\alpha \in L'(x) \cap L'(y)$, where L' is the quotient structure on B/F induced by the map $q : B \to B/F$ that identifies F with a point $*$ and is the identity on $B \backslash F$. If α is improper, then, by Corollary 3.4, we are done. Suppose α is proper. Since q is the quotient map this implies (see Remark 2.3) that $\exists \beta \in L(a)$ and $\exists \delta \in L(b)$ such that $q(\beta) \subset \alpha$, $q(\delta) \subset \alpha$, and $qa = x$, $qb = y$. It follows that $q(\beta) \cup q(\delta)$ is proper and, by Lemma 2.7, either $\beta \cup \delta$ is proper or $\beta \cup [F]$ and $\delta \cup [F]$ are proper. The first case cannot occur since (B, L) is ST_2. Since $x \neq y$, we may assume $a \notin F$. Since F is strongly closed, by Corollary 3.4, $\beta \cup [F]$ is improper. This shows that the second case also cannot hold. Therefore, α must be improper and by Corollary 3.4, we have the result.

(3). Suppose (B, L) is ST_3'. Since (B, L) is T_1, by Corollary 3.4, for all $x \neq y$ in B, $[x] \notin L(y)$. If $\alpha \in L(x)$, where $x \in B$, then $q(\alpha) \in L'(qx)$. Since B/F is PreT_2', (F is a non-empty strongly closed subset of B) by Corollary 3.4, $q(\alpha) = [qx]$ (since α is proper). If $x \notin F$, then, by Lemma 2.6, $[x] = q^{-1}(x) = q^{-1}q(\alpha) \subset \alpha$ and consequently $\alpha = [x]$. If $x \in F$, it follows easily that $q(\alpha) = [*]$ iff $F \in \alpha$. Conversely, suppose the conditions hold. By Corollary 3.4, clearly, (B, L) is T_1. We now show that B/F is PreT_2' for all nonempty strongly closed subsets F of X. If $x \in B/F$ and $\alpha \in L'(x)$, it follows that there exists $\beta \in L(a)$ such that $q(\beta) \subset \alpha$ and $qa = x$. If β is improper, then so is α. If β is proper, then by assumption $\beta = [a]$ or $F \in \beta$. If the first case holds, then $[qa] = q(\beta) \subset \alpha$ and thus $\alpha = [qa]$. If the second case holds, then $\{*\} = q(F) \in q(\beta) \subset \alpha$ and consequently $\alpha = [*]$. Hence, by Lemma 3.3, B/F is PreT_2' and by Definition 3.1, (B, L) is ST_3'.

The proof of (4) is similar to the proof of (3), on using Definition 3.1, Lemma 3.3 and Corollary 3.4. □

Remark 4.4. For the category **FCO** (resp., **LFCO**), we have :

(1) By Theorem 4.3, $ST_3' \Rightarrow T_3' \Rightarrow ST_3 \Rightarrow \Delta T_3$, but the converse of each implication is not true in general.
(2) By Lemma 3.3 and Theorem 4.3, $ST_3' \Rightarrow T_2' \Rightarrow ST_3 \equiv ST_2 \Rightarrow \Delta T_3 = \Delta T_2$, but the converse of each implication is not true in general.
(3) By Corollary 3.4 and Theorem 4.3, if (B, L) is ST_3' or T_3', then all subsets of X are both closed and strongly closed.
(4) By Corollary 3.4 and Theorem 4.3, if (B, L) is ΔT_3, then F is always closed and F is strongly closed iff $\forall x \in B$ if $x \notin F$ and $\alpha \in K(x)$, then $\alpha \cup [F]$ is improper.

5. T_4-objects

We now recall various generalizations of the usual T_4 separation axiom to arbitrary set based topological categories that are defined in [2] and [8], and characterize each of them for the topological categories **FCO** and **LFCO**.

Definition 5.1. Let $U : \mathbf{E} \to \mathbf{Set}$ be topological and X an object in $\mathbf{E}$ with $UX = B$. Let F be a non-empty subset of B.

1. X is ST_4' iff X is T_1 and X/F is ST_3' if it is T_1, where F is any non-empty subset of $U(X)$.
2. X is T_4' iff X is T_1 and X/F is T_3' if it is T_1, where F is any non-empty subset of $U(X)$.
3. X is ΔT_4 iff X is T_1 and X/F is ΔT_3 if it is T_1, for all $F \neq \varnothing$ in $U(X)$.
4. X is ST_4 iff X is T_1 and X/F is ST_3 if it is T_1, for all $F \neq \varnothing$ in $U(X)$.

Remark 5.2. (1). For the category **Top** of topological spaces, all of the T_4's reduce to the usual T_4 separation axiom by the Introduction, [2], and [8].

(2). If $U : \mathbf{E} \to \mathbf{B}$, where $\mathbf{B}$ is a topos [23], then Definition 5.1 still makes sense since each of these notions requires only finite products and finite colimits in their definitions.

Theorem 5.3. *Let (B, L) be in* **FCO** *(resp.,* **LFCO***).*

(1) *(B, L) is ΔT_4 iff (B, L) is T_1.*
(2) *(B, L) is ST_4 iff (B, L) is ST_2.*
(3) *(B, L) is ST_4' (T_4') iff the following two conditions hold:*
   (i) *For all $x \neq y$ in B, we have $[x] \notin L(y)$.*
   (ii) *For any $x \in B$ and for any proper filter $\alpha \in L(x)$, and for any non-empty disjoint strongly closed (resp., closed) subsets F and F' of B, we have either condition (I) or (II) below:*
      (I) *$\alpha = [x]$;*
      (II) *$F \in \alpha$ or $F' \in \alpha$.*

Proof. (1). This follows from Definition 5.1 and Theorem 4.3.

(2). The proof has the same form as that of Theorem 4.3 (2). One has only to replace the term ST_3 by ST_4 and the numbers 3.3, 3.4, 3.4, 3.4, 3.4, 2.3, 3.4, 3.4 respectively by 3.1, 3.3, 4.3, 3.4, 4.3, 2.3, 3.3, 4.3.

(3). Suppose (B, L) is ST_4'. Since (B, L) is T_1, by Corollary 3.4, for all $x \neq y$ in B, $[x] \notin L(y)$. If $\alpha \in L(x)$, where $x \in B$, then $q(\alpha) \in L'(qx)$, where L' is the quotient structure on B/F induced by the map q of Definition 2.1. Since B/F is ST_3', (F is a non-empty strongly closed subset of B, i.e., B/F is T_1) by Corollary 3.4, we have either $q(\alpha) = [qx]$ (since α is proper) or $F' \in q(\alpha)$, for any non-empty strongly closed subset F' of B/F not containing the point $*$ (Note that $q^{-1}(F') = F'$ and F' is disjoint from F). Suppose that $q(\alpha) = [qx]$. If $x \notin F$, then, by Lemma 2.6, $[x] = q^{-1}(x) = q^{-1}q(\alpha) \subset \alpha$, and consequently $\alpha = [x]$. If $x \in F$, it follows easily that $q(\alpha) = [*]$ iff $F \in \alpha$.

If $F' \in q(\alpha)$ for any non-empty strongly closed subset F' of B/F not containing the point $*$, then it follows easily that $F' \in \alpha$.

Conversely, suppose the conditions hold. By Lemma 3.3, clearly, (B, L) is T_1. We now show that B/F is ST_3' for all non-empty strongly closed subsets F of B. If $x \in B/F$ and $\alpha \in L'(x)$, it follows that there exists $\beta \in L(a)$ such that $q(\beta) \subset \alpha$ and $qa = x$. If β is improper, then so is α. If β is proper, then by assumption either $\beta = [a]$ or $F \in \beta$, or $F' \in \beta$ for any strongly closed subset F' of B disjoint from F. If the first case holds, then $[qa] = q(\beta) \subset \alpha$ and thus $\alpha = [qa]$. If the second case holds, then $\{*\} = q(F) \in q(\beta) \subset \alpha$, and consequently $\alpha = [*]$ or $F' = q(F') \in q(\beta) \subset \alpha$ and consequently $F' \in \alpha$. Hence, by Theorem 4.3, B/F is ST_3' and by Definition 5.1, (B, L) is ST_4'.

The proof for T_4' is similar to the proof for ST_4'. □

Remark 5.4. For the category **FCO** (resp., **LFCO**), we have :

(1). By Theorem 4.3, $ST_4' \Rightarrow T_4' \Rightarrow ST_4 \Rightarrow \Delta T_4$, but the converse of each implication is not true in general.

(2). By Lemma 3.3, Theorem 4.3, and Theorem 5.3, $ST_4'(T_4') \Rightarrow ST_3'(T_3') \Rightarrow T_2' \Rightarrow ST_4 = ST_3 = ST_2 \Rightarrow \Delta T_4 = \Delta T_3 = \Delta T_2$, but the converse of each implication is not true in general.

(3). By Remark 4.4 and Theorem 5.3, if (B, L) is ST_4' or T_4', then all subsets of X are both closed and strongly closed.

(4). By Remark 4.4 and Theorem 5.3, if (B, L) is ΔT_4, then all subsets F of X are closed and F is strongly closed iff $\forall x \in B$, if $x \notin F$ and $\alpha \in L(x)$, then $\alpha \cup [F]$ is improper.

Corollary 5.5. *Let (B, L) be in* **FCO** *(resp.,* **LFCO***). If (B, L) is $\Delta T_4, ST_4$, ST_4' or T_4', then any subspace of (B, L) is $\Delta T_4, ST_4, ST_4'$ or T_4', respectively.*

Proof. This follows from Remark 2.4, Theorem 5.3, and Remark 5.4 (3). □

6. Compact objects

Recall that each of the notions of (strongly) closed morphism and (strongly) compact object in a topological category **E** over **Set** are introduced in [7].

Definition 6.1. Let $U : \mathbf{E} \to \mathbf{Set}$ be topological, X and Y objects in **E**, and $f : X \to Y$ a morphism in **E**.

1. f is said to be *closed* iff the image of each closed subobject of X is a closed subobject of Y.
2. f is said to be *strongly closed* iff the image of each strongly closed subobject of X is a strongly closed subobject of Y.
3. X is *compact* if and only if the projection $\pi_2 : X \times Y \to Y$ is closed for each object Y in **E**.
4. X is *strongly compact* if and only if the projection $\pi_2 : X \times Y \to Y$ is strongly closed for each object Y in **E**.

Remark 6.2. (1). For the category **Top** of topological spaces, the notions of closed morphism and compactness reduce to the usual ones ([12] p. 97 and

103). Furthermore, by Remark 2.2 and Definition 6.1, one can show that the notions of compactness and strong compactness are equivalent.

(2). If $U : \mathbf{E} \to \mathbf{B}$ is topological, where $\mathbf{B}$ is a topos with infinite products and infinite wedge products, then Definition 6.1 still makes sense.

(3). Since the notions of closedness and strong closedness are, in general, different (see [4] p. 393), it follows that the notions of compactness and strong compactness are different, in general.

(4). For an arbitrary topological category, it is not known in general whether the closure used in 2.1 is a closure operator in the sense of Dikranjan and Giuli [17] or not. However, it is shown, in [10], that the notions of closedness and strong closedness that are defined in 2.1 form appropriate closure operators in the sense of Dikranjan and Giuli [17] in case the category is one of the categories **FCO** and **LFCO**. The same two facts are proved in [11] for the categories **Lim** (limit spaces) and **PrTop** (pretopological spaces).

Theorem 6.3. *Let* $\mathbf{E}$ *be one of the categories* **FCO** (*resp.* **LFCO**).

(1) *Every* $(B, L) \in \mathbf{E}$ *is compact.*
(2) $(B, L) \in \mathbf{E}$ *is strongly compact iff every ultrafilter in* B *converges.*

Proof. (1). By Definition 5.1 (3) we need to show that, for all $(A, S) \in \mathbf{E}$, $\pi_2 : (B, L) \times (A, S) \to (A, S)$ is closed. Suppose $M \subset B \times A$ is closed. Suppose that for any $a \in A$ there exists $c \in \pi_2 M$ such that $[a] \in S(c)$. It follows that $\exists x \in B$ such that $(x, c) \in M$. Note that $[(x, a)] \in L^2((x, c))$, where L^2 is the product structure on $B \times A$, (since $[x] \in L(x)$ and $[a] \in S(c)$). Since M is closed, $(x, a) \in M$ and consequently $a = \pi_2(x, a) \in \pi_2(M)$. Hence, by Lemma 3.3, $\pi_2(M)$ is closed and consequently, (B, L) is compact.

(2). Suppose every ultrafilter in B converges. We show that (B, L) is strongly compact, i.e., by Definition 6.1 (4), we need to show that, for all $(A, S) \in \mathbf{E}$, $\pi_2 : (B, L) \times (A, S) \to (A, S)$ is strongly closed. Suppose that $M \subset B \times A$ is strongly closed. To show that $\pi_2 M$ is strongly closed, we assume the contrary and apply Lemma 3.3 (4). Thus for some point $a \in A$ with $a \notin \pi_2 M$, we have either $[a] \in S(c)$ for some $c \in \pi_2 M$ or $[\pi_2 M] \cup \alpha$ is proper for some $\alpha \in S(a)$. If the first case holds, that is for some $a \in A$ we have $a \notin \pi_2 M$ and $[a] \in S(c)$ for some $c \in \pi_2 M$, then it follows that $\exists x \in B$ such that $(x, a) \notin M$. Note that $[(x, a)] \in L^2((x, c))$, a contradiction, since M is strongly closed.

In the second case, suppose that for some $a \in A$ with $a \notin \pi_2 M$ and $\alpha \in S(a)$, $[\pi_2 M] \cup \alpha$ is proper. Let $\sigma = [M] \cup \pi_2^{-1}\alpha$. Note that σ is proper and $\pi_1(\sigma)$ is a filter on B. It follows that there exists an ultrafilter β on B with $\beta \supset \pi_1(\sigma)$. In view of the assumption on (B, L), there exists $x \in B$ such that $\beta \in L(x)$. Let $\gamma = \pi_1^{-1}\beta \cup \pi_2^{-1}\alpha$. Note that $\gamma \in L^2(x, a)$ since $\pi_1(\gamma) = \beta \in L(x)$ and $\pi_2(\gamma) = \alpha \in S(a)$. Since $a \notin \pi_2 M$, we have $(x, a) \notin M$. It follows from $\beta \supset \pi_1(\sigma)$ that $[M] \cup \gamma$ is proper, a contradiction since M is strongly closed, by Lemma 3.3 (4). Hence, by Lemma 3.3 (4), $\pi_2(M)$ must be strongly closed and consequently, by Definition 6.1, (B, L) is strongly compact.

Conversely, assume that (B, L) is strongly compact and α is a non convergent ultrafilter of B, i.e., for all $x \in B$, $\alpha \notin L(x)$. Let A be the set obtained by adjoining a new element, say ∞, to B, i.e., $A = B \cup \{\infty\}$. Let (A, S), where S is defined by $S(x) = \{[\varnothing], [x]\}$ for each $x \neq \infty$ of A, and $\beta \in S(\infty)$ iff $\alpha = \beta \cup [B]$, i.e., the trace of β on B coincides with α. Note that $(A, S) \in$ **FCO** (resp., **LFCO**). Let $\Delta = \{(x, y) \in B \times A \mid x = y\} \subset B \times A$. Let $\sigma = \pi_1^{-1}[x] \cup \pi_2^{-1}\alpha$. Since $\pi_1\sigma = [x] \in L(x)$ and $\pi_2\sigma = \alpha \in S(\infty)$, $\sigma \in L^2((x, \infty))$, where L^2 is the product structure on $B \times A$. Note that $\sigma \cup [\Delta]$ is improper (let $V = A \backslash \{x\} \in \alpha$ and $V \cap \Delta = \varnothing$). Since $[\infty] \notin S(c)$ for all $c \in B$, it follows that $[(x, \infty)] \notin L^2(c, c)$. Hence, by Lemma 3.3, Δ is strongly closed in $B \times A$. Note that $\alpha \cup [\pi_2(\Delta)]$ is proper for $\alpha \in S(\infty)$, a contradiction since (B, L) is strongly compact. □

Remark 6.4. Results akin to Theorem 6.3 have been proved for the categories **Lim** (limit spaces) and **PrTop** (pretopological spaces) in ([11], Lemma 4.3).

7. Tychonoff objects

We now define various forms of Tychonoff objects for an arbitrary set-based topological category. Furthermore, we characterize each of them for the categories that are mentioned in Section 2 and investigate the relationships among them.

Definition 7.1. Let $U : \mathbf{E} \to \mathbf{Set}$ be topological and X an object in $\mathbf{E}$.

1. X is $\Delta T_{3\frac{1}{2}}$ iff X is a subspace of ΔT_4.
2. X is $ST_{3\frac{1}{2}}$ iff X is a subspace of ST_4.
3. X is $T'_{3\frac{1}{2}}$ iff X is a subspace of T'_4.
4. X is $ST'_{3\frac{1}{2}}$ iff X is a subspace of ST'_4.
5. X is $C\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a compact ΔT_2.
6. X is $CST_{3\frac{1}{2}}$ iff X is a subspace of a compact ST_2.
7. X is $LT_{3\frac{1}{2}}$ iff X is a subspace of a compact T'_2.
8. X is $S\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ΔT_2.
9. X is $SST_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ST_2.
10. X is $SLT_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact T'_2.

Remark 7.2. (1). For the category **Top** of topological spaces, all ten of the properties defined in Definition 7.1 are equivalent and reduce to the usual $T_{3\frac{1}{2}} =$ Tychonoff, i.e, completely regular T_1, spaces [27], Remark 5.2, and Remark 6.2.

(2). For an arbitrary set-based topological category, properties (3–4) and (5–7) are defined in [8] and [7], respectively.

(3). For the categories **FCO** and **LFCO**, it is shown in [10] that the notions of closedness and strong closedness form appropriate closure operators in the sense of [16]. As a consequence, properties (5) and (9) of Definition 6.1 reduce to Definition 8.1 of [13].

Theorem 7.3. *Let X be in* **FCO** *(resp., LFCO).*

1. *X is $\Delta T_{3\frac{1}{2}}$ ($C\Delta T_{3\frac{1}{2}}$) iff X is T_1.*
2. *X is $ST_{3\frac{1}{2}}$ ($CST_{3\frac{1}{2}}$) iff X is ST_2.*
3. *X is $T'_{3\frac{1}{2}}$ iff X is T'_4.*
4. *X is $ST'_{3\frac{1}{2}}$ iff X is ST'_4.*
5. *X is $S\Delta T_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact T_1.*
6. *X is $SST_{3\frac{1}{2}}$ iff X is a subspace of a strongly compact ST_2.*
7. *X is $SLT_{3\frac{1}{2}}$ iff X is a finite discrete space.*

Proof. (1)-(6) follow from Lemma 3.3, Theorem 5.3, Corollary 5.5, Theorem 6.3 and Definition 7.1. Note that every strongly compact discrete space is finite. Thus, (7) follows from Remark 2.4, Lemma 3.3, Theorem 6.3, Definition 7.1, and this fact. □

Remark 7.4. For the Category **FCO** (resp. **LFCO**), one has: By Theorem 5.3, Theorem 6.3 and Theorem 7.3, $SLT_{3\frac{1}{2}} \Rightarrow ST'_{3\frac{1}{2}} \Rightarrow T'_{3\frac{1}{2}} \Rightarrow ST_{3\frac{1}{2}} = CST_{3\frac{1}{2}} \Rightarrow \Delta T_{3\frac{1}{2}} = C\Delta T_{3\frac{1}{2}}$ and $SLT_{3\frac{1}{2}} \Rightarrow SST_{3\frac{1}{2}} \Rightarrow S\Delta T_{3\frac{1}{2}}$ but the converse of each implication is not true, in general.

References

[1] J. Adamek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories* (John Wiley and Sons, New York, 1990).

[2] M. Baran, *Separation properties*, Indian J. Pure Appl. Math. **23** (1992), 333–341.

[3] M. Baran, *Stacks and filters*, Doğa Mat. (Turkish J. Math.) **16** (1992), 95–108.

[4] M. Baran, *The notion of closedness in topological categories*, Comment. Math. Univ. Carolinae **34** (1993), 383–395.

[5] M. Baran, *Separation properties in category of filter convergence spaces*, Bull. Calcutta Math. Soc. **85** (1993), 249–254.

[6] M. Baran and H. Altındis, *T_2-objects in topological categories*, Acta Math. Hungar. **71** (1996), 41–48.

[7] M. Baran, *A notion of compactness in topological categories*, Publ. Math. Debrecen **50** (1997), 221–234.

[8] M. Baran, *T_3 and T_4-objects in topological categories*, Indian J. Pure Appl. Math. **29** (1998), 59–69.

[9] M. Baran, *Completely regular objects and normal objects in topological categories*, Acta Math. Hungar. **80** (1998), 211–224.

[10] M. Baran, *Closure operators in convergence spaces*, Acta Math. Hungar., **87** (2000), 33–45.

[11] M. Baran, *Compactness, perfectness, separation, minimality and closedness with respect to closure operators*, Applied Categorical Structures, **10** (2002), 403–415.

[12] N. Bourbaki, *General Topology* (Addison-Wesley Publ. Co., 1966).

[13] A. S. Davis, *Indexed systems of neighbourhoods for general topological spaces*, Amer. Math. Monthly, **68** (1961), 886–893.

[14] M. M. Clementino, E. Giuli, and W. Tholen, *Topology in a category: Compactness*, Port. Math. **53** (1996), 397–433.

[15] D. Dikranjan and W. Tholen, *Categorical structure of closure operators* (Kluwer Academic Publishers, Dordrecht, 1995).

[16] D. Dikranjan and E. Giuli, *Epis in categories of convergence spaces*, Acta Math. Hungar. **61** (1993), 195–201.

[17] D. Dikranjan and E. Giuli, *Closure operators I*, Topology Appl. **27** (1987), 129–143.
[18] D. Dikranjan and E. Giuli, *Compactness, minimality and closedness with respect to a closure operator*, In: J. Adamek and S. MacLane (eds), Categorical Topology (Prague 1988), (World Scientific Publ., Singapore, 1990), 284–295.
[19] D. Dikranjan, E. Giuli and A. Tozzi, *Topological categories and closure operators*, Quaestiones Math. **11** (1988), 323–337.
[20] H. Herrlich and G. E. Strecker, *Category Theory*, 2nd ed. (Heldermann Verlag, Berlin, 1979).
[21] H. Herrlich, *Topological functors*, Gen. Topology Appl. **4** (1974), 125–142.
[22] H. Herrlich, G. Salicrup, and G. E. Strecker, *Factorizations, denseness, separation, and relatively compact objects*, Topology Appl. **27** (1987), 157–169.
[23] P. T. Johnstone, *Topos Theory*, L. M. S Mathematics Monograph: No. 10 (Academic Press, New York, 1977).
[24] E. Lowen-Colebunders, *Function classes of Cauchy continuous maps* (Marcel Dekker Inc., New York, 1989).
[25] E. G. Manes, *Compact hausdorff objects*, Gen. Topology Appl. **4** (1974), 341–360.
[26] M. V. Mielke , *Geometric topological completions with universal final lifts*, Topology Appl. **9** (1985), 277–293.
[27] J. R. Munkres, *Topology: A First Course* (Prentice Hall, 1975).
[28] L. D. Nel , *Initially structured categories and Cartesian closedness*, Canad. J. Math. **27** (1975), 1361–1377.
[29] G. Preuss, *Theory of Topological Structures, An Approach to Topological Categories* (D. Reidel Publ. Co., Dordrecht, 1988).
[30] F. Schwarz, *Connections between convergence and nearness*, Lecture Notes in Math. No. 719, (Springer-Verlag, 1978), 345–354.

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