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# Representations of ordered semigroups and the Physical concept of Entropy

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Abstract. The abstract concept of entropy is interpreted through the concept of numerical representation of a totally preordered set so that the concept of composition of systems or additivity of entropy can be analyzed through the study of additive representations of totally ordered semigroups.

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## 1. Introduction

In an interesting paper Cooper [?] studies the foundations of thermodynamics and the existence of entropy functions on state spaces of thermodynamic systems. Three different formulations of the second law of thermodynamics due to Clausius, Kelvin and Caratheodory are considered and it is proved that Caratheodory' s axiom is not sufficient for the existence of an entropy function even in simple spaces. Cooper studies this problem by formulating a concept of an accesibility relation on the state space $\mathcal{S}$ of a thermodynamical system. This accessibility relation on $\mathcal{S}$ is a total preorder and an entropy function is defined to be a real-valued function on $\mathcal{S}$ that preserves the accessibility relation. If, in addition, the state space $\mathcal{S}$ has an additive structure then an entropy function is also required to preserve the algebraic structure, i.e. it is an order-preserving function which is also an (algebraic) homomorphism. Moreover if the phase space of the thermodynamic system is a topological space and

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the accessibility relation is continuous then the problem of the existence of a continuous entropy function is equivalent to that of proving the existence of a continuous order-preserving real-valued function defined on a topological space equipped with a continuous total preorder.

Originally the problem of the existence of an order-preserving real-valued function defined on a totally ordered set was posed and solved by Cantor (see [?], [?]). Subsequent generalizations were made by Birkhoff (see [?]) and Debreu (see [?], [?]) among others, including those generalizations that deal with continuity. For more recent discussions concerning the new contributions to this framework, see [?]. For an account of the mathematical aspects and foundations of thermodynamics the reader is referred to the book [?] which appeared before the paper of Cooper. See also the recent account of the second law of thermodynamics in [?].

Cooper's paper has the merit of independently discovering or anticipating, at least implicitly, some deep concepts and results that are to be found in the vast literature on the existence of order-preserving functions in mathematics, mathematical economics, measurement theory and other related fields. In particular, this is true for the concept of additivity of entropy. Cooper's ideas about additive entropy closely parallel the modern theory of order-preserving functions on ordered semigroups and algebraic utility theory. One of the objectives of this paper is to establish this fact.

It is a remarkable fact that, with the exception of a few quotations (see, e. g., [?]), Cooper's paper appears to have gone largely unnoticed by researchers in mathematical economics, in the theory of measurement, or in the representation theory of totally ordered sets and semigroups by real valued functions.

It transpires that there are some mathematical mistakes in Cooper's paper. In the present paper we pay special attention to the algebraic aspects, and show how these errors may be rectified or extenuated. We feel strongly that Cooper's paper should be highly commended for introducing several crucial ideas, enabling us to discover the astonishing similarity between the structure of the entropy representation problem and that of the utility representation problem. To this end we propose certain mathematical interpretations of Cooper's modelling of entropy.

The article is organized as follows: Section **??** contains definitions, notations, and necessary background. In section **??** the concept of entropy is compared with the concept of utility function. In Section **??** the algebraical aspects of entropy functions are considered.

## 2. Definitions and previous results

Let $X$ be a nonempty set. A binary relation "$\precsim$" defined on $X$ is a *total preorder* if it is reflexive, transitive and total. If in addition "$\precsim$" is antisymmetric, then it is said to be a *total order.*

Associated to "$\precsim$" we define the *strict preference* and the *indifference* relations, respectively denoted by "$\prec$" and "$\sim$", given by $x \prec y \iff \neg(y \precsim x)$ and $x \sim y \iff x \precsim y,\ y \precsim x$ $(x, y \in X)$.

A total preorder "$\precsim$" on $X$ is said to be *representable* (respectively: *pseudorepresentable*) if there is a real-valued function $u : X \longrightarrow \mathbb{R}$ such that $x \precsim y \iff u(x) \leq u(y)$ (respectively: $x \precsim y \implies u(x) \leq u(y)$) $(x, y \in X)$. Such a function is said to be an *order-preserving* function or a *strictly isotone* function. If the set $X$ is a set of alternatives of some economic agent on which is defined a preference relation then such an order-preserving function is said to be a *utility* (respectively: *pseudoutility*) function in the economics literature.

On a totally ordered set $X$ it is possible to define a natural topology, called the *order topology*, a subbasis of which is given by the family of subsets: $(-\infty, a) = \{x \in X \ : \ x \prec a\}$, $(b, +\infty) = \{y \in X \ : \ b \prec y\}$ $(a, b \in X)$.

Let $(X, \tau)$ be a topological space endowed with a total preorder "$\precsim$". Then "$\precsim$" is said to be *$\tau$-continuous* if for every $x \in X$ the subets $(-\infty, x)$ and $(x, +\infty)$ are $\tau$-open.

A total preorder "$\precsim$" on $X$ is said to be *separable in the sense of Debreu* if there is a countable subset $C \subseteq X$ such that for every $x, y \in X$ with $x \prec y$, there exists $c \in C$ with $x \precsim c \precsim y$. It turns out that this property characterizes the representability of a total preorder "$\precsim$" by means of an order-preserving function. (See, e.g., [**?**], pp. 14 and ff.)

In general an order-preserving function may or may not be continuous with respect to the order topology of $X$ and the Euclidean topology of $\mathbb{R}$. The problem of the existence of a continuous order-preserving function on an ordered topological space was solved by Debreu (see [**?**], [**?**]) in two classical papers. To that end, Debreu introduced the concept of a gap.

**Definition 2.1.** *Let $\bar{\mathbb{R}}$ denote the extended real line. A* degenerate set *in $\bar{\mathbb{R}}$ is one having at most one element. A* gap *of a subset $S$ of $\bar{\mathbb{R}}$ is a maximal nondegenerate interval disjoint from $S$ and with a lower bound and an upper bound in $S$. An interval of $\bar{\mathbb{R}}$ of the form $(a, b]$ or $[a, b)$ is said to be* half-open half-closed.

**Theorem 2.2.** *(Debreu' s Open Gap Lemma): If $S$ is a subset of $\bar{\mathbb{R}}$, there is an increasing function $g : S \longrightarrow \mathbb{R}$ such that all the gaps of $g(S)$ are open.*

**Theorem 2.3.** *(Debreu' s Representation Theorem): If there is a real-valued order-preserving function on a totally preordered topological space $X$ then there is a continuous real-valued order-preserving function on $X$.*

Finally, we mention that the abstract study of the relationship between order and topology was initiated by Nachbin (see [**?**]).

## 3. Entropy and utility theory

In this section we present the approach given by Cooper in [**?**], relative to the existence of a continuous utility function on a totally preordered topological space.

First we recall some nomenclature and notations used in Cooper's work. (See [**?**]).

The *state space* $\mathcal{S}$ of a thermodynamic system is a separable topological space. There is a relationship, called *accesibility relation* and denoted "$\underline{\rightarrow}$" among the elements of $\mathcal{S}$. The fact $s_1\underline{\rightarrow}s_2$ may be read "a transition from $s_1$ to $s_2$ is possible". We write $s_1 \nrightarrow s_2$ for the negation of the statement $s_1\underline{\rightarrow}s_2$, and we write $s_1 \rightleftarrows s_2$ if both $s_1\underline{\rightarrow}s_2$ and $s_2\underline{\rightarrow}s_1$ hold. For $s_1 \rightarrow s_2$ we shall understand $s_1\underline{\rightarrow}s_2$ and $s_2 \nrightarrow s_1$. A function $f : S \longrightarrow \mathbb{R}$ is an *entropy function* for an accesibility relation "$\underline{\rightarrow}$" whenever $s_1\underline{\rightarrow}s_2 \iff f(s_1) \leq f(s_2)$ for every $s_1, s_2 \in S$.

**Remark 3.1.** Observe the analogy with the classical framework coming from Economics in which a consumer defines a *preference relation* "$\precsim$" on a nonempty set of goods $X$, usually called *consumption set*. As defined above, in this context a *utility function* is a map $u : X \longrightarrow \mathbb{R}$ such that $x \precsim y \iff u(x) \leq u(y)$ for every $x, y \in X$. Utility functions and entropy functions (and other similar concepts such as *scale* in measurement theory) are examples of order-preserving functions.

Henceforth we will not use Cooper's notation and refer to all order-preserving functions as utility functions.

Coming again to Cooper's work, Theorem 1 in [**?**] can be stated as follows:

**Theorem 3.2.** *(Cooper's Theorem 1): Let $(X, \tau)$ be a separable topological space equipped with a continuous total preorder "$\precsim$". Then there is a continuous utility function for "$\precsim$".*

Unfortunately Cooper's result is not correct in the general case. This can be easily seen in the next example.

**Example 3.3.** Let $X = [0, 1] \times \{0, 1\} \subseteq \mathbb{R}^2$ endowed with the lexicographic order "$\precsim_L$" given by $(x, y) \precsim_L (a, b) \iff x < a$, or $x = a$ , $y \leq b$. Let $\tau$ be the order topology on $X$ relative to "$\precsim_L$". Observe that $(\mathbb{Q} \cap [0, 1]) \times \{0, 1\}$ is topologically dense in $X$. However $X$ is not order separable in the sense of Debreu because $X$ has uncountably many jumps. (See [**?**], Proposition 1.6.11 on p. 23). Therefore, there is no utility function for "$\precsim_L$". (See, e.g., [**?**], pp. 14-15). Here a *jump* in $X$ is defined as a pair of points $x, y \in X$ with $x \prec y$ such that there is no $z \in X$ with $x \prec z \prec y$.

**Remark 3.4.** In view of this example some additional condition must be added to separability in order to get the desired result. A possible such condition is *connectedness*.

Since Cooper's paper deals with problems in physics, perhaps it was taken for granted that the state spaces that do arise in thermodynamics satisfy the connectedness assumption. In Cooper's words:

<<*The physical model for a thermodynamic system is a system isolated from the external world by barriers impassible to heat but through which mechanical, electromagnetic, gravitational or other interactions with the external world are possible: these interactions will be summed up under the term interactions of the ground theories. The ground theories are the parts of physics established*

*independently of thermodynamics such as mechanics, electromagnetic theory. Within the thermodynamic system, subsystems capable of being isolated by barriers impassible to heat may exist: but it must be assumed that these internal barriers can be removed.>>*

We can interpret here that the existence of one of the "barriers" there mentioned would carry a disconnection of the space. So that if "the barriers were removed", the space would become connected. Here we should recall Newton's words: *"Natura non facit saltum"*.

Debreu's theorem (see [**?**], [**?**]) states that being $(X, \tau)$ a second countable space, every $\tau$-continuous total preoder "$\precsim$" defined on $X$ is continuously representable. In the particular case of $X$ being a metric space, separability and second countability are equivalent conditions. So, another way to correct Cooper's Theorem 1 is assuming that the system space is metric.

## 4. Entropy and the theory of ordered semigroups

Section 4 in Cooper's paper [**?**] is devoted to the study of *"Composition of systems: additivity of entropy"*. Having a positive perspective in mind we only want here to give a possible *interpretation* of Cooper's arguments, and to establish the results in a more rigorous setting. In our opinion, the ideas contained in section 4 of [**?**] are very rich and deep, and have clear analogies with powerful items concerning the representation of *totally ordered semigroups* through *additive utility functions*.

A glance at the beginning of section 4 in [**?**] shows that the main idea object of study is the possibility of finding thermodynamic systems that interact to get a new system. Consequently, a natural question arises: *"What relationships, if any, appear between the entropy of the new system after interaction, and the entropies before the interaction of the systems involved?"* Cooper establishes some axioms to deal with this kind of problem. Such axioms lean on additivity properties. The objective is finding entropy functions that are unique up to linear transformations.

As in the previous sections in [**?**], the validity of several lemmata and theorems in section 4 of Cooper's paper require additional assumptions and considerations for the physical models studied that are not explicitly mentioned by Cooper. For instance, if we understand the composition of systems as a binary operation defined on the set of all possible systems, it is *natural* that this binary operation be associative and commutative as well. Cooper does not mention the above conditions in [**?**], in spite that in Cooper's arguments such properties seem to be implicitly assumed.

With such interpretations, our framework will be the theory of totally ordered semigroups. Useful references here for further reading are [**?**], [**?**], [**?**], [**?**], [**?**].

**Remark 4.1.**

(1) Despite it is greatly at variance with the notation common in semigroup theory, along this section **??** we will keep additive notation, much more

familiar to reserchers on algebraic utility. We have already chosen that notation in previous works as [?], [?] and [?], and it fits better with the notation used in section ?? of the present paper.

(2) There are interpretations of several axioms encountered in other mathematical theories that could have some similarities with Cooper's axioms and ideas in [?]. The reader could investigate analogies with expected utility theory (see [?]) and axiomatical treatment of statistical means (see [?], [?]).

In order to deal with the algebraic setting used in Cooper's work (see [?]), we introduce some previous concepts about ordered semigroups.

A *semigroup* $(S,+)$ is a set $S$ endowed with a binary operation $+$ that is associative. A semigroup $S$ having a *null element* $e$ such that $x+e=x=e+x$ for every $x \in S$ is said to be a *monoid*. If each element $x$ of a monoid $S$ has a converse $-x$ such that $x+(-x)=(-x)+x=e$ then $S$ is said to be a *group*. A semigroup $(S,+)$ endowed with a total ordering $\precsim$ is said to be a *totally ordered semigroup* if the ordering $\precsim$ is *translation-invariant* (i.e.: $x \precsim y \iff x+z \precsim y+z \iff z+x \precsim z+y$ for every $x,y,z \in S$). In particular, a totally ordered semigroup $S$ is always *cancellative*, i.e.: $x+z = y+z \iff x=y \iff z+x=z+y$ for every $x,y,z \in S$.

Given a totally ordered semigroup $(S,+,\precsim)$, an element $x \in S$ is said to be *positive* (respectively: *negative*) when $y \prec x+y$ and also $y \prec y+x$ (respectively: when $x+y \prec y$ and also $y+x \prec y$) for every $y \in S$. Notice that an element $x \in S$ is positive (respectively: negative) if and only if $x \prec x+x$ (respectively: $x+x \prec x$). The set of positive (respectively: negative) elements of $S$ is said to be the *positive cone* of $S$, denoted $S^+$ (respectively: $S^-$). A simple exercise shows that these cones are *stable* in the following sense: If $x,y \in S^+$ (respectively: $S^-$) then $x+y, y+x \in S^+$ (respectively: $S^-$). Notice also that $S$ may have an element $e$ that is neither positive nor negative. In this case $e$ must be the null element for the operation $+$, and $S$ is, a fortiori, a monoid. Moreover, in this case it is clear that an element $x$ is positive (respectively: negative) if and only if $e \prec x$ (respectively: $x \prec e$).

A totally ordered semigroup $(S,+,\precsim)$ is said to be:

(1) *positive* (respectively: *negative*) if it consists only of positive (respectively: negative) elements.
(2) *additively representable* (respectively: *pseudo-representable*) if there exists a utility (respectively: pseudo-utility) function $u$ for $\precsim$ that is an homomorphism (i.e.: $u(x+y)=u(x)+u(y)$, for every $x,y \in S$). The associated function $u$ is said to be an *additive utility* (respectively: *pseudo-utility*) *function.*

A positive semigroup $(S,+,\precsim)$ is said to be:

(1) *Archimedean* if for every $x,y \in S$ with $x \prec y$, there exists $n \in \mathbb{N}$ such that $y \prec n \cdot x$, $(n \cdot x = \overbrace{x+\cdots+x}^{n \text{ times}})$,

(2) *super-Archimedean* if for every $x, y \in S$ such that $x \prec y$ there exists $n \in \mathbb{N}$ such that $(n+1) \cdot x \prec n \cdot y$.

A totally ordered group is said to be *Archimedean* if its positive cone is Archimedean.

A totally ordered semigroup $(S, +, \precsim)$ is said to be *super-Archimedean* if its positive cone $(S^+, +, \precsim)$ is super-Archimedean and also the negative cone $(S^-, + \precsim_{op})$, endowed with the *converse ordering* "$\precsim_{op}$" defined by $x \precsim_{op} y \iff y \precsim x$ $(x, y \in S)$, is super-Archimedean.

*In the case of totally ordered groups, the Archimedean condition is equivalent to the existence of an additive representation.* This is a key result stated by Hölder early in 1901. (See [**?**], or [**?**], p. 300.)

**Remark 4.2.**

(1) *Even in the case of positive semigroups Archimedeaness is not good enough to guarantee the additive representability.*
An *example* is the strictly positive cone $(0, \infty) \times (0, \infty)$ of the lexicographic plane $(\mathbb{R}^2, +, \precsim_L)$ where the sum $+$ is defined coordinatewise and the ordering $\precsim_L$ is given by $(a, b) \precsim_L (c, d)$ if $a < c$ or else $a = c$ , $b \leq d$. It is well-known that this ordered set does not admit a utility representation, even non-additive. (See [**?**], pp. 200-201).

(2) In [**?**] it is proved that *super-Archimedean implies Archimedean:* For instance, in the particular case of $S$ being a commutative and positive semigroup, it holds that if $x, y$ are positive, then $y \prec x + y$, so that $(n+1) \cdot y \prec n \cdot (x + y) \implies y \prec n \cdot x$.
*The converse is not true:* In the strictly positive cone of the lexicographic plane we have that $(1, 1) \prec (1, 2)$ but, for any positive $n \in \mathbb{N}$, $(n+1, n+1) \prec (n, 2n)$, so that it is not super-Archimedean.

(3) As also shown in [**?**], *Archimedean totally ordered groups are super-Archimedean:* For instance, if $G$ is Abelian and $e \prec x \prec y$, we have that $y - x \prec y$. Thus if $y \prec n \cdot (y - x) \implies (n+1) \cdot x \prec n \cdot y$.

In this framework of semigroups, there is also a characterization of additive representability:

**Theorem 4.3.**

(1) *The following statements are equivalent for a positive totally ordered semigroup* $(S, +, \precsim)$*:*
*(i)* $(S, +, \precsim)$ *is additively representable,*
*(ii)* $(S, +, \precsim)$ *is super-Archimedean.*

(2) *A semigroup* $(S, +, \precsim)$ *is additively representable if and only if its positive and negative cones are additively representable.*

*Proof.* The proof may be seen in [**?**]. For the sake of completeness let us see its main ideas: In order to prove the key implication $(ii) \implies (i)$ of part (1), fix an element $x_0 \in S$ and, given $x \in S$, set $u(x) = \sup\{m/n \ : m, n \in \mathbb{N} \ , m \cdot x_0 \prec n \cdot x\}$. Following the proof of Hölder's theorem that appears in [**?**], pp. 300-301, we obtain that $u$ is an additive pseudo-utility. So, it only remains to check the injectivity of $u$, and this comes from the fact of $S$ being super-Archimedean: Observe that being $x, y \in S$ such that $x \prec y$, there exists $n \in \mathbb{N}$ for which $(n+1) \cdot x \prec n \cdot y \implies (n+1) \cdot u(x) \leq n \cdot u(y) \implies u(x) \leq (n/(n+1)) \cdot u(y) < u(y)$. To prove part (2) we must give a construction of a *global* utility function $u : S \longrightarrow \mathbb{R}$ from *partial* utility functions $u^+, u^- : S^+, S^- \longrightarrow \mathbb{R}$. The key step consists in proving that for any $x \in S$ one can find an element $y \in S^+$, that depends on $x$, such that $x + y \in S^+$. Then set $u(x) = u^+(x+y) - u^+(y)$, and test that $u$ is the required additive utility function. □

Hölder's main result can be improved taking into account the *continuity* of the additive utilities involved.

**Proposition 4.4.** *Let $(G, +, \precsim)$ be a totally ordered group. Then $(G, +, \precsim)$ is representable through a continuous and additive utility function if and only if $(G, +, \precsim)$ is Archimedean.*

*Proof.* See Theorem 1 in [**?**]. □

**Remark 4.5.**

(1) One may expect that the key property of Archimedeaness established in Proposition **??**, that in the case of totally ordered groups guarantees the existence of a continuous additive utility function, will be maintained for *semigroups*. Unfortunately things are no longer the same in this case. It follows from Remark **??** that Archimedeaness is not good enough to obtain additive utility representations (*continuous or not*) for totally ordered semigroups. Actually, we have that: *Even being representable by an additive utility function, a semigroup could not admit a continuous additive utility representation.* An *example* is the semigroup $S = [2, 3) \cup [4, \infty)$ with the usual addition and ordering of the reals. The crux for the non-existence of a continuous and additive utility function in this example, is the discontinuity as regards the order topology of the algebraic operation $+$. In other words, $S$ is not a *"topological"* totally ordered semigroup in the sense that the algebraic operation is not continuous as regards the order topology.

(2) The result established in Debreu's open gap lemma cannot be extended to the framework of additive utility functions on semigroups. The last example shows that there exists positive semigroups that admit additive utility functions, but none of such additive utility representations is continuous. Of course, by Theorem **??**, any such ordered semigroup will admit continuous utility representations, but now none of those continuous utilities is additive.

Let $(S, +, \precsim)$ be a totally ordered semigroup. First we might notice that there is no topology given a priori on $S$, except maybe the order topology. But, even endowed with the order topology, we do not know whether $(S, +, \precsim)$ is a *topological* semigroup or not, in the sense of the following definition.

**Definition 4.6.** *A* topological semigroup $(S, +, \tau)$ *is a semigroup* $(S, +)$ *endowed with a topology "$\tau$" that makes continuous the binary operation "+":* $(x, y) \in S \times S \longmapsto x + y \in S$. *A totally ordered semigroup* $(S, +, \precsim)$ *is said to be a* topological totally ordered semigroup *if the binary operation "+" is continuous with respect to the order topology on* $S$. *Similarly, a* topological group $(G, +, \tau)$ *is a group* $(G, +)$ *endowed with a topology "$\tau$" that makes continuous the binary operations "+"* : $G \times G \longrightarrow G$, *and "inv"* : $G \longrightarrow G$, *given by* $inv(x) = -x$, *for every* $x \in G$. *So a* topological totally ordered group *is a totally ordered group* $(G, +, \precsim)$ *such that "+" and "inv" are both continuous as regards the order topology.*

**Remark 4.7.** It is known that totally ordered groups are topological as regards the order topology (see [**?**]), so that Theorem **??** can be extended to the framework of totally ordered groups. As was pointed out in Remark **??**, the above property is no longer true for totally ordered semigroups.

The condition of being topological will be necessary for the existence of a continuous additive representation on totally ordered semigroups. In addition, the following main question arises now : Let $(S, +, \precsim)$ be a super-Archimedean topological totally ordered semigroup. Is $S$ representable by a *continuous* utility function? The answer is positive, as next result states.

**Theorem 4.8.**
(1) *Let* $(S, +, \precsim)$ *be a totally ordered semigroup, additively representable by a continuous utility function* $u : S \longrightarrow \mathbb{R}$. *Then* $(S, +, \precsim)$ *is a topological semigroup as regards the order topology.*
(2) *Let* $(S, +, \precsim)$ *be a super-Archimedean topological totally ordered semigroup. Then* $S$ *is representable by a continuous additive utility function.*

*Proof.* See Proposition 1 and Theorem 2 in [**?**]. □

Coming back to Cooper's work, we observe that the nub of the reasoning in section 4 of [**?**] seems to be in the Lemma 1, in which Cooper justifies the existence of an state that is the *mid point* between two given states that interact. Cooper's proof is essentially based on the *connectedness* of the system and the *continuity* of the relation "$\underline{\rightarrow}$". Cooper's result presents an evident analogy with the following result that appears in [**?**].

**Proposition 4.9.** *Let* $(S, +, \precsim)$ *be a topological totally ordered semigroup that is positive and connected. Then given* $u, v \in S$ *there exists* $s \in S$ *such that* $s + s = u + v$. *(Such point* $s$ *is said to be the* mid point *between* $u$ *and* $v$*).*

*Proof.* Just consider the sets $A = \{z \in S \ : \ z + z \prec u + v\}$ and $B = \{y \in S \ : \ u + v = y + y\}$ and use an standard argument of connectedness. (For details, see Lemma 6 in [**?**]). □

One of the keys in Proposition **??** is the condition of *translation-invariance*. Cooper, in section 4 of [**?**], uses a very similar condition, denoted "$Int(a)$". In Cooper's words:

$<<$ *A system* $\Xi$ *is called the composition of the systems* $\Xi^1$, $\Xi^2$, $\ldots \Xi^n$ *and is written* $\Xi = \{\Xi^1, \Xi^2, \ldots \Xi^n\}$ *if there is a homeomorphism of the product space* $S^1 \times S^2 \times \ldots \times S^n$ *onto the state space of* $\mathcal{S}$ *which is such that if* $\{s^1, s^2, \ldots s^n\}$ *is the state corresponding to* $(s^1, s^2, \ldots s^n)$ *then* $\{s^1, s^2, \ldots s^{r-1}, s^r_1, s^{r+1}, \ldots, s^n\}$ $\to \{s^1, s^2, \ldots s^{r-1}, s^r_2, s^{r+1}, \ldots, s^n\}$ *if and only if* $s^r_1 \prec s^r_2$. $>>$

Observe that if we understand the composition of systems as being commutative, and denote $a+b$ the composition of states $a$ and $b$ (i.e.: $a+b$ is $\{a,b\}$ in Cooper's notation) then by $Int(a)$ we have that $s_1 \to s_2 \iff s_1 + t \to s_2 + t$ ($\iff t + s_1 \to t + s_2$, by commutativity of the composition), for any states $s_1, s_2$ and $t$. Thus we recover the translation-invariance of the operation "+". Moreover, using henceforward the usual notation "$\prec$" instead of Cooper's "$\to$", it follows from $Int(a)$ that $p \prec s \prec q \implies p+p \prec s+s \prec q+q$ for any states $p, s, q$. This fact is used by Cooper to justify the *uniquenesss* of the mid point.

As a matter of fact, in Cooper's arguments the property $Int(a)$ is not used in such justification. Apparently Cooper only uses $Int(a)$ for compositions in which at least four states are involved.

Let us analyze now the *topological* condition that is imposed on the semigroup in the statement of Proposition **??**. Observe that the binary operation "+" is required to be *continuous* as regards the order topology. This requirement is essential to obtain the desired result. Coming back to the example introduced in Remark **??**, we notice that "+" is not continuous there: Indeed $(7,9)$ is a neighbourhood of $8 = 4+4$, but every neighbourhood of 4 as regards the order topology must contain some element $\alpha$ smaller than 3 so that $\alpha + \alpha \notin (7,9)$. In particular, the existence of a mid point fails to be true in such example because there is no $s \in S$ such that $s + s = 2 + 5$.

Therefore we may assert that Cooper should have noticed that *not only a continuous order is necessary, but also the composition of systems must be continuous.* In Cooper's proof it is said that, given two states $u$ and $v$, the sets $L(u,v) = \{s \ : \ s+s \prec u+v\}$ and $R(u,v) = \{s \ : \ u+v \prec s+s\}$ are open, due to the continuity of "$\precsim$". But *this is not true in general:* In the example given in Remark **??**, we see that $L(2,5) = [2,3)$ is open, but $R(2,5) = [4,+\infty)$ fails to be open. This anomalous behaviour cannot appear when the operation "+" is continuous.

The result stated in Proposition **??** is used in [**?**] to prove that, under such conditions, if there exists an additive utility function that represents $(S,+,\precsim)$, then it must be continuous. Later in [**?**] it is proved that an additive utility *must* always exist, as a consequence of the connectedness and the fact of $(S,+,\precsim)$ being topological. Actually it is proved that, under the conditions of Proposition **??**, connected implies super-Archimedean. This fact is used to get an additive utility function, in view of Theorem **??**.

Cooper's reasoning in section 4 of [**?**] follows a different path: Cooper starts by assuming that there is a continuous entropy (constructed in [**?**], Theorem 1). Then Cooper tries to achieve a new entropy, now continuous and additive, by modifying the (not necessarily additive) original entropy. In Cooper's method, given any entropy for the system, the set of states can be embedded in a segment of the real line, since it is the continuous image of a connected set. So any other entropy will be the composition of the given entropy with a strictly increasing function from the real line into itself. Interpreting Cooper's arguments, it seems that any two states $s_0$ and $s_1$, such that $s_0 \to s_1$, are identified with the real numbers 0 and 1. Then Cooper applies the Lemma 1 in [**?**] again and again, to obtain all the states that are in correspondence with a dyadic number in $(0, 1)$. By continuity, and density of the dyadic numbers in $(0, 1)$, Cooper obtains the converse of an additive entropy whose range is the whole $[0, 1]$. It seems that Cooper is arguing that, being $u$ an entropy, the expression $u^{-1}(u(a) + u(b))$ taken as the converse entropy of the composition $a + b$, defines an additive entropy $v$ such that $v(a + b) = u^{-1}(u(a) + u(b))$, $a, b$ being any two systems. Such construction is not clear, however: Actually the so defined function $v$ may or may not be additive. Indeed, the original system could fail to have a *null element* as regards "+", whereas $v^{-1}(0)$ should act as a null element. All along this construction, the only true fact concerning additivity is that the original entropy $u$ is additive as regards a new binary operation "$*$" defined by $a * b = u^{-1}(u(a) + u(b))$, $a, b$ being any two states. Unfortunately, this new composition of states, "$*$", could have no connection with the original one "+".

Anyways, Cooper's arguments in section 4 of [**?**] are by no means worthless: On the one hand in Cooper's proof of Lemma 1 the density of the dyadic numbers in $(0, 1)$ is considered in order to get a suitable entropy. This argument has been used to analyze the structure of nontrivial totally ordered connected topological semigroups $(S, +, \precsim)$, and prove that they are homeomorphic, algebraically isomorphic and isotonic to some unlimited interval of the totally ordered group of additive real numbers endowed with the usual Euclidean topology. (For further details see Corollary 1 and Theorem 5 in [**?**]). On the other hand, it is also noticeable that in Theorem 2 in section 4 of [**?**], Cooper says that: << *The entropy function is uniquely determined for any one system by its values for two particular systems and, when so defined for one system, is defined uniquely for any other system by its value for one state of that system. Any two possible choices of the entropy function are related linearly.*>> Cooper's argument follows from Lemma 1 in [**?**] whose proof is not clear as we have already mentioned. However, we feel that while Cooper's argument is not completely rigorous from a mathematical point of view, it does demonstrate that Cooper's intuition was correct and provides a basis for some deep results in fields that are apparently far removed from concepts and theories of thermodynamics such as, for example, algebraic utility theory.

**Proposition 4.10.** *Given two additive pseudo-utilities $u, v$ defined on a totally ordered semigroup $(S, +, \precsim)$ there exists a positive constant $\alpha$ such that $v = \alpha \cdot u$.*

*Proof.* Let us see a proof for the particular case of *positive* semigroups and $u, v$ being pseudoutilies that take values in $(0, +\infty)$. (For a complete proof, see Lemma 1 in [**?**]): Fix $a \in S$, and consider an element $s \in S$. Put $k = \frac{u(s)}{u(a)}$ and $\alpha = \frac{v(a)}{u(a)}$. Approximate $k$ by a strictly increasing sequence $(r_n)_{n \in \mathbb{N}}$ of rational numbers. Being $r_n = \frac{p_n}{q_n}$ with $p_n, q_n \in \mathbb{Z}$, $q_n \neq 0$, $(n \in \mathbb{N})$, it follows that $p_n u(a) < q_n u(s) \implies p_n \cdot a \prec q_n \cdot s \implies p_n v(a) < q_n v(x)$ $(n \in \mathbb{N})$. Taking limits as $n$ tends to infinity, we have that $v(s) \geq [\frac{u(s)}{u(a)}] \cdot v(a) = \alpha \cdot u(s)$. A similar argument shows that $u(s) \geq \alpha^{-1} \cdot v(s)$. Therefore $v(s) = \alpha \cdot u(s)$. □

## References


[1] A. Ben-Tal, A. Charles and M. Teboulle, *Entropic means*, Journal of Mathematical Analysis and Applications **139** (1989), 537–551.

[2] G. Birkhoff, *Lattice theory (Third edition)*, American Mathematical Society (Rhode Island, 1967)

[3] D. S. Bridges and G. B. Mehta, *Representation of preference orderings*, (Springer Verlag, Berlin, 1995).

[4] J. C. Candeal, J. R. De Miguel and E. Induráin, *Extensive measurement: Continuous additive utility functions on semigroups*, Journal of Mathematical Psychology **40 (4)** (1996), 281–286.

[5] J. C. Candeal, J. R. De Miguel and E. Induráin, *Topological additively representable semigroups*, Journal of Mathematical Analysis and Applications **210** (1997), 375–389.

[6] J. C. Candeal, J. R. De Miguel, E. Induráin and E. Oloriz, *Associativity equation revisited*, Publicationes Mathematicae Debrecen **51 (1-2)** (1997), 133–144.

[7] G. Cantor, *Beiträge zur Begründung der transfinite Mengenlehre I*, Mathematische Annalen **46** (1895), 481–512.

[8] G. Cantor, *Beiträge zur Begründung der transfinite Mengenlehre II*, Mathematische Annalen **49** (1897), 207–246.

[9] J. H. Carruth, J. F. Hildenbrandt, and R. J. Koch, *The theory of topological semigroups*, (Marcel Dekker, New York, 1983).

[10] J. H. Carruth, J. F. Hildenbrandt, and R. J. Koch, *The theory of topological semigroups, 2*, (Marcel Dekker, New York. 1986).

[11] J. L. B. Cooper, *The foundations of Thermodynamics*, Journal of Mathematical Analysis and Applications **17** (1967), 172–193.

[12] J. R. De Miguel, J. C. Candeal, and E. Induráin, *Archimedeaness and additive utility on totally ordered semigroups*, Semigroup Forum **52** (1996), 335–347.

[13] G. Debreu, *Representation of a preference ordering by a numerical function*, in *Decision processes*, R. M. Thrall et al. (eds.) (John Wiley, New York, 1954).

[14] G. Debreu, *Continuous properties of Paretian utility*, International Economic Review **5** (1964), 285–293.

[15] P. C. Fishburn, *The foundations of expected utility*, (D. Reidel, Dordrecht, The Netherlands, 1982).

[16] L. Fuchs, *Partially ordered algebraic structures*, (Pergamon Press, Oxford, 1963).

[17] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. Scott, *A compendium of continuous lattices*, (Springer Verlag, Heidelberg, 1980).

[18] R. Giles, *The mathematical foundations of thermodynamics*, (Pergamon Press, Oxford, 1964).

[19] B. Girotto and S. Holzer, *On the axiomatic treatment of the* Φ*-mean*, Journal of the Italian Statistical Society **4 (3)** (1995), 299–336.

[20] K. H. Hofmann and J. D. Lawson, *Linearly ordered semigroups: Historical origins and A. H. Clifford's influence*, in pp. 15–39 of *Semigroup theory and its applications*, edited by K. H. Hofmann and M. W. Mislove (Cambridge University Press, Cambridge, U. K. 1996).

[21] O. Hölder, *Der Axiome der Quantität und die Lehre von Mass*, Leipziger Berichte Math. Phys. C1. **53** (1901), 1–64.

[22] D. H. Krantz, R. D. Luce, P. Suppes and A. Tversky, *Foundations of measurement*, (Academic Press, New York and London, 1971).

[23] E. H. Lieb and J. Yngvason, *A guide to entropy and the second law of thermodynamics*, Notices of the American Mathematical Society **5** (1975), 195–204.
[24] A. A. J. Marley, *Abstract one-parameter families of commutative learning operators*, Journal of Mathematical Psychology **4** (1967), 414–429.
[25] L. Nachbin, *Topologia e ordem*, (University of Chicago Press, 1950).
[26] P. J. Nyikos and H. C. Reichel, *Topologically orderable groups*, General Topology and Applications **45 (5)** (1998), 571–581.

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