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On functionally θ-normal spaces

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Abstract. Characterizations of functionally θ-normal spaces including the one that of Urysohn's type lemma, are obtained. Interrelations among (functionally) θ-normal spaces and certain generalizations of normal spaces are discussed. It is shown that every almost regular (or mildly normal $\equiv \kappa$-normal) θ-normal space is functionally θ-normal. Moreover, it is shown that every almost regular weakly θ-normal space is mildly normal. A factorization of functionally θ-normal space is given. A Tietze's type theorem for weakly functionally θ-normal space is obtained. A variety of situations in mathematical literature wherein the spaces encountered are (functionally) θ-normal but not normal are illustrated.

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1. INTRODUCTION

Normality is an important topological property and hence it is of significance both from intrinsic interest as well as from applications view point to obtain factorizations of normality in terms of weaker topological properties. First step in this direction was taken by Viglino [18] who defined seminormal spaces, followed by the work of Singal and Arya [13] who introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a seminormal space and an almost normal space. A search for another decomposition of normality motivated us to introduce in [6] the class of (weakly) θ-normal spaces and (weakly) functionally θ-normal spaces. These weak forms of normality serve as a necessary ingredient towards a decomposition of normality.

In [6] functionally θ-normal spaces are defined in terms of the existence of certain continuous real-valued functions. In this paper, in analogy with the normal spaces, we obtain a characterization of functionally θ-normal spaces in terms of separation of certain closed sets by θ-open sets. The resulting characterizations are then used to investigate the interrelations that exist among certain generalizations of normal spaces such as (weakly) (functionally) θ-normal spaces, δ-normal spaces defined by Mack [10], δ-normally separated spaces initiated by Mack [10] and Zenor [19], mildly normal spaces ($\equiv \kappa$-normal spaces) studied by Stchepin [15], and Singal and Singal [14], and almost normal spaces studied by Singal and Arya [13]. Moreover, we obtain a decomposition of functional θ-normality in terms of weak functional θ-normality. In the process we obtain improvements of several known results in the literature. Furthermore, a Tietze's type theorem for weakly functionally θ-normal spaces is obtained.

Section 2 is devoted to basic definitions and preliminaries. Characterizations of functionally θ-normal spaces are obtained in Section 3. A Tietze's type extension theorem is included in Section 4. Interrelations among (weakly) (functionally) θ-normal spaces and certain other weak variants of normality are investigated in Section 5. Moreover, a factorization of functional θ-normality in terms of weak functional θ-normality is obtained in Section 5. Finally, an appendix (Section 6) is included which exhibits a wide variety of situations encountered in mathematical literature wherein a space may be (functionally) θ-normal but not normal.

Throughout the paper, no separation axioms are assumed unless explicitly stated otherwise. The closure of a set A will be denoted by $\overline{A}$ and interior by int$A$.

2. BASIC DEFINITIONS AND PRELIMINARIES

Definition 2.1 ([17]). *Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a **θ-limit point** of A if every closed neighbourhood of x intersects A. Let A_θ denote the set of all θ-limit points of A. The set A is called **θ-closed** if $A = A_\theta$.*

*The complement of a θ-closed set is referred to as a **θ-open set**.*

Lemma 2.2 ([17]). *An arbitrary intersection of θ-closed sets is θ-closed and the union of finitely many θ-closed sets is θ-closed.*

In general the θ-closure operator is not a Kuratowski closure operator, since θ-closure of a set may not be θ-closed (see [4]). However, the following modification yields a Kuratowski closure operator.

Definition 2.3. *Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a **$u\theta$-limit point** of A if every θ-open set U containing x intersects A. Let $A_{u\theta}$ denote the set of all $u\theta$-limit points of A.*

Lemma 2.4. *The correspondence $A \to A_{u\theta}$ is a Kuratowski closure operator.*

It turns out that the set $A_{u\theta}$ is the smallest θ-closed set containing A.

Definition 2.5 ([2]). *A function $f : X \to Y$ is said to be **θ-continuous** if for each $x \in X$ and each open set U containing $f(x)$, there exists an open set V containing x such that $f(\overline{V}) \subset \overline{U}$.*

The concept of θ-continuity is a slight generalization of continuity and is useful in studying spaces which are not regular.

Lemma 2.6. *Let $f : X \to Y$ be a θ-continuous function and let U be a θ-open set in Y. Then $f^{-1}(U)$ is θ-open in X.*

Lemma 2.7 ([6, 7]). *A subset A of a topological space X is θ-open if and only if for each $x \in A$ there exists an open set U such that $x \in U \subset \overline{U} \subset A$.*

Definition 2.8. *A subset A of a topological space X is called **regular G_δ-set** if A is the intersection of countably many closed sets whose interiors contain A.*

The following lemma seems to be known and is easily verified.

Lemma 2.9. *In a topological space, every zero set is a regular G_δ-set and every regular G_δ-set is θ-closed.*

Definition 2.10 ([13]). *A topological space X is said to be **almost normal** if every pair of disjoint closed sets one of which is regularly closed are contained in disjoint open sets.*

Definition 2.11 ([14, 15]). *A topological space X is said to be **mildly normal** ($\equiv$ **κ-normal**) if every pair of disjoint regularly closed sets are contained in disjoint open sets.*

3. FUNCTIONALLY θ-NORMAL SPACES

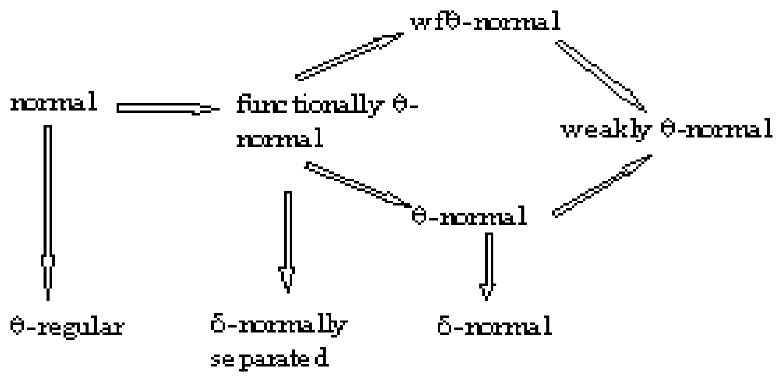
Definition 3.1 ([6]). *A topological space X is said to be*

- *(i) **θ-normal** if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;*
- *(ii) **Weakly θ-normal** if every pair of disjoint θ-closed sets are contained in disjoint open sets;*
- *(iii) **Functionally θ-normal** if for every pair of disjoint closed sets A and B one of which is θ-closed there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$;*
- *(iv) **Weakly functionally θ-normal (wf θ-normal)** if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$; and*
- *(v) **θ-regular** if for each closed set F and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.*

Definition 3.2. *A topological space X is said to be*

- *(i) **δ-normal** [10] if every pair of disjoint closed sets one of which is a regular G_δ-set are contained in disjoint open sets; and* *(ii)* ***δ-normally separated*** [10, 19] *if for every pair of disjoint closed sets A and B one of which is a zero set there exists a continuous function* $f : X \to [0,1]$ *such that* $f(A)=0$ *and* $f(B) = 1$.

The following implications are immediate in view of definitions and Lemma 2.9 and well illustrate the interrelations that exist among generalizations of normality outlined in Definitions 3.1 and 3.2.



None of the above implications is reversible as is shown by Examples 3.6, 3.7, 3.8 in [6], Examples [10, p. 267, p.270] and Example 3.4 in the sequel.

It is shown in [6] that in the class of Hausdorff spaces, the notions of θ-normality and functional θ-normality coincide with normality and that in the class of θ-regular spaces all the four variants of θ-normality characterize normality. Furthermore, it is shown in [6] that every Lindelöf space as well as every almost compact space is weakly θ-normal. In contrast the class of functionally θ-normal space is much larger than the class of normal spaces (see Section 6).

Our next result shows that a Urysohn type lemma holds for functionally θ-normal spaces.

Theorem 3.3. *For a topological space X, the following statements are equivalent.*

(a) *X is functionally θ-normal.*

(b) *For every pair of disjoint closed sets one of which is θ-closed are contained in disjoint θ-open sets.*

(c) *For every θ-closed set A and every open set U containing A there exists a θ-open set V such that $A \subset V \subset V_{u\theta} \subset U$.*

(d) *For every closed set A and every θ-open set U containing A there exists a θ-open set V such that $A \subset V \subset V_{u\theta} \subset U$.*

(e) *For every pair of disjoint closed sets A and B, one of which is θ-closed there exist θ-open sets U and V such that $A \subset U$, $B \subset V$ and $U_{u\theta} \cap V_{u\theta} = \phi$.*

Proof. To prove the assertion (a) $\Rightarrow$ (b), let X be a functionally θ-normal space and let A, B be disjoint closed sets in X, where B is θ-closed. By functional θ-normality of X there exists a continuous function f: $X \to$[0,1] such that $f(A) = 0$ and $f(B) = 1$. Since every continuous function is θ-continuous, by Lemma 2.6, $f^{-1}[0, 1/2)$ and f^{-1} $(1/2, 1]$ are disjoint θ-open sets containing A and B respectively.

To Prove (b) $\Rightarrow$ (c), let A be a θ-closed set in X and let U be an open set containing A. Since A and $X - U$ are disjoint, by hypothesis there exist disjoint θ-open sets V and W such that $A \subset V$ and $X - U \subset W$. So $A \subset V \subset X - W \subset U$. Since $X - W$ is θ-closed and $V_{u\theta}$ is the smallest θ-closed set containing V, $A \subset V \subset V_{u\theta} \subset U$.

To prove (c) $\Rightarrow$ (d), let A be a closed set contained in a θ-open set U. Then $X - U$ is a θ-closed set contained in the open set $X - A$. By hypothesis, there exists a θ-open set W such that $X - U \subset W \subset W_{u\theta} \subset X - A$. Let $V = X - W_{u\theta}$. Then $A \subset V \subset X - W \subset U$. Since $X - W$ is θ-closed and $V_{u\theta}$ is the smallest θ-closed set containing V, $A \subset V \subset V_{u\theta} \subset U$.

To prove (d) $\Rightarrow$ (e), let A be a closed set disjoint from a θ-closed set B. Then $X - B$ is a θ-open set containing A. So there exists a θ-open set W such that $A \subset W \subset W_{u\theta} \subset X - B$. Again by hypothesis there exists a θ-open set U such that $A \subset U \subset U_{u\theta} \subset W \subset W_{u\theta} \subset X - B$. Let $V = X - W_{u\theta}$, then U and V are θ-open sets containing A and B respectively and $U_{u\theta} \cap V_{u\theta} = \varnothing$.

The assertion (e) $\Rightarrow$ (c) is easily verified.

Finally to prove the implication (c) $\Rightarrow$ (a), let A be a θ-closed set disjoint from a closed set B. Then $A \subset X - B = U_1$ (say). Since U_1 is open, there exists a θ-open set $U_{1/2}$ such that $A \subset U_{1/2} \subset (U_{1/2})_{u\theta} \subset U_1$. Again, since $(U_{1/2})_{u\theta}$ is a θ-closed set, there exist θ-open sets $U_{1/4}$ and $U_{3/4}$ such that $A \subset U_{1/4} \subset (U_{1/4})_{u\theta} \subset U_{1/2}$ and $(U_{1/2})_{u\theta} \subset U_{3/4} \subset (U_{3/4})_{u\theta} \subset U_1$. Continuing the above process, we obtain for each dyadic rational r, a θ-open set U_r satisfying $r < s$ implies $(U_r)_{u\theta} \subset U_s$. Let us define a mapping $f : X \to$[0,1] by

$$f(x) = \begin{cases} \inf \{\ x : x \in U_r\ \} & \text{if } x \text{ belongs to some } U_r, \\ 1 & \text{if } x \text{ does not belongs to any } U_r. \end{cases}$$

Clearly f is well defined and $f(A) = 0$, $f(B) = 1$. Now it remains to prove that f is continuous. To this end we first observe that if $x \in U_r$, then $f(x) \leq r$. Similarly $f(x) \geq r$ if $x \notin (U_r)_{u\theta}$. To prove continuity, let $x \in X$ and (a, b) be an open interval containing $f(x)$. Now choose two dyadic rationals p and q such that $a < p < f(x) < q < b$. Let $U = U_q - (U_p)_{u\theta}$. Then U is an open set containing x. Now for $y \in U$, $y \in U_q$. So $f(y) \leq q$. Also as $y \in U$, $y \notin (U_p)_{u\theta}$. Thus $f(y) \geq q$. And so $f(y) \in [p, q]$. Therefore $f(U) \subset [p, q] \subset (a, b)$. Hence f is continuous. □

Example 3.4. *A θ-normal space which is not functionally θ-normal.* Let X be the set of positive integers. Define a topology T on X, where every odd integer is open and a set U is open if for every even integer $p \in U$, the successor and the predecessor of p also belongs to U. Let $Y = X \cup \{\infty\}$ be the one point compactification of the space X. Since every paracompact space is θ-normal

[6, Theorem 3.12], Y is θ-normal. The space Y is not functionally θ-normal since the θ-closed set $\{\infty\}$ and the closed set {1,2} cannot be separated by disjoint θ-open sets in Y.

The above example shows that even a compact θ-normal space need not be functionally θ-normal, and so fills a gap left in [6].

4. A TIETZE TYPE THEOREM

In this section we formulate a Tietze's type extension theorem for weakly functionally θ-normal spaces. First we introduce the notion of a θ-embedded subset which is instrumental in the formulation of Tietze type theorem.

Definition 4.1. *A subset A of a topological space X is said to be **θ-embedded** in X if every θ-closed set in the subspace topology of A is the intersection of A with a θ-closed set in X; equivalently $A \subset X$ is θ-embedded in X if every θ-open set in the subspace topology of A is the intersection of A with a θ-open set in X.*

Remark 4.2. A θ-closed subset of a topological space X need not be θ-embedded in X. For example, let us consider the closed unit interval $X =$ [0,1] with Smirnov's deleted sequence topology [16, p. 86]. Let $A = K \cup \{0\}$, where $K = \{1/n : n \in \mathrm{N}\}$. Here A is a θ-closed subset of X which is not θ-embedded in X.

Theorem 4.3. *Let X be a weakly functionally θ-normal space and let A be a θ-closed, θ-embedded subset of X. Then*

(a) *every continuous function $f : A \to [0, 1]$ defined on the set A can be extended to a continuous function $g : X \to [0, 1]$.*
(b) *every continuous function $f : A \to R$ can be extended to a continuous function $g : X \to R$.*

Proof. We shall prove the theorem only in case (a). Let X be a weakly functionally θ-normal space and let A be a θ-closed set in X. Let $A_1 = \{x \in A : f(x) \geq 1/3\}$ and $B_1 = \{x \in A : f(x) \leq -1/3\}$. Then A_1 and B_1 are two disjoint θ-closed sets in A. Since A is θ-embedded in X, $A_1 = F_1 \cap A$ and $B_1 = F_2 \cap A$, where F_1 and F_2 are θ-closed sets in X. Since A is θ-closed in X, by Lemma 2.2, A_1 and B_1 are disjoint θ-closed sets in X. By weak functional θ-normality of X, there exists a continuous function $f_1 : X \to [-1/3, 1/3]$ such that $f_1(A_1) = 1/3$ and $f_1(B_1) = -1/3$. Now, for each $x \in A$, it is clear that $|f(x) - f_1(x)| \leq 2/3$. So $f - f_1$ is a mapping of A into [-2/3, 2/3].

Let $g_1 = f - f_1$, Then $A_2 = \{x \in A : g_1(x) \geq 2/9\}$ and $B_2 = \{x \in A : g_1(x) \leq -2/9\}$ are two disjoint θ-closed sets in A. As argued earlier, A_2 and B_2 are θ-closed sets in X. Again, by weak functional θ-normality of X there exists a continuous function $f_2 : X \to [-2/9, 2/9]$ such that $f_2(A_2) = 2/9$ and $f_2(B_2) = -2/9$. Clearly, $|(f - f_1) - f_2| \leq (2/3)^2$ on A.

Continuing this process, we obtain a sequence of continuous functions $\{f_n\}$ defined on A such that $|f - \sum_{i=1}^{n} f_i| \leq (2/3)^n$ on A. It is routine to verify that the function $g : X \to R$ defined by $g(x) = \sum_{i=1}^{\infty} f_i(x)$ for every $x \in X$, is the desired continuous extension of f. □

Corollary 4.4. *Let X be a functionally θ-normal space. Then every continuous function $f : A \to [0,1]$ ($f : A \to R$) defined on a θ-closed, θ-embedded subset A of X can be extended to a continuous function $g : X \to [0,1]$ ($g : X \to R$).*

5. INTERRELATIONS

In this section we exhibit the relationships that exists among (functionally) θ-normal spaces, mildly normal spaces and δ-normally separated spaces etc. in the presence of additional mild conditions. This in turn yields improvements of certain known results in the literature.

Definition 5.1 ([12]). *A topological space X is said to be* ***almost regular*** *if every regularly closed set and a point out side it are contained in disjoint open sets.*

The following characterization of almost regular spaces obtained in [8] will be useful in the sequel.

Theorem 5.2 ([8]). *A topological space X is almost regular if and only if, for every open set U in X, int $\overline{U}$ is θ-open.*

Theorem 5.3 ([6]). *A topological space X is θ-normal if and only if for every pair of disjoint closed sets A, B one of which is θ-closed, there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \varnothing$.*

Theorem 5.4. *An almost regular, θ-normal space is functionally θ-normal.*

Proof. Let X be an almost regular, θ-normal space. Let A be a θ-closed set disjoint from a closed set B. Since X is θ-normal, by Theorem 5.3, there exist disjoint open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \varnothing$. Now $A \subset U \subset \text{int}\overline{U} \subset \overline{U}$ and $B \subset V \subset \text{int}\overline{V} \subset \overline{V}$. Since X is almost regular, by Theorem 5.2, int$\overline{U}$ and int$\overline{V}$are disjoint θ-open sets containing A and B respectively. So in view of Theorem 3.3, the space X is functionally θ-normal. □

Corollary 5.5. *An almost regular, θ-normal space is δ-normally separated.*

Proof. Every functionally θ-normal space is δ-normally separated. □

Corollary 5.6. *An almost regular paracompact space is functionally θ-normal.*

Proof. It is shown in [6] that, every paracompact space is θ-normal. Hence the result follows by Theorem 5.4. □

Theorem 5.7. *A mildly normal, θ-normal space is functionally θ-normal.*

Proof. Suppose X is a mildly normal, θ-normal space and let A and B be disjoint closed sets in X, where A is θ-closed . Since X is θ-normal, by Theorem 5.3, there exists disjoint open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \varnothing$. Then $\overline{U}$ and $\overline{V}$ are disjoint regularly closed sets in X. Since X is mildly normal, by [13, Theorem 3], there exists a continuous function $f : X \to [0,1]$ such that $f(\overline{U}) = 0$ and $f(\overline{V}) = 1$. Consequently $f(A) = 0$ and $f(B) = 1$ and so X is a functionally θ-normal space. □

Corollary 5.8. *A mildly normal paracompact space is functionally θ-normal.*

Proof. Every paracompact space is θ-normal [6]. □

Corollary 5.9. *A mildly normal, θ-normal space is δ-normally separated.*

Proof. Every functionally θ-normal space is δ-normally separated. □

Remark 5.10. The example of Smirnov's deleted sequence topology [16, p.86] shows that the hypothesis of θ-normal space in Theorem 5.7 can not be weakened to even "weakly functionally θ-normal space".

Remark 5.11. Example 3.4 shows that the hypothesis of almost regularity in Theorem 5.4 can not be omitted. The same example also shows that the hypothesis of mild normality in Theorem 5.7 can not be deleted. Moreover, simple examples can be given to show that even an almost regular compact space need not be normal.

Theorem 5.12. *An almost regular weakly θ-normal space is mildly normal.*

Proof. Suppose X is an almost regular weakly θ-normal space and let A and B be disjoint regularly closed sets in X. Since A is regularly closed, $X - A$ is regularly open and so $X - A = \text{int}\overline{(X - A)}$. By Theorem 5.2, $X - A$ is θ-open and hence A is θ-closed. Similarly, B is θ-closed. Since X is weakly θ-normal, there exist disjoint open sets U and V containing A and B, respectively. □

Corollary 5.13 ([14])**.** *An almost regular almost compact space is mildly normal.*

Proof. It is observed in [6] that every almost compact space is weakly θ-normal. □

Corollary 5.14 ([14])**.** *An almost regular Lindelöf space is mildly normal.*

Proof. Every Lindelöf space is weakly θ-normal [6]. □

Corollary 5.15 ([11])**.** *An almost compact Urysohn space is mildly normal.*

Proof. Every almost compact Urysohn space is almost regular [11]. □

Theorem 5.16. *An almost regular θ-normal space is almost normal.*

Proof. Suppose X is an almost regular, θ-normal space. Let A and B be disjoint closed sets in X such that A is regularly closed. In view of Theorem 5.2 (as also shown in the proof of Theorem 5.12), A is θ-closed. So by θ-normality, there exist disjoint open sets U and V containing A and B, respectively. Thus X is almost normal. □

Corollary 5.17 ([13])**.** *An almost regular paracompact space is almost normal.*

Proof. Every paracompact space is θ-normal [6]. □

Theorem 5.18. *An almost regular weakly θ-normal space is weakly functionally θ-normal.*

Proof. Let X be an almost regular weakly θ-normal space and let A and B be disjoint θ-closed sets in X. Then there exist disjoint open sets U and V containing A and B, respectively. It is easily verified that the sets int$\overline{U}$ and int$\overline{V}$ are disjoint. By Theorem 5.2, each of the sets int$\overline{U}$ and int$\overline{V}$ is θ-open. Thus every pair of disjoint θ-closed sets in $X$are separated by disjoint θ-open sets and so by [7, Theorem 8], X is weakly functionally θ-normal. □

Corollary 5.19. *An almost regular almost compact space is weakly functionally θ-normal.*

Corollary 5.20. *An almost regular Lindelöf space is weakly functionally θ-normal.*

Next we give a factorization of functionally θ-normal space in terms of weakly functionally θ-normal space.

Definition 5.21. *A topological space X is said to be* ***weakly θ-regular*** *if for each θ-closed set $F$and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.*

Every θ-regular space is weakly θ-regular. However, the cofinite topology on an infinite set yields a weakly θ-regular space (vacuously) which is not θ-regular.

Theorem 5.22. *A topological space X is functionally θ-normal if and only if it is both a weakly θ-regular space and a wf θ-normal space.*

Proof. Necessity is immediate in view of Theorem 3.3 and the fact that every functionally θ-normal space is wf θ-normal. To prove sufficiency suppose that X is a weakly θ-regular space and a wf θ-normal space. Let A and B be disjoint closed sets in X such that A is θ-closed. Then $A \subset X - B$ and $X - B$ is open. So by weakly θ-regularity of X, there exists a θ-open set U such that $A \subset U \subset X - B$. Then A and $X - U$ are disjoint θ-closed sets and $B \subset X - U$. Since X is wf θ-normal, there exists a continuous function $f : X \to [0, 1]$ such that $f(A)$ =0 and $f(X - U) = 1$. Since $B \subset X - U$, X is functionally θ-normal. □

Theorem 5.23. *A topological space X which is both a weakly θ-regular space and a weakly θ-normal space is θ-normal.*

Remark 5.24. Example 3.4 is a wf θ-normal space which is not weakly θ-regular. Similarly, the Smirnov's deleted sequence topology [16, p.86] is a weakly θ-normal space which is not weakly θ-regular.

We conclude this section with the following characterizations of normality in Hausdorff spaces.

Theorem 5.25. *For a Hausdorff space X, the following statements are equivalent.*
(a) X is normal.
(b) X is functionally θ-normal.
(c) X is θ-normal.
(d) X is weakly θ-regular and wf θ-normal.
(e) X is weakly θ-regular and weakly θ-normal.

Proof. Equivalence of (a)-(c) is given in [6] and the equivalence of (a), (d) and (e) is immediate in view of Theorems 5.22 and 5.23. □

6. APPENDIX (EXAMPLES)

Theorem 5.25 shows that the notions of θ-normality and functional θ-normality assume significance in non-Hausdorff spaces. Modern applications of topology in algebraic geometry, spectral theory of commutative rings and lattices, and theoretical computer science have advanced the point of view that interesting topological spaces need not be Hausdorff. In this section we exhibit a few situations arising in spectral theory of commutative rings, topologies on partially ordered sets and lattices and in algebraic geometry wherein the spaces involved are (functionally) θ-normal but not necessarily normal.

- Every finite topological space is functionally θ-normal which need not be normal.
- Cofinite topology on an infinite set as well as the co-countable topology on an uncountable set is a functionally θ-normal space which is not normal.
- One point compactification of a T_1-space which is not a locally compact Hausdorff space is a θ-normal space which is not normal [5].
- The Wallman compactification of a non-normal T_1-space is a θ-normal space which is not normal [5].

6.1. **PRIME SPECTRUM OF A RING.** Let R be a commutative ring with unity. Let $X = Spec(R)$ be the set of all prime ideals of R. For each ideal I of R, let $V(I) = \{P \in Spec(R) : I \subset P\}$. The collection $\{Spec(R)\backslash V(I) : I$ is an ideal of $R\}$ is a topology on X and the collection of all sets $X_f = \{P \in Spec(R) : f \notin P\}$, $f \in R$ constitutes a base for this topology. The topology on $SpecR$ described above is called the ***Zariski topology.*** $Spec(R)$ endowed with Zariski topology is called prime spectrum of the ring R and is always a compact T_0-space. So by [6, Theorem 3.12], Spec(R) is a θ-normal space which is not necessarily a normal space.

For example, the spectrum of the ring Z of integers is homeomorphic to a countably infinite space X in which, apart from X, only finite subsets are closed. This space is a compact T_1-space which is not Hausdorff and in which any two nonempty open sets intersects. Hence X is a functionally θ-normal space which is not normal.

6.2. **TOPOLOGICAL REPRESENTATION OF LATTICES.** Details of the definitions and results quoted here may be found in [1]. Let L be a distributive lattice. Let $P(L)$ denotes the set of all prime ideals of L. For each $x \in I$,
$\hat{x} = \{I \in P(L) : x \notin I\}$. Then
(1) $\hat{x} \cup \hat{y} = \widehat{x+y}$ for all $x, y \in L$.
(2) $\hat{x} \cap \hat{y} = \widehat{xy}$ for all $x, y \in L$.

Definition 6.1. *A Stone space is a topological space X satisfying:*

(i) $X$is a T_0-space.
(ii) Compact open subsets of X constitute a base for X and ring of sets.
(iii) If $(X_s)_{s\in S}$ and $(Y_t)_{t\in T}$ are nonempty families of nonempty compact open sets and $\bigcap_{s\in S} X_s \subset \bigcap_{t\in T} Y_t$, then there exist finite subsets S' and T' of S and T respectively, such that $\bigcap_{s\in S'} X_s \subset \bigcap_{t\in T'} Y_t$.

*For a distributive lattice L, let $R(L)$ denotes the **representation space** of L, whose points are members of $P(L)$ with the topology induced by taking the collection $\{\varnothing\}\cup\{\hat{x} : x \in L\}$ as a base. Then the space $R(L)$ is a Stone space.*

- Let L be a distributive lattice with 1. Then its representation space $R(L)$ is a compact space and so it is a θ-normal space which need not be normal (see [1, p.78]).
- A distributive lattice is relatively complemented if and only if its representation space $R(L)$ is a T_1-space. Hence if L is a relatively complemented distributive lattice with 1 which is not a Boolean algebra, then its representation space $R(L)$ is a compact T_1-space which is not Hausdorff (see [1, p.78]). So $R(L)$ is a θ-normal space which is not normal.

6.3. **TOPOLOGIES ON LATTICES AND POSETS.** Details of the results and definitions quoted below may be found in [3].

Definition 6.2. *Let L be a lattice. The topology generated by the complements $L\setminus \uparrow x$ of all filters is called the **lower topology** on L and is denoted by $w(L)$. A base for the lower topology $w(L)$ is given by $\{L\setminus \uparrow F : F$is a finite subset of $L\}$.*

Definition 6.3. *Let L be a lattice. The **upper topology** on L is generated by the collection of all sets $L\setminus \uparrow x$ and is denoted by $v(L)$.*

Definition 6.4. *Let L be a poset closed under directed sups. A topology on L is said to be **order consistent** if*

(i) $\overline{\{x\}} = \downarrow x$ *for all* $x \in L$
(ii) If x = *sup* I *for an ideal, then* x = *lim* I.

Definition 6.5. *Let* L *be a complete lattice. A subset* U *of* L *is said to be* ***Scott open*** *if*
(i) $U = \uparrow U$, *i.e.,* U *is an upper set.*
(ii) sup $D \in U$ *implies* $D \cap U \neq \varnothing$ *for all directed* $D \subseteq L$.
The collection of all Scott open subsets of L *constitutes a topology on* L *called Scott topology and is denoted by* $\sigma(L)$.

Definition 6.6. *Let* L *be a complete lattice. Then the common refinement of Scott topology and the lower topology is called the* ***Lawson topology*** *and is denoted by* $\lambda(L)$.

Definition 6.7. *A topology on a poset is called* ***compatible*** *if the directed nets converge to their sups and filtered nets converge to their infs.*

- For a complete lattice L, the space (L, $\lambda(L)$) is a compact T_1-space and hence a θ-normal space which may fail to be normal (unless it is Hausdorff) (see [3, p.146]).
- Let (X, $\leq$) be a complete lattice equipped with a topology such that the relation $\leq$ is lower semicontinuous (i.e. each $\downarrow x$ is closed in X). Then the set of open upper sets is an order consistent topology on X and hence X is a compact space [3, p.307]. So X is a θ-normal space.

6.4. **AFFINE ALGEBRAIC VARIETIES.** Details of the definitions and results quoted below may be found in [9]. Let $A^n(L)$ be n-dimensional affine space over a field L and let $K \subset L$ be a subfield. Let $K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K.

Definition 6.8. *A Subset* $V \subset A^n(L)$ *is called an* ***affine algebraic K-variety*** *if there are polynomials* $f_1, \ldots, f_m \in K[x_1, \ldots x_n]$ *such that* V *is the solution set of the equations* f_i $(x_1, \ldots, x_n)=0$ $(i = 1, \ldots, m)$.
A K-variety is called ***irreducible*** *if* $V = V_1 \cup V_2$ *with K-varieties* V_1, V_2, *then* $V = V_1$ *or* $V = V_2$.

The finite unions and arbitrary intersections of K-varieties in $A^n(L)$ are K-varieties. Thus K-varieties form the closed sets of a topology on $A^n(L)$ called the Zariski topology with respect to K. If $V \subset A^n(L)$ is a K-variety, then V carries the relative Zariski topology on V.

Definition 6.9. [16] *A topological space* X *is said to be* ***hyperconnected*** *if every nonempty open set is dense in* X.

Definition 6.10. *A topological space* X *is called* ***Noetherian*** *if every descending chain* $F_1 \supset F_2 \supset \ldots$ *of closed subsets* $F_i \subset X$ *is stationary.*

- An algebraic K-variety endowed with Zariski topology is a hyperconnected space if and only if it is irreducible, and every hyperconnected space is functionally θ-normal (which need not be normal). For example, co-finite topology on an infinite set is a hyperconnected space which is not normal. Thus it turns out that every irreducible algebraic K-variety is a functionally θ-normal space which need not be normal.

- It turns out that every open subset of a Noetherian topological space is compact and hence θ-normal by [6, Theorem 3.12]. Since every algebraic K-variety V endowed with Zariski topology is a Noetherian topological space, so every open subset of V is a θ-normal space which is not necessarily normal.

6.5. **PROJECTIVE VARIETIES.** The n-dimensional projective space $P^n(L)$ over a field L is the set of all lines through the origin in L^{n+1}. A point $x \in P^n(L)$ can be represented by an (n+1)-tuple $(x_0,\ldots,x_n) \neq (0,\ldots,0)$ in L^{n+1} and $(x_0',\ldots,x_n') \in L^{n+1}$ defines the same point if and only if there is $\lambda \in L$, with $(x_0,\ldots,x_n)=(\lambda x_0',\ldots,\lambda x_n')$. An (n+1)-tuple representing x is called the system of homogeneous co-ordinates of x. We write $x = < x_0,\ldots,x_n >$.

If K is a subfield of L and $f \in K[x_0,\ldots,x_n]$, then $x \in P^n(L)$ is called a *zero of f* if $f(x_0,\ldots, x_n)=0$ for every system $(x_0,\ldots,x_n)$ of homogeneous co-ordinates of x.

Definition 6.11. *A subset $V \subset P^n(L)$ is called a **projective algebraic K-variety** if there are homogeneous polynomials $f_1,\ldots,f_m \in K[x_0,\ldots,x_n]$ such that V is the set of all common zeros of f_i in $P^n(L)$.*

The finite unions and arbitrary intersections of projective K-varieties in $P^n(L)$ are projective K-varieties. Thus projective K-varieties in $P^n(L)$ constitute the closed sets of a topology on $P^n(L)$, called the K-Zariski topology. It turns out $P^n(L)$ (and hence any projective K-variety) endowed with K-Zariski topology is a Noetherian topological space and so every open subset of $P^n(L)$ is a θ-normal space which need not be normal. Moreover, an irreducible projective K-variety endowed with K-Zariski topology is a hyperconnected space and so it is a functionally θ-normal space which need not be normal.

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