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Computational Differential Topology

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Abstract. Some of the more differential aspects of the nascent field of computational topology are introduced and treated in considerable depth. Relevant categories based upon stratified geometric objects are proposed, and fundamental problems are identified and discussed in the context of both differential topology and computer science. New results on the triangulation of objects in the computational differential categories are proven, and evaluated from the perspective of effective computability (algorithmic solvability). In addition, the elements of innovative, effectively computable approaches for analyzing and obtaining computer generated representations of geometric objects based upon singularity/stratification theory and obstruction theory are formulated. New methods for characterizing complicated intersection sets are proven using differential analysis and homology theory. Also included are brief descriptions of several implementation aspects of some of the approaches described, as well as applications of the results in such areas as virtual sculpting, virtual surgery, modeling of heterogeneous biomaterials, and high speed visualizations.

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1. Introduction

The field of computational topology (CT) is a relatively new one - roughly a decade old - that invites and actually relies upon contributions from several more well established disciplines such as algebraic topology, computer science, differential topology, and computational geometry (see in particular [19] and [150], and also [27], [32], [34], [43], [44], [49], [70], [87], [126], [129], [147], and [154], which are representative of the roots of this new field). CT is about ten years old in terms of being a reasonably well defined discipline that has generated its own specialized research, but the name was apparently first coined by Mäntylä [103] a little over twenty years ago. Although still in its very early stages, CT has experienced a significant spurt of growth and maturity in the last seven years.

This rapid development and expansion can be traced in large measure to the important role it plays in many applications in such areas as computer aided design and manufacturing, (CAD/CAM), the life sciences, and virtual reality. However, another major component of the current attraction and vitality of CT lies in the wealth, depth and diversity of its approaches, which are based upon or incorporate modern techniques in algorithm and representation theory, and image processing from computer science, as well as mathematical fields such as algebraic geometry, algebraic topology, differential geometry, differential topology, dynamical systems theory, general topology, and singularity and stratification theory such as in [2] - [6], [25], [26], [31] and [139]. With regard to the singularity/stratification theory aspects of CT, one should also add the interesting and useful related work of Damon and his collaborators, examples of which are to be found in [37], [38], and [41] (see also [36] and [46]).

Owing in part to its tender age, CT is an attractive area of research for mathematicians, computer scientists, and computer aided geometric designers: There is much work to be done in laying the foundations for the discipline, there are a wide variety of challenging open problems - many of fundamental importance - and there are a host of anticipated applications waiting to be discovered and exploited. The field is by its very nature interdisciplinary, and has in recent years begun to attract the attention of a growing number of computer aided geometric designers, computational biologists, computer scientists, mathematicians, and also engineers interested in CAD/CAM and virtual reality applications. In contrast, the far more well developed discipline of computational geometry (CG) has been around much longer, and has established itself in the course of the last forty years as a virtually indispensable tool for solving difficult problems arising in CAD/CAM and other contexts that rely on computationally powerful methods for analysis and accurate representation of geometric objects (cf. [50], [56], [58], [79], [80], [104], and [130]).

CG will no doubt continue to be an active and important field, and is likely to play a significant complementary role in the development of CT. The significance and essential nature of CG in the modeling and computer generated representation of geometric objects is now well established, but the role of CT in such applications has only recently risen to a comparable level of prominence (cf. [59]). One can view the difference between these two disciplines as being roughly analogous to the difference between geometry and topology, and can be rather effectively summarized in the following terms: The field of CG is primarily concerned with algorithmic (computer implementable) methods for analyzing and producing representations of geometric objects that are close - usually in some Whitney-like (piecewise) C^2 sense - to the actual object, while the fundamental goal of CT is to algorithmically guarantee that a computer generated representation of an object is equivalent to the actual object in an appropriate topological sense. Simply stated, CG is concerned with insuring the (differential geometric) closeness of the representation of an object to the original, while CT takes care of the topological consistency of the rendering.

Remarkable advances in computer technology - coupled with impressive progress in CG and computer aided geometric design - have made it possible to algorithmically analyze and render geometric configurations of dazzling complexity. But with this complexity, it has become increasingly more difficult to avoid very small scale errors that can have a dramatic impact on the topological type of the representations. It is therefore not surprising that those questions that lie at the very heart of the nascent discipline of computational topology have attained a far greater level of importance in applications over the last few years.

For example, suppose one wants to produce a computer generated representation of a water-tight container to be used in an automated manufacturing process. We can view this container in idealized form as a (deformed) sphere in three-dimensional space, thus rendering it as an object in a standard differential geometry or topology category. An algorithm can readily be found that produces a representation that is as close as desired (in some suitable Whitney-type topology) to the designed container, but still has one or more very small holes. Insofar as the specified tolerances are met in terms of the position and derivatives associated to the generated representation of the desired geometric object, this may be considered satisfactory from the perspective of CG. However, it is certainly not acceptable from the CT viewpoint, and the result obviously would produce serious shortcomings in the manufactured article (cf. [45]).

We shall, in this paper, attempt to delineate a suitable context in which to formulate many of the fundamental concepts, questions, and techniques of CT, which may help to develop the rigorous foundations that are necessary for this relatively new discipline to take its place among related, but more mature fields such as CG. Our perspective will be a decidedly differentiable one; focusing on approaches that borrow liberally from the basic elements of differential and manifold topology (see e.g. [15], [66], [78], [82], [96], [99], [106], [109], [110], [112], [115], [133], [146], [151], and [156]), algebraic topology and geometry (see e.g. [73], [86], [105], [116], [140], and [160]), differential and computational geometry (see e.g. [50] and [91]), and dynamical systems theory (see e.g. [69], [72], [88], and [142]).

As CT is still an emerging discipline and is largely unknown to many in the computer aided geometric design, computer sciences, and mathematics communities, before embarking on a more thorough description of the frontiers of this nascent field, we shall present a brief outline of the elements of computational differential topology (in the form of a primer), including a description of the relevant categories, identification of some of the fundamental questions and problems, and a few new results, in Section 2. This, we hope, will serve to introduce the reader to most of the basic tools that are to be used in the sequel, and provide a reasonably clear description of the entelechy of computational differential topology. Next, in Section 3, we present a rather detailed, wide ranging description of the state-of-the-art of the classification problem for geometric objects - both manifolds and non-manifolds - of various dimensions. What sets our treatment apart from the usual ones in differential topology is the focus on the algorithmic nature (effective computability) of the methods for determining the isomorphism type of the objects.

The treatment in Section 3 is followed in Section 4 with a brief description of an effectively computable singularity/stratification theory for a class of varieties that we refer to as sweep-like. Then we sketch the elements of an effectively computable obstruction theory for this same class of sweep-like varieties in Section 5. We then follow this in Section 6 with several new analytical and homology based results on the identification and topological characterization of certain types of intersections of geometric objects. In Section 7, we present a brief discussion of some of the applications of the methods and approaches described in this paper. The applications range from virtual design, manufacturing and surgery, to modeling of heterogeneous biomaterials such as bones, to new approaches for the kinds of complex high speed visualizations that one is apt to employ or encounter in dynamic genetic modeling. Finally, in Section 8, we make some remarks concerning the results presented in the paper, and identify several related research directions that ought to be pursued.

2. Computational Differential Topology Primer

A sure sign of a well developed mathematical or scientific subdiscipline is the establishment and general acceptance of well defined mathematical categories that characterize and circumscribe the field (cf. [129] and [140]). As far as we can gather, the field of computational (differential) topology has not yet matured to the point where its fundamentals are widely accepted, so we first describe the categories in which we shall work in order to frame the rest of this paper.

2.1. **Categories.** The sets of interest in computational differential topology (CDT) are geometric objects in Euclidean space, usually having certain differentiability properties, but they need not and should not be restricted to manifolds. Examples such as the locus of $x^2 + y^2 - z^2 = 0$ in $\mathbb{R}^3$, which defines a cone, and geometric objects with self-intersections show that we need to include varieties. A possible approach to describing the objects in an appropriate category is to introduce special varieties (s-varieties) having the property that there are at most finitely many local regular (topological manifold) branches at each of the singular points (cf. [26]). However, a perfectly plausible and potentially more efficient - although more restrictive - way to describe the objects in the computational topology (CT) categories is to employ *Whitney regular stratifications* (see e.g. [15], [26], [66], [99], [106], [146], and [156]). To begin, we fix Euclidean space $\mathbb{R}^N$ as the ambient space for the geometric objects and an order of differentiability k ($0 \leq k \leq \omega$), where $k = \infty$ represents continuous derivatives of all orders, and $k = \omega$ stands for (real) analyticity. Then we have the following definition of the objects in the CDT category:

Definition 2.1. *A subset V of $\mathbb{R}^N$ is a cdt_N^k object if it can be represented in the form*

$$V = M_1 \cup M_2 \cup \cdots \cup M_s, \tag{2.1}$$

where the collection $\mathfrak{S} := \{M_i : 1 \leq i \leq s\}$ is a Whitney regular stratification of V. This stratification is comprised of a finite disjoint set of strata M_i, which are open or closed C^k submanifolds of $\mathbb{R}^N$, called the strata of the stratification, and the strata have dimensions that can range from 0 (points) to N (open solid regions). The dimension of V in cdt_N^k is defined as $\dim V := \max\{\dim M_i : M_i \in \mathfrak{S}\}$.

Remark 2.2. The cdt_N^k objects in the above definition are clearly subvarieties of $\mathbb{R}^N$, and in the sequel we shall use the terminology objects, subvarieties, and varieties more or less interchangeably when we refer to these entities.

Remark 2.3. Note that the cone described above is in cdt_3^ω, as is a closed cube. As we shall be concentrating in this paper mainly on geometric objects that have some differential structure, most of our attention shall be directed to cases where $k \geq 1$.

Remark 2.4. We note for future use that the regular stratification gives rise to a natural ordering of the strata defined as follows:

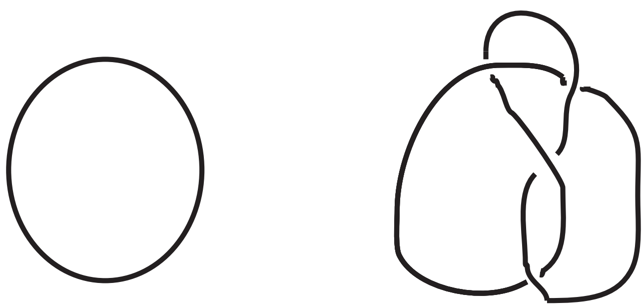
$$V_i \prec V_j \Longleftrightarrow V_i \subset \mathrm{cl}\,(V_j)\,, \tag{2.2}$$

where $\mathrm{cl}\,(V_j)$ denotes the (topological) closure of V_j. With this, it follows from Thom-Mather theory (cf. [66], [99], [106], and [146]) that if $F : \mathfrak{S} \to \tilde{\mathfrak{S}}$ is an order preserving bijection between regular stratifications of subvarieties V and $\tilde{V}$, respectively, such that the corresponding strata are homeomorphic, then V and $\tilde{V}$ are homeomorphic. Accordingly the question of homeomorphism type of objects in any CDT category can be simplified by reduction to the individual strata.

Remark 2.5. Now that we have suitable objects for our categories, it remains to define appropriate morphisms. The more usual choice leading to homeomorphic or diffeomorphic equivalence obviously will not do as is illustrated in the following example.

Example 2.6. A circle S^1 embedded in a hyperplane of $\mathbb{R}^3$ provides a convenient representation of the *unknot* K_0, which we wish to compare with the figure-eight knot K embedded in $\mathbb{R}^3$, both of which are shown (in standard over and under form) in Fig. 1. The embedded sets K_0 and K are obviously C^ω-diffeomorphic, 1-dimensional submanifolds, but can certainly not be viewed as the same in any reasonable CT sense since they are not equivalent as embeddings in the ambient space $\mathbb{R}^3$. In particular, the knot group for the circle is $\pi_1\left(\mathbb{R}^3 \setminus K_0\right) = \mathbb{Z}$, while the knot group for the figure-eight knot $\pi_1\left(\mathbb{R}^3 \setminus K\right)$ is the group with two generators α and β and one relation, $\beta\alpha^{-1}\beta\alpha\beta^{-1} = \alpha^{-1}\beta\alpha\beta^{-1}\alpha$, where $\pi_1(X)$ denotes the fundamental group of a topological space X (cf. [105] and [140]).

As is clear from the preceding example, morphisms must be equivalent in some sense as embeddings in the ambient space, as well has having certain differentiability properties. However, if the knots are embedded in $\mathbb{R}^4$, they are embedding equivalent if and only if they are C^ω-diffeomorphic (see [133]).



(a) K_0 (b) K

Figure 1. (a) Unknot (b)Figure-eight knot

Definition 2.7. *A morphism between two objects V and W in cdt_N^k is a continuous mapping $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following properties:*

(i) *Φ maps the strata of V into the strata of W.*
(ii) *The restriction $\Phi_{|V}$ of Φ to V is of class C^k.*

Recall that, as usual, to say that $\varphi := \Phi_{|V}$ is of class C^k means that φ can be extended to a C^k function in an open neighborhood of V in $\mathbb{R}^N$. With this we have the last piece necessary for the definition of our CT categories for objects embedded in Euclidean space $\mathbb{R}^N$.

Definition 2.8. *For the Euclidean space $\mathbb{R}^N$ and order of differentiability $0 \leq k \leq \omega$, the CDT category, denoted as CDT_N^k, is comprised of all the objects in cdt_N^k as in Definition 2.1, and the morphisms as in Definition 2.2, with the usual composition of morphisms. We denote the objects in CDT_N^k as Obj(CDT_N^k) and the morphisms by Morph(CDT_N^k), and the set of all morphism from V to W as $hom(V, W)$.*

The above definition leads directly to the notion of an isomorphism in the CDT categories.

Definition 2.9. *Two objects V and W in Obj(CDT_N^k) are isomorphic, denoted as*

$$V \underset{N}{\overset{k}{\approx}} W, \tag{2.3}$$

iff there is a homeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Phi(V) = W$, and the restrictions of Φ to V and its inverse Φ^{-1} to W are both of class C^k.

Remark 2.10. In most cases when the ambient space and differentiability class are fixed, we simplify the above notation by omitting the subscript and superscript in the isomorphism notation, and simply write $V \approx W$. In the sequel we shall, for convenience, indulge in the harmless abuse of notation of referring to both objects and morphisms as being members of the category CDT_N^k rather than distinguishing between the set of objects and set of morphisms comprising this category.

Equivalence (isomorphism) in the categories CDT_N^k (which is sometimes referred to as embedding equivalence - cf. [133]) is obviously more restrictive than homeomorphic equivalence in the standard topological category TOP, or in the standard differential category DIFF^k when applied to manifolds. To be more specific, in addition to the usual homeomorphism type invariants such as homotopy, cohomotopy, homology, and cohomology that one needs to consider for equivalence in TOP, one must also verify the invariance of quantities such as linking numbers to check equivalence in the CDT categories. For future reference, we denote isomorphism (homeomorphism) in the TOP category as

$$V \simeq W, \tag{2.4}$$

and isomorphism (diffeomorphism) in the category DIFF^k as

$$V \overset{k}{\simeq} W. \tag{2.5}$$

Remark 2.11. In the CT literature, when an algorithmically generated representation V of a prototype object V_0 satisfies $V \simeq V_0$, V or the algorithm itself is often said to be *topologically consistent*.

There is another useful equivalence that is even stronger than (2.2), and has been effectively employed by T. J. Peters and his collaborators in an interesting and useful series of papers in computational topology (see e.g. [7]-[8], [12], [13], [14], [20], [114], [124], [125], [127], [135], and [136]).

Definition 2.12. *Two objects V and W in CDT_N^k are ambient isotopic if there exists a continuous map (called an ambient isotopy)*

$$\Theta : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$$

such that

(a) $\Theta_0 = \mathrm{id}$,
(b) $\Theta_t : \mathbb{R}^N \to \mathbb{R}^N$ *is a homeomorphism for all* $0 \le t \le 1$,
(c) $\Theta_1(V) = W$,

where $\Theta_t := \Theta(\cdot, t)$ *for all* $t \in [0,1]$.

Remark 2.13. We note here that for the case of smooth knotted and unknotted circles in $\mathbb{R}^3$, standard knot equivalence, ambient isotopy, and isomorphism in CDT_3^0 are all equivalent to one another (cf. [75], [105]).

A fundamental goal in CT is to create computer generated procedures for obtaining representations of objects having the same shape as a given geometric object - at least in some acceptable approximate sense. Accordingly one must understand precisely what is meant by shape, a question that we address in the next subsection.

2.2. **Shape of geometric objects.** What does it mean to say that two objects, V and W in $\mathrm{Obj}(\mathsf{CDT}_N^k)$ have the same shape? The shape of a geometric object and its preservation for various types of algorithmic representations has been the subject of several investigations, such as [26], [32], [34], [41], [87], [95], [118], and [119], yet there still seems to be no consensus on the definition of shape of geometric objects in the CT community. Naturally, to have the same shape, V and W ought to at least be isomorphic in the CT category, but intuition suggests that quite a bit more is required. The following appears to be a viable definition of shape.

Definition 2.14. *The objects V and W in CDT_N^k have the same shape if the following properties obtain:*

(i) *There exists an isomorphism* $\varphi : V \to W$
(ii) *There exists a constant* $c > 0$ *such that* $c^{-1}\varphi$ *is an isometry. More particularly, recall that for* φ *to be an isomorphism in* CDT_N^k *it must be extendable to a homeomorphism* $\Phi : \mathbb{R}^N \to \mathbb{R}^N$. *Consequently, the restriction of* Φ *to* V *(which is* φ*) must be a* C^k *map such there exist a* $c > 0$ *and an isometric* C^k*-embedding* $\psi : V \to \mathbb{R}^N$ *(in the metric induced on* V *by the Euclidean metric on* $\mathbb{R}^N$ *) with* $\Phi(x) = c\psi(x)$ *for all* $x \in V$. *We denote this property of having the same shape by*

$$V \overset{k}{\underset{N}{\equiv}} W, \tag{2.6}$$

and omit the subscript and superscript for simplicity whenever the context is clear.

Computational representation of geometric objects usually involves some approximation error, which necessitates the use of the following definition, or something of the same sort, for computational topology applications.

Definition 2.15. *Given $\epsilon > 0$, we say that V and W in CDT_N^k have the same shape* $(\mathrm{mod}\,\epsilon)$ *if they are isomorphic in this category via $\varphi : V \to W$, and there are a positive number c and an isometric C^k-embedding $\psi : V \to \mathbb{R}^N$ such that φ is ϵ-close to $c\psi$ in the Whitney C^k-topology, which means that derivatives of all orders less than or equal to k of φ and $c\psi$ differ by less than ϵ (in the appropriate operator norm) over all of V. Having the same shape* $(\mathrm{mod}\,\epsilon)$ *is denoted as*

$$V \overset{k}{\underset{N}{\equiv}} W \ (\mathrm{mod}\,\epsilon), \tag{2.7}$$

where as usual we shall suppress the subscript and superscript when the context is clear.

We note that all the material that one needs to understand the Whitney topology in the above definition may be found in [26], [66], [99], or [106]. At this stage, we are now are in possession of all the notation that we require for our formulation of CDT.

2.3. **Effective computability of geometric objects.** We have accumulated all of the background material necessary to describe some of the primary challenges facing computational differential topologists. The challenges all revolve around the process of being given a prototype object V_0 in CDT_N^k, which must be represented by computer generated means based upon an algorithm $\mathcal{A}$. The word 'given' here is not as simple as one would wish, and can in some cases assume a rather loose interpretation. This derives from the reality that the prototype object may be an actual physical object, or it may be defined exactly in terms of equations, or a precisely specified, completely developed model of a geometric object, or represented by data sampled from an existing physical object such as a statue or building, or - in the worst case - may be only partially and imprecisely known simply in terms of representative data, such as point-clouds, sampled according to some scheme (cf. [134]).

From the perspective of a computational topologist, an algorithm for representing and analyzing a geometric object must include a subroutine for verifying that the computed object has the same isomorphism type as the given object - assuming that this much is known about the object to be represented. For cases in which one has only incomplete topological knowledge of the prototype object, an algorithm designed to produce computer generated representations, say at various levels of (metric) accuracy, should at least be capable of verifying that the isomorphism type (or some of its key invariants) remains constant as the accuracy is refined. When it is possible to devise a suitable algorithm of this type, such a constant 'limit' may serve as a good educated guess of the actual isomorphism type of the partially known prototype object.

Before elaborating further on the quest for solutions of some fundamental problems in CDT, we shall find it useful to introduce a few more concepts and notation. The next definition concerns effectively computable recognition of objects in CDT_N^k and the related categories TOP and DIFF^k.

Definition 2.16. *Let V_0 be a given object in CDT_N^k, DIFF^k, or TOP, and let V be another such object. Then V is said to be $\mathsf{CDT}_N^k\,(V_0)$-decidable, $\mathsf{DIFF}^k\,(V_0)$-decidable, or $\mathsf{TOP}(V_0)$-decidable, respectively, if there exists an algorithm $\mathcal{A}$ to determine if $V \approx V_0$, $V \overset{k}{\simeq} V_0$, or $V \simeq V_0$. Such an algorithm is called, respectively, a $\mathsf{CDT}_N^k\,(V, V_0)$-decider, $\mathsf{DIFF}^k\,(V, V_0)$-decider, or a $\mathsf{TOP}(V, V_0)$-decider.*

A closely related and useful notion is the following:

Definition 2.17. *If there exists an algorithm that determines the isomorphism type of an object V in CDT_N^k, DIFF^k, or TOP, we say this object is, respectively, CDT_N^k - decidable, DIFF^k - decidable, or TOP- decidable. Such an algorithm is called, respectively, a CDT_N^k - decider, DIFF^k - decider, or a TOP - decider.*

It is convenient to also have an idea of what is meant by stability or robustness of algorithms with respect to their ability to determine the isomorphism classes of the geometric objects that they generate.

Definition 2.18. *An algorithm $\mathcal{A}$ for generating representations V of an object V_0 in CDT_N^k, DIFF^k, or TOP, is $\mathsf{CDT}_N^k\,(V_0)$ -stable (robust), $\mathsf{DIFF}^k\,(V_0)$ -stable (robust), or $\mathsf{TOP}(V_0)$ -stable (robust), respectively, if for small changes in the input data to $\mathcal{A}$, there is no change in the isomorphism type of the output representation V in the respective categories.*

Remark 2.19. With regard to the above definitions, it is obvious that we have the following sequences of implications:

$$\begin{gathered}
\mathsf{CDT}_N^k\,(V_0)\,\text{−decidable} \Longrightarrow \mathsf{DIFF}^k\,(V_0)\,\text{−decidable} \Longrightarrow \mathsf{TOP}\,(V_0)\,\text{−decidable},\\
\mathsf{CDT}_N^k\text{−decidable} \Longrightarrow \mathsf{DIFF}^k\text{−decidable} \Longrightarrow \mathsf{TOP}\text{−decidable},\\
\mathsf{CDT}_N^k\,(V_0)\,\text{−stable} \Longrightarrow \mathsf{DIFF}^k\,(V_0)\,\text{−stable} \Longrightarrow \mathsf{TOP}\,(V_0)\,\text{−stable}.
\end{gathered}$$

There is a lack of rigor in our definition of stability insofar as what is meant by ‘small changes’. We shall return to this briefly in the sequel, but it should be a reasonably straightforward matter to fit a rigorous definition to any particular environment or context arising in applications of CDT.

We are now in a position to adumbrate a grand challenge for CDT (including the requirement of a good metric approximation consistent with the expectations of computational geometry), which amounts to what one may call the holy grail for the discipline, or possibly an ultimate wish list for those working in the field.

Grand Challenge Problem of CDT (GCPCDT)

Determine a subset $\mathcal{C}_{\circledast}$ of the objects in CDT_N^k, ample enough to include most geometric objects arising in applications, and such that $V_0 \in \mathcal{C}_{\circledast}$ and all sufficiently small $0 < \epsilon$, there is an algorithm $\mathcal{A}(V_0, \epsilon)$ satisfying the following properties:

(a) *It generates a representation $V(\epsilon)$ such that* $V(\epsilon) \equiv V_0 \pmod{\epsilon}$.
(b) *It is $\mathsf{CDT}_N^k(V_0)$ -stable.*
(c) *It is optimally efficient in that its computational complexity, denoted as $CC(\mathcal{A}(V_0, \epsilon))$, is minimal in some reasonable sense.*

Remark 2.20. Observe that (a) above implies that the algorithm $\mathcal{A}(V_0, \epsilon)$ is a $\mathsf{CDT}_N^k(V, V_0)$-decider.

Remark 2.21. It should be noted that, although not specifically included in the above definition of the GCPCDT, ease of implementation with regard to producing user-friendly software based on the algorithm is also an important consideration, especially when it comes to applications. For cases in which the isomorphism type of V_0 is not *a priori* completely known, (a) above would have to be adjusted; perhaps along the following lines: (a$'$) $V(\epsilon_1) \equiv V(\epsilon_2) \pmod{\epsilon_0}$ for all $0 < \epsilon_1, \epsilon_2 < \epsilon_0$. We shall have more to say about this in the sequel.

In general, a completely satisfactory solution of the GCPCDT as stated may be extremely difficult - or even impossible - to achieve, so simplified versions of this problem, such as those we describe in the next section, are highly desirable. It should be noted that if the GCPCDT is viewed from a computational geometry rather than a CT viewpoint, one should choose the differentiability class k to be greater or equal to two, so that the representations produced are acceptable in terms of differential geometry; where second derivatives (wherever they exist) manifested in curvature tensors (or differential forms) are essential elements - at least in the classical sense - in determining good approximations.

3. Equivalence and Decidability

In this section we focus on effective procedures for the classification of compact subvarieties V in CDT_N^k, with $k \geq 1$. We shall obtain a few new results concerning the structure of the objects under investigation, and look at known and developing results from differential topology with an eye toward algorithmic implementation. Throughout we assume reasonably good knowledge of the material in our references on algebraic topology, piecewise linear topology, and differential topology, such as [63], [78], [86], [99], [105], [108], [109], [110], [112], [115], [116], [133], [140], [146], [151], [156], and [160]. We begin our treatment with a closer look at the structure and associated structures of the subvarieties, which we shall find useful in our investigation of the decidability of the varieties to be studied.

3.1. **Structures associated to the subvarieties.** Our first result here shows that compact objects in the CDT category have smooth (and *a fortiori*, topological) triangulations in the sense of Munkres [115]. Triangulations, of course, are very useful in solving problems in computational topology (see e,g. [28], [35], [42], [50], [71], and [127]).

Theorem 3.1. *If V is a compact, connected object in CDT_N^k, with $1 \leq k < \infty$, then it has a finite C^k triangulation.*

Proof. Let $\mathfrak{S} := \{M_i : 1 \leq i \leq s\}$ be the regular stratification of the compact subvariety V. If the stratum M_i is closed, it follows from [115] that it has a C^k triangulation. This triangulation can be extended to all strata M_j satisfying $M_i \prec M_j$ by making minor adjustments in the proof of Munkres' theorem on the extension of C^k triangulations of the boundary of a manifold to the whole manifold. It then follows from compactness, connectedness and Munkres' refinement theorems that the finite number of various triangulations of the strata can be refined in a consistent manner that produces a finite C^k triangulation of all of V.

□

The following property is an immediate consequence of this theorem.

Corollary 3.2. *A compact, connected object V in CDT_N^k, with $1 \leq k < \infty$, has the structure of a finite CW-complex.*

Another useful construct for the compact subvarieties is an analog of a tubular neighborhood for a manifold. We start with the usual distance function

$$d\,(x,V) := \min\left\{|x-y| : y \in V\right\}, \tag{3.1}$$

where $|\cdot|$ is the standard Euclidean norm induced by the standard inner product $\langle\cdot,\cdot\rangle$.

Definition 3.3. *Let $\lambda : V \to \mathbb{R}$ be a positive (tolerance) function of class C^r, with $r > k$. The λ-tubular neighborhood $\tau_\lambda = \tau_\lambda\,(V)$ of V is defined as*

$$\tau_\lambda = \tau_\lambda\,(V) := \left\{x \in \mathbb{R}^N : d\,(x,V) \leq \lambda\,(\sigma(x))\right\}, \tag{3.2}$$

where the function $\sigma : \mathbb{R}^N \to V$ is given as

$$\sigma(x) := y \in V$$

such that $y = (y_1, y_2, ..., y_N)$ is the unique point of V satisfying $d\,(x,V) = |x-y|$ and y_1, $y_1 + y_2$,..., and $y_1 + y_2 + \cdots + y_N$ are all minimized.

If V is a submanifold (without boundary, i.e $\partial M = \varnothing$), then τ_λ has the usual nice disk-bundle structure if the maximum of λ is chosen to be sufficiently small (cf. [78] and [82]). When V is not a manifold, the bundle structure breaks down on the boundary and singularities of the subvariety, yet the utility of this tubular neighborhood is not unduly affected for many applications. In fact, it appears that one could, with appropriate modifications, proper choice of the tolerance function (along the lines of having the smallest values near the singularities and boundaries) and perhaps restrictions on the tolerance of derivatives, recover analogs of most of the tubular neighborhood based, ambient isotopy results such as that of Chazal and Cohen-Steiner [30], and of Peters *et al.*, such as those in [7] -[8], [12], [124], [125] and [135]. This leads us to make the following conjecture.

Conjecture 3.4. *Let V be a compact, connected subvariety in CDT_N^k, with $1 \leq k < \infty$. Then there exists a choice of tolerance function λ and positive ϵ such that if W is another compact, connected subvariety in CDT_N^k satisfying $W \subset \tau_\lambda(V)$, and nearby tangent spaces (when defined) of V and W are ϵ-close in an appropriate Grassmannian sense, then the two subvarieties are isomorphic in the category CDT_N^0.*

3.2. **Fundamental problems of CDT.** We return here to the GCPCDT with more rigor than in the preceding section, reformulate it as a 'fundamental problem', and then show how this problem can be greatly simplified. Of course, it always seems a bit presumptuous to decide to refer to anything in mathematics or computer science as fundamental, but we feel that the name is apt.

3.2.1. *Fundamental problem.* The GCPCDT as presented in the preceding sections is somewhat lacking in rigor. Moreover, as H. Edelsbrunner pointed out when a version of this problem was unveiled recently, it also is deficient in scope - especially as regards the wide range of possibilities in knowledge of the prototype object, means of obtaining data from the object for the algorithm, and methods available for rendering the computational representations.

In order to pose this problem in more depth and with greater specificity, we shall first present a more detailed version of the GCPCDT that will enable us to better formulate just what aspects require further work. It is clear that we need more precise notation concerning the computational procedures embodied in the algorithm $\mathcal{A}$ devised to produce an approximate representation $V(\epsilon)$ of the prototype geometric object V_0 in CDT_N^k for a given error bound ϵ. We emphasize here that the error bound is on the geometry - not the topology, as invariance of the isomorphism type is a *sine qua non* for the algorithm from the perspective of CDT. The input data from V_0, which we denote as $D\left(V_0\right)$, may assume any one of several possible forms such as the vertex points and connection relations for the elements of a triangulation of the prototype object, a global functional representation or a set of local functional expressions arising from exact mathematical models, an approximate nonuniform rational B-spline (NURBS) decomposition of V_0, or points forming a point-cloud sampled in a manner designed to provide a good approximation of the given object, which is often the case when V_0 is not completely known.

One can already see here that there is a problem in formulating an adequate characterization of the space $\mathfrak{D}$ in which the data obtained from the prototype object resides. A good definition of this data space is required so that we can consider D as a function from (the object set of) CDT_N^k to $\mathfrak{D}$, which can be expressed as

$$D : \mathsf{CDT}_N^k \to \mathfrak{D}.$$

Naturally, the tolerance (geometric accuracy) ϵ must also be counted as an argument of the algorithm. Now we may regard the algorithm as a recursive map of the form

$$\begin{aligned} \mathcal{A} : D\left(\mathsf{CDT}_N^k\right) \times \mathbb{R}_+ &\longrightarrow \mathsf{CDT}_N^k \\ (D(V_0), \epsilon) &\longmapsto V(\epsilon) \end{aligned} \tag{3.3}$$

where $\mathbb{R}_+$ is the set of positive real numbers, and $V(\epsilon)$ is a representation of a computer generated (geometric) approximation of V_0 - or more precisely, an algorithm for producing an approximate representation of the prototype object. With this more rigorous foundation established, we are now in a position to give a reasonably precise account of one of the main problems in CDT:

Fundamental Problem of CDT

Given an object V_0 from an ample subset of interest $\mathcal{O}$ of the class of objects of CDT_N^k , a data function D as in (3.10), *and a small positive number ϵ_*, devise an algorithm $\mathcal{A} = \mathcal{A}\left(D(V_0), \epsilon\right)$ defined for all $V_0 \in \mathcal{O}$ and $0 < \epsilon < \epsilon_*$, for which the following obtain*:

(i) *$\mathcal{A}$ generates an approximate representation $V(\epsilon)$ of V_0.*
(ii) $V(\epsilon) \equiv V_0 \pmod{\epsilon}$ for all $\epsilon \in (0, \epsilon_*)$.
(iii) *$\mathcal{A}$ is CDT_N^k - stable for all $V_0 \in \mathcal{O}$.*
(iv) *The algorithm has minimal computational complexity $CC(\mathcal{A})$ in some sense.*

Once again, for practical purposes it may be appropriate to include the requirement that the algorithm be user-friendly in the conventional sense. Thus it would be necessary to give a more detailed definition of just what user-friendly means. Although more precise than the GCPCDT, which was presented in preceding section, the Fundamental Problem (FPCDT) described above is clearly still beset with deficiencies in several respects, which we briefly address. The above description of the FPCDT is lacking in detail with regard to the wide range of methods that can be used in the development of the algorithm, and the degree to which the prototype object is known. Moreover, it would probably benefit from a more thorough and detailed exposition of the representation approach used to produce the output object $V(\epsilon)$.

Remark 3.5. In a case where the isomorphism class of the prototype object V_0 in CDT_N^k is not completely known, it becomes necessary to revise the requirement (ii). One possibility is a direct modification to something like

(ii)′ The outputs $V(\epsilon_1)$ and $V(\epsilon_2)$ with $0 < \epsilon_1, \epsilon_2 < \epsilon_0$ satisfy $V(\epsilon_1) \equiv V(\epsilon_2) \pmod{\epsilon_0}$ for all sufficiently small ϵ_0.

This suggests a possible notion of persistence of isomorphism type analogous to the basic ideas used to formulate persistent homology and other related homology-based approaches (cf. [55], [86], [117], [160], and [161]). There also are several other ways in which ambiguity can be manifested, such as

- only certain isomorphism invariants of the prototype are known, in which case we would like the approximations to have the same values for these invariants,
- we could demand that the approximations agree in some global statistical sense - say in terms of statistical tests of randomly sampled point-cloud data,
- we could require that our knowledge of the object is increased (manifold learning) by analyzing a sequence of approximations.

Additional insights on the analysis for prototype objects whose isomorphism type is not *a priori* specified, may be found in [10], [28], [50], [57], [67], [77], [81], [92], [102], [134], and [154].

The exposition of the fundamental problem also is inadequate owing to the imprecision of the minimality statement for computational efficiency. It appears that no single definition of what constitutes an appropriate minimum is going to be possible - the criteria will most likely depend on the context in which the algorithms are used, and the applications for which it is implemented. At any rate, the question of acceptable notions of minimality for computational costs of algorithms that generate isomorphically consistent representations of geometric objects, appears to be one that should lead to fertile grounds for continued research.

Any resolution of the minimality of computational complexity, even if restricted to specific applications oriented contexts, is bound to be quite challenging, partly owing to the extensive array of minimality criteria available for applications, but more likely to stem from the difficulty of actually proving minimality for an algorithm in most reasonable, nontrivial senses. As algorithms developed to render approximations of geometric objects possessing only a fair degree of complexity are usually rather intricate, verifying minimality of computational complexity tends to be difficult.

Researchers with some experience in solving problems in CT will have also observed that part of the inherent ambiguity of the fundamental problem would be ameliorated if some of the techniques for determining isomorphism type (at least approximately) were included in the above description. Most of the methods currently employed to analyze isomorphism type involve the algorithmic computation, where feasible, of isomorphism invariants such as characteristic classes (e.g. the Euler class, Stiefel-Whitney classes, Pontryagin classes, and Chern classes for complex manifolds) homology groups, and cohomology rings, along with approaches based upon tubular neighborhoods, Morse theory, Morse-Floer theory, singularity/stratification theory, and obstruction theory, examples of which can be found in [2]-[6], [7]-[8], [9], [12], [16], [25], [26], [18],

[30], [31], [50], [52], [61], [86], [90], [111], [117], [125], [136], [139], [147], [157], [160], and [161].

Among the more interesting recent approaches to employing Morse theory in an effectively computable way are the cell complex method of Forman [61], and the related discrete strategy of King & Knudson [90]. We should also mention current efforts to employ Morse theory in an algorithmic way using Morse-Smale complexes and Reeb graphs such as in Cole-McLaughlin *et al.* [33], and Edelsbrunner *et al.* [51, 53, 54].

Remark 3.6. The notion of a Reeb graph, which connects critical points of a Morse function associated to a handlebody decomposition of a differentiable manifold, can be extended to varieties in the category CDT_N^k by including singularities as additional vertices of the graph. This extension, which we call the *Reeb* graph* of the variety, will then play a role analogous to that of Reeb graphs for manifolds. We shall show in a forthcoming paper that there are effective procedures for using Reeb* graphs to determine the isomorphism type of subvarieties in CDT_N^k.

Another interesting direction that shows some promise for the algorithmic classification of isomorphism type involves characterization of the medial axis of a geometric object (see e.g. [11], [36], [38], and [46]). As for more surprising approaches, there also is a fairly recent spate of publications employing innovative methods from general (T_0) topology, such as [65], [77], [92], [93] and [102], that appear to be applicable to the (complete or partial) recursive computation of isomorphism type.

3.2.2. *Simplified fundamental problem.* The last twenty years have produced impressive advances in the realm of computational geometry leading to the creation of several algorithms for generating very (metrically) accurate representations of geometric objects. By marrying these developments with the derivation of new tubular neighborhood based theorems, it now appears possible to recast the fundamental problem in the following far more tractable form.

Simplified Fundamental Problem of CDT

Given a compact object V_0 from an ample subset of interest $\mathcal{O}$ of the class of objects of CDT_N^k , a data function D as in (3.10), *and a small positive number ϵ_*, devise an algorithm $\mathcal{A} = \mathcal{A}\left(D(V_0), \epsilon\right)$ for all $V_0 \in \mathcal{O}$ and $0 < \epsilon < \epsilon_*$ such that the following obtain*:

- (i) *$\mathcal{A}$ generates an approximation $V(\epsilon)$ of V_0, which is ϵ-close in a suitable Whitney-type topology.*
- (ii) *$V(\epsilon) \simeq V_0$ for all $\epsilon \in (0, \epsilon_*)$.*
- (iii) *$\mathcal{A}$ is TOP - stable for all $V_0 \in \mathcal{O}$.*
- (iv) *The algorithm has minimal computational complexity $CC\left(\mathcal{A}\right)$ in some sense.*

The basis for the above simplification is what has been called the *Self-intersection Precedes Knotting Principle* (SIPKP), which can be intuitively argued to be plausible as follows: Using methods from computational geometry, it is possible to algorithmically generate an approximate representation V, which is an arbitrarily small perturbation in position and derivatives (where they exist) of a given compact prototype object $V_0 \in \mathrm{Obj}(\mathsf{CDT}_N^k)$, with $1 \leq k$. Thus, in particular, we can assume that V is contained in as thin a tubular neighborhood of V_0 (of the type described in Def. 3.3) as desired, and has tangent hyperplanes on its strata that are arbitrarily close to the corresponding ones on the strata of V in the appropriate Grassmannian spaces.

Under suitably restrictive hypotheses on the perturbation in terms of these kinds of measures of closeness, it should follow that a homeomorphism from V to V_0 can be confined to the tubular neighborhood of V that contains it, so as to enable its extension to all of τ_λ such that it coincides with the identity map in a neighborhood of the boundary of τ_λ. Then, it only remains to extend the homeomorphism to all of $\mathbb{R}^N$ by defining it to be the identity in the complement of τ_λ. Finally, the differentiability of the homeomorphism can be bootstrapped up to class C^k using the inherent differentiable structures of V_0 and V. The methods suggested here should be compared with those employed in such papers as [7]-[8], [13], [64], and [136].

Of course, our argument here is only suggestive of a verification of the SIPKP; not a real proof. We shall prove a version of this principle in a forthcoming paper, but until then, we shall continue to treat the Simplified Fundamental Problem of CDT (SFPCDT) as a viable premise. However, we make the following speculation.

Conjecture 3.7. *The SIPKP is valid under suitable hypotheses on the proximity of V to V_0 in an appropriate Whitney-type topology.*

It goes almost without saying that the simplified fundamental problem has weaknesses that are analogous to those of the fundamental problem, and we leave it to the reader to draw these analogies.

3.3. **Decidability of Isomorphism Type.** Our discussion of the fundamental problem and its simplified version in the previous section raises the question of just what types of objects in CDT_N^k, with $k \geq 1$, are amenable to algorithmic determination of their isomorphism types. We shall concentrate on this question in this section. It is assumed here that the reader has some familiarity with the basics of differential topology, as well as a reasonably good grasp of those aspects of computer science and logic related to recursive functions and effective (algorithmic) procedures.

The focus of this section is upon the properties that render a geometric object (embedded in Euclidean space) decidable in the relevant categories for CDT, where we recall that decidability of an object in a particular category is tantamount to the existence of an effective procedure for determining its isomorphism type. In the process, we shall present a fairly wide ranging survey of classical and current results on the topological and differential classification of manifolds. We begin with compact submanifolds and submanifolds-with-boundary, as they are typically easier to classify in terms of the categories of interest here, namely TOP, DIFF^k and CDT_N^k. It should be noted that we follow the convention of sometimes grouping submanifolds (which are usually defined to have empty boundaries) with submanifolds-with-boundary.

3.4. **Decidability of compact submanifolds.** Our discussion, which proceeds in the order of increasing dimension N of the ambient Euclidean space, although not exhaustive, will touch upon many of the most important aspects of the (algorithmic) classification of compact objects in CDT_N^k, with $k \geq 1$. In the process, we shall occasionally make comparisons with results for noncompact objects. We shall assume throughout that our objects are connected. This will entail no loss of generality inasmuch as compact objects can have at most finitely many components.

3.4.1. *Compact submanifolds of* $\mathbb{R}$. We begin with connected, *closed* (= compact, boundaryless) submanifolds M of $\mathbb{R}$. These are particulary simple - M is a point in the zero-dimensional case, and there are no closed one-dimensional submanifolds of $\mathbb{R}$. If we remove the compactness assumption, one has connected, codimension-0, C^k submanifolds; namely open intervals. In any case, these are all obviously trivially decidable by stable algorithms in minimal (linear) time. In particular, the (ranks of the) integral homology provides a complete, effectively computable set of isomorphism invariants in all of the categories TOP, DIFF^k, CDT_1^k. Recall that a *complete set of isomorphism invariants* is set of quantities that are invariant under isomorphism in the particular category, and have the property that two objects are isomorphic if and only if these invariants assume the same values for both objects.

The connected, compact, C^1 submanifolds-with-boundary of $\mathbb{R}$ are also easy to classify algorithmically in CDT_N^k, for they are comprised of closed intervals. Moreover, the homology provides a complete set of recursively computable invariants for an algorithm that is manifestly stable, and has minimal (linear) computational complexity.

3.4.2. *Compact submanifolds of* $\mathbb{R}^2$. The situation in $\mathbb{R}^2$ is also essentially trivial, with the decidability of the homeomorphism type or isomorphism type in CDT_3^k being a simple matter indeed. All connected, closed, zero-dimensional submanifolds of $\mathbb{R}^2$ are just points, and the codimension-1, submanifolds are simple as well. In particular, it follows from the Jordan curve theorem and other basic principles, that every connected, closed, C^k submanifold M of codimension-1 must be equivalent to the circle S^1 in CDT_2^k and *a fortiori* in DIFF^k and TOP. Moreover this can be determined by a single effectively computable invariant, which is the condition $H_1(M, \mathbb{Z}) = \mathbb{Z}$ for the first integral homology group, or equivalently described in terms of the Euler-Poincaré characteristic as

$$\chi(M) = \sigma_0 - \sigma_1 = \operatorname{rank} H_0(M, \mathbb{Z}) - \operatorname{rank} H_1(M, \mathbb{Z}) = 0, \tag{3.4}$$

where σ_j is the number of j-dimensional simplices in a triangulation, and the rank is defined in the usual way (cf. [78], [105], [112], [116], [140], and [160]). Note also that if we choose an algorithm $\mathcal{A}$ based on computation of χ, we readily find that $CC(\mathcal{A}) = O(n_s)$, where n_s is the number of top (=1)-dimensional simplices in a triangulation of M, and one cannot do much better than this with respect to computational efficiency. As a matter of fact, it follows readily that both the full and simplified fundamental problems are completely solved for compact submanifolds of $\mathbb{R}^2$ including the establishment of computational minimality for the algorithm assuming that the prototype submanifold is completely simplicially defined in terms of triangulations.

As for the connected, compact, C^k submanifolds-with-boundary ($\partial M \neq \varnothing$) of $\mathbb{R}^2$, they can be readily characterized as follows: There are no such zero-dimensional submanifolds; the one-dimensional submanifolds are C^k diffeomorphs of closed, finite intervals in $\mathbb{R}$; and the two-dimensional submanifolds are closed disks with at most finitely many open disks removed from their interior. In all of these cases, the homology yields a complete set of invariants for deciding isomorphism type in CDT_2^k, and the algorithm can be chosen to be stable and executable in linear time.

These simple results already provide an indication of the usefulness of algebraic topology in dealing with the decidability problem for submanifolds. In this vein, we include the following result for future reference. It can be readily proved using the C^1 triangulation theorems of Munkres [115], and some basic results on the effective (algorithmic) computability of homology and cohomology for finite simplicial complexes (see [86], [116], and [160]).

Theorem 3.8. *Let M be a compact submanifold in CDT_N^k ($k \geq 1$). Then M has a finite C^1 triangulation, and the homology $H_*(M,F)$, cohomology $H^*(M,F)$, and all of the applicable characteristic classes such as the Euler, Stiefel-Whitney, and Pontryagin classes for M are effectively computable in polynomial time, where the coefficient ring F can be the integers $\mathbb{Z}$, or the integers mod 2 denoted as $\mathbb{Z}_2$.*

3.4.3. *Compact manifolds in Euclidean 3-space.* We now show that it is in $\mathbb{R}^3$ that both the isomorphism classification and the decidability problem first assume nontrivial proportions. Let M be a compact, connected submanifold (possibly with $\partial M \neq \varnothing$) in CDT_3^k with $k \geq 1$. When $\dim M = 0$, both the classification and decidability problem are trivial in TOP, DIFF^k and CDT_3^k. For $\dim M = 1$, things begin to get very complicated. If M is closed, it must be diffeomorphic to a circle, but it can be embedded in $\mathbb{R}^3$ as a very complicated knot. Decidability in TOP is easy - in fact it is completely decidable via homology as in $\mathbb{R}^2$, so there exists a stable algorithm for deciding homeomorphism type in linear time. In CDT_3^k, the isomorphism classes correspond to knot types. It follows from [74] and [75] that M is CDT_3^k-decidable, but the problem of determining the isomorphism class appears to be quite computationally expensive, and is likely to be NP-complete (see also [9] and [83]). This dramatic contrast is a very effective demonstration of how much more difficult it can be to solve the complete fundamental problem (FPCDT) than the simplified fundamental problem (SFPCDT).

An embedded closed surface M, must be orientable, and an easy solution of the decidability problem follows from the simple and elegant classical result (see e.g. [105] and [140]) that the homeomorphism and diffeomorphism types of such a submanifold are completely determined by the Euler-Poincaré characteristic

$$\chi = \sigma_0 - \sigma_1 + \sigma_2 = \operatorname{rank} H_0(M, \mathbb{Z}) - \operatorname{rank} H_1(M, \mathbb{Z}) + \operatorname{rank} H_2(M, \mathbb{Z}) . \tag{3.5}$$

Accordingly the problem for DIFF^k- and TOP-decidability is stably solvable in linear time. Again, there is a striking difference in the degree of difficulty of the TOP- and CDT_3^k-decidability problems, as one can see by considering the thin toral surface of a smoothly thickened knotted curve. Once again, it appears that M is CDT_3^k-decidable - although there seems to be no proof of this in the literature - but the computational complexity of any associated algorithm would appear to be quite high.

The homeomorphism or diffeomorphism types of compact submanifolds-with-boundary M of codimension-1 in $\mathbb{R}^3$ - which may be nonorientable as in the case of a Möbius strip - is completely determined by $\chi(M)$, the orientability, and the number of boundary components (cf. [105]). We note here that the orientability of a submanifold can be completely determined via homology (cf. [140]). Therefore, M is TOP-decidable in linear time. On the other hand, M is apparently CDT_3^k-decidable, but the computational complexity of the problem is very high and largely unknown. Some of these observations can be conveniently summarized in the following problem.

Problem 3.9. *Prove that every compact, connected, C^1-submanifold (possibly with a nonempty boundary) of $\mathbb{R}^3$ of dimension less than or equal to 2 is stably CDT_3^k-decidable, and obtain estimates for the computational complexity of algorithms that can determine isomorphism type.*

A compact, connected, 3-dimensional, C^1-submanifold M of $\mathbb{R}^3$ must have a nonempty boundary ∂M. It is easy to see that if ∂M is connected, it completely determines M; hence, M is decidable in both TOP and CDT_3^k. An analog of this ought to be true in the case when ∂M is not connected, but this still appears to be an open problem.

Problem 3.10. *Prove that every compact, connected, three-dimensional C^1-submanifold-with-boundary of $\mathbb{R}^3$ is both TOP- and CDT_3^k-decidable, and obtain estimates for the computation complexity of algorithms that can determine isomorphism type in these categories.*

3.4.4. *Compact manifolds in Euclidean 4-space.* There is a far more diverse and interesting range of compact C^k submanifolds of $\mathbb{R}^4$ than $\mathbb{R}^3$, but we shall confine our attention to just some of those of dimension one or higher. Any connected, closed, 1-dimensional, C^k submanifold M of $\mathbb{R}^4$ must be a diffeomorph of the circle S^1. In $\mathbb{R}^4$, as compared with $\mathbb{R}^3$, the extra dimension renders classification in CDT_4^k equivalent to that in DIFF^k, so it follows from our preceding analysis that M is CDT_4^k-decidable by a stable algorithm executable in linear time, where the homology provides a complete set of isomorphism invariants. In this and the higher dimensional cases in the sequel, we concentrate mainly on TOP-decidability, which is associated with the simplified fundamental problem.

It is well known that all closed surfaces and compact surfaces-with-boundary, including the nonorientable ones such as the Klein bottle, projective plane, and the Möbius strip, can be embedded in $\mathbb{R}^4$ (cf. [105]). We showed above how the decidability problem for oriented compact surfaces can be easily and efficiently solved. This is also true for the nonorientable surfaces, all of which can be realized as two-dimensional, closed submanifolds and compact submanifolds-with-boundary of $\mathbb{R}^4$. For these cases the TOP, DIFF^k and CDT_4^k isomorphism types also are completely determined by the orientability, or lack thereof, the Euler-Poincaré characteristic, and the number of boundary components. Moreover, the isomorphism type can be stably computed in linear time. To summarize decidability for compact surfaces: they represent the lowest dimensional nontrivial submanifolds for which the fundamental problem becomes interesting, yet is easily solvable by simple classical means expressed, modulo orientability and possible boundary components, in terms of a single invariant that is computable in linear time. Consequently, surfaces - both orientable and nonorientable and with or without boundary - represent excellent examples to illustrate how the fundamental problem may be solved, and also the ideal in that the algorithm is just about as simple, efficient and stable as could be hoped for.

The 3-sphere S^3 is the simplest closed, connected, three-dimensional, C^k submanifold of $\mathbb{R}^4$. Unlike most of the examples considered so far, its homeomorphism type is not completely determined by homology. In fact, Poincaré, while in the process of formulating his famous conjecture, produced his *homology 3-sphere example*, defined as $M_\pi := \mathrm{SO}(3)/\mathcal{I}$, where SO(3) is the Lie group known as the special orthogonal group (rotations) of $\mathbb{R}^3$, and $\mathcal{I}$ is the icosahedral group of order 60. M_π has the same homology as S^3, but it is not simply-connected (i.e. $\pi_1(M) \neq 0$), so it cannot be homeomorphic with S^3. The Poincaré Conjecture has been dominating the mathematical news of late, owing to the excitement created by the work of Perelman [120, 121, 122, 123]. This long-standing conjecture of Poincaré states that a connected, simply-connected, three-dimensional manifold M having the homology of a 3-sphere must, in fact, be homeomorphic with S^3. Perelman's results, which rely heavily upon Hamilton's Ricci flow methods, actually prove Thurston's Elliptization Conjecture [149], which implies the Poincaré Conjecture (cf. [29]). We should consider the impact of his work in the context of decidability questions.

Although Ricci flow methods do not naturally lend themselves to algorithmic computation, Perelman's approach, does suggest a very straightforward effective procedure for determining if a closed, three-dimensional, C^1-manifold M is a 3-sphere: First, using the C^1 triangulation guaranteed by Theorem 3.1, show that the fundamental group is trivial, which can be accomplished algorithmically by computing the edge-path group of a triangulation of M (cf. [140]). Employing the same triangulation, it follows from Theorem 3.8 that the integral homology of M is effectively computable. Then if one computes that $H_0(M,\mathbb{Z}) = H_3(M,\mathbb{Z}) = \mathbb{Z}$, and $H_1(M,\mathbb{Z}) = H_2(M,\mathbb{Z}) = 0$, it follows that M is diffeomorphic, and *a fortiori* homeomorphic with S^3. But there already is an effective procedure [147], namely the Rubinstein-Thompson algorithm, for deciding if a manifold is homeomorphic with S^3, which requires at most exponential time. These considerations lead naturally to the following problem, which is comprised of several parts.

Problem 3.11. *Develop an algorithm based on the computation of the edge-path group and the integral homology as described above for deciding whether a closed manifold is homeomorphic (or diffeomorphic) with S^3. Compare the computational complexity of this new algorithm with that of the Rubinstein-Thompson algorithm. In addition, devise an alternative algorithm, if possible, employing Ricci flow techniques, and compare it with the other algorithms.*

Based upon our analysis up to this point, we make the following conjecture.

Conjecture 3.12. *Every closed, connected, simply-connected, three-dimensional C^k submanifold of $\mathbb{R}^4$ is stably TOP-decidable in polynomial time.*

Remark 3.13. If we drop the compactness assumption, $\mathbb{R}^4$ naturally is a connected, simply-connected, open, C^∞ submanifold of itself, and fair game for the FPCDT and SFPCDT. Although TOP-decidability is relatively easy, recent results of Donaldson, Freedman, Gompf and others show that DIFF^k-decidability may be hopeless: It has been proved that there are uncountably many *fake* $\mathbb{R}^4$'s, which are 4-manifolds homeomorphic with $\mathbb{R}^4$, but all of which have different diffeomorphism types (see eg. [47], [62], and [63]). It is interesting to compare this with the case of compact submanifolds of dimension less than or equal to three, where classification up to isomorphism is equivalent in the categories TOP and DIFF^1.

3.4.5. *Submanifolds of higher dimensional Euclidean spaces.* The Whitney Embedding Theorem (see e.g. [15], [66], and [99]) implies that every closed, four-dimensional C^1-manifold M can be embedded in $\mathbb{R}^N$ with $N \geq 9$. Four-manifolds provide some of the most interesting and complicated DIFF^k- and TOP-decidable examples available, and they also yield important insights into the limitations of decidability. It follows from the work of Freedman, Donaldson, *et al.* (as in [47], [62], and [63]) that all closed, simply-connected, orientable, four-dimensional, C^1-manifolds M can be classified up to homeomorphism type. As a corollary, one obtains a proof of the Generalized Poincaré Conjecture for 4-spheres; namely, every simply-connected, homology 4-sphere is homeomorphic with the 4-sphere S^4.

A remarkable aspect of this classification theory is the particularly simple criteria for determining the homeomorphism type, which comes out of the following observations. Elementary algebraic topology, Poincaré duality and the universal coefficient theorem for homology imply that $H_0(M,\mathbb{Z}) = H_4(M,\mathbb{Z}) = \mathbb{Z}$, $H_1(M,\mathbb{Z}) = H_3(M,\mathbb{Z}) = 0$, and $H_2(M,\mathbb{Z})$ is a free abelian group. This leads one to at least predict the important role in classification of 4-manifolds played by the bilinear, unimodular *intersection form*

$$\omega : H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}. \tag{3.6}$$

Freedman's classification theorem states that the closed, simply-connected, four-dimensional C^1 manifolds are completely classified by their intersection forms. Consequently, we infer from Theorem 3.8 that these manifolds are also TOP-decidable. However, this result has, as far as we know, not appeared in the literature, so we include it here as a conjecture.

Conjecture 3.14. *All closed, connected, simply-connected, four-dimensional C^1 submanifolds of Euclidean space $\mathbb{R}^N$ are TOP-decidable by an algorithm that requires no more than polynomial time.*

Remark 3.15. If M is merely a connected, closed, topological 4-manifold, another invariant besides ω is required for topological classification; namely the Kirby-Siebenmann invariant κ, which is defined to be zero or one, according as $M \times \mathbb{R}$ has a differentiable structure or not.

We see then 4-manifolds can lead to what may be considered as ideal examples of geometric objects when it comes to topological decidability, but they also can produce undecidable objects. In particular, it can be shown using simple manifold surgery that every finitely presented group G can be realized as the fundamental group of a closed, connected, four-dimensional C^∞ manifold. Using this fact, and certain undecidability results for the isomorphism problem for groups, A. Markov proved that 4-manifolds are in general not TOP-decidable (cf. [105] and [140]). There are limits to the topological decidability of manifolds after all, and one need not look higher than four dimensions to find them.

Higher dimensions provide more room for the techniques of differential topology to perform their mathematical legerdemain, so it is not surprising that the Generalized Poincaré Conjecture and the classification of closed, simply-connected, differentiable manifolds were actually disposed of by Smale [141], Stallings [143], Wallace [152], Zeeman [159], and others for closed manifolds of dimension greater than four more than a decade before Freedman's four-dimensional tour de force. The earlier breakthroughs of Smale, Stallings, Wallace and Zeeman employed a variety of differential topological techniques such as Morse Theory, cobordism theory, and obstruction theory, all of which appear to be accessible to algorithmic formulations for manifolds in CDT_N^k . Combining these results with those of Freedman, Donaldson *et al.*, we are emboldened enough to make the following speculation, which promises to be difficult to verify. Actually, we suggest that it might be prudent to first try to prove it for the case of simply-connected, homology n-spheres.

Conjecture 3.16. *Every closed, connected, simply-connected, n-dimensional submanifold in CDT_N^k, where $k \geq 1$ and $n \geq 4$, is stably TOP-decidable in polynomial time.*

Observe that in these last results we have considered only decidability in TOP, and we did so with good reason. Isomorphism type in TOP and DIFF^k, which are equivalent for dimensions less than four, are demonstrably different in higher dimensions, thus establishing the intrinsic difference between topology and differential topology. This was first demonstrated by Milnor [107], who showed that there are several differentiable manifolds (*exotic spheres*) - realized as S^3-bundles over S^4 - that are homeomorphic, but not diffeomorphic, with the 7-sphere. Additional work by Milnor, Kervaire, and Brieskorn led to methods for calculating the number of exotic spheres in all dimensions greater than four; for example, there are 28 exotic 7-spheres if orientation is considered (see e.g. [89] and [108]). It took the work of Freedman to settle the case of the 4-sphere; namely, there are no exotic 4-spheres. On the other hand, for some closed, connected, 4-dimensional manifolds, it is quite possible - based upon what is known about fake $\mathbb{R}^4$'s - that there may exist uncountably many diffeomorphism classes in a single homeomorphism class.

3.5. **Decidability of compact nonmanifolds.** Most of the decidability results delineated in the preceding sections for compact submanifolds in CDT_N^k have easily formulated analogs - which tend to be more challenging - for compact varieties V that are not submanifolds. Owing to the relative ease with which the decidability problems for manifolds embedded in Euclidean spaces of dimensions less than or equal to three can be solved, and expecting Thom-Mather theory (see e.g. [26], [66], [99], [106] and [146]) to reduce much of the work to submanifold strata in (2.1), it is reasonable to assume that the analogous decidability problems can be resolved with roughly the same degree of effort. To this end, we need to make use of the following analog of Theorem 3.8, which can be proved in essentially the same way as that theorem, except we shall have to use the extension of Munkres' differentiable triangulation results to the subvarieties in the CDT category embodied in Theorem 3.1.

Theorem 3.17. *Let V be a compact subvariety in the object class of CDT_N^k $(k \geq 1)$. Then M has a finite C^1 triangulation, and the homology $H_*(V,F)$, cohomology $H^*(V,F)$, and all of the applicable characteristic classes such as the Euler, Stiefel-Whitney, and Pontryagin classes for M are effectively computable in polynomial time, where the coefficient ring F can be either the integers $\mathbb{Z}$, or the integers mod 2 denoted as $\mathbb{Z}_2$.*

Armed with this result, which guarantees the effective computability of most of the key isomorphism invariants, and given the insights provided by our previous observations, it is reasonable to make the following conjecture regarding the solvability of the simplified fundamental problem.

Conjecture 3.18. *Every connected, compact subvariety V in CDT_N^k with $N \leq 3$ and $k \geq 1$ is stably TOP-decidable via an algorithm executable in at most polynomial time.*

It may well be possible to also prove a version of this result in all higher dimensions, but clearly not without some further restrictions on the homotopy type. Simple-connectedness might work, but this would severely restrict the types of nonmanifolds available for applications, as many of the excluded ones would be apt to arise in problems related to computer aided geometric design and the modeling and visualization of complex configurations encountered in life science related research. For example, consider a thickened self-intersecting curve in the shape of a figure eight embedded in a Euclidean space of dimension four or higher. We surmise that the following result can be proved.

Conjecture 3.19. *Every connected, compact subvariety V in CDT_N^k with $k \geq 1$ is stably TOP-decidable via an algorithm executable in at most polynomial time, assuming that the fundamental group $\pi_1(V)$ is recursively computable in polynomial time.*

Another direction that one can pursue is to consider nonmanifolds obtained in a simple fashion from a compact manifold that is TOP-decidable. It is precisely this tack that we briefly follow in the remainder of this section.

3.5.1. *Sweep-like projective varieties.* Swept volumes represent an important class of objects in computer aided geometric design, which play key roles in any number of applications (see e.g. [3]-[6], [21]-[26], [97], [98], and [153]). One of the most useful features of swept volumes is that if they are generated by a manifold, they can be obtained as the projection of a manifold - embedded in one higher dimension - called the *extended swept volume* of the object. Therefore, we are motivated to make the following definition of a class of varieties that may yield to algorithmic classification of isomorphism types.

Definition 3.20. *A compact subvariety V of $\mathbb{R}^N$ is a* ***sweep-like projective subvariety*** *if there exists a compact submanifold M of $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ such that $P(M) = V$, where P is the standard projection of $\mathbb{R}^N \times \mathbb{R}$ onto $\mathbb{R}^N = \mathbb{R}^N \times 0$, in which case V is said to be the projection of M.*

We shall routinely abbreviate sweep-like projective subvarieties by referring to them as sweep-like projective varieties. A sweep-like projective variety - actually a triangulated piecewise linear approximation of such a geometric object - is illustrated in Fig. 2 in the next section. By studying this figure, we see the projected variety appears to be completely determined by the projection map restricted to its 'covering' manifold (suggesting a singularity theory connection), and that the self-intersection cell in the projection of the manifold has the appearance of an obstruction to lifting the variety to its regular pre-image manifold. These observations indicate that we can use the projection to characterize the variety, and employ a triangulation of the variety to identify this cell, in the manner of obstruction theory (cf. [140] and [151]), in an algorithmic way. Accordingly if the projecting manifold itself is topologically decidable, it appears that the same should be true of its image, which makes the following assertion quite plausible.

Conjecture 3.21. *Every connected, compact, sweep-like projective subvariety V that is an object of CDT_N^k, and is the projection of a compact, TOP-decidable or CDT_{N+1}^k-decidable C^1-submanifold M of $\mathbb{R}^{N+1}$ is also, respectively, TOP-decidable or CDT_N^k-decidable. Moreover, the computational complexity of deciding the homeomorphism type or CDT_N^k-isomorphism type of V is no greater than the square of that of M.*

4. Framework for a Computable Singularity Theory

In this section we show how to create a framework for an effectively computable singularity theory for differentiable maps of piecewise linear objects in CDT_N^k with $k \geq 1$. Our attention here will be confined to some specific cases to illustrate the nature of this piecewise linear theory and its use in developing an effective procedure for determining the isomorphism types of geometric objects. Details of the theory shall be provided in a forthcoming paper.

One of the drawbacks of using singularity theory for general differentiable objects, such as those arising in the analysis of swept volumes, is the fact that the singularities are usually described in terms of zero sets of rather complicated nonlinear functions, and are therefore not particularly well suited to algorithmic characterization (cf. [2]-[6], and [25]). Of course there are numerical procedures for approximating the solutions of these nonlinear functions, but convergence issues - tending to militate against effective computation - can arise in certain cases. One way of circumventing these computational difficulties, is to obtain various approximate normal forms for singularities, such as those described in [25], which simplify the computations. But this can also be rather tricky and hard to implement algorithmically. These problems can be avoided if the objects have piecewise linear structure, but adjustments have to be made to include points where the derivatives fail to exist. Notwithstanding the modifications required in a piecewise linear setting, it is not difficult to show that there is an effective procedure (algorithm) for determining the singularities. Moreover, if the isomorphism type of the domain of the mapping is decidable, this procedure can be used to decide the isomorphism type of the codomain. We shall indicate how this works for sweep-like projective varieties and their generalizations.

Let us consider a connected, compact sweep-like projective variety V in CDT_N^k. By definition, there is a C^k submanifold M, possibly with $\partial M \neq \varnothing$, embedded in $\mathbb{R}^{N+1}$ such that we may consider V to be embedded in the hyperplane $\mathbb{R}^N \times 0$ of $\mathbb{R}^{N+1}$ in a way that guarantees that $P(M) = V$, where P is the standard linear projection of $\mathbb{R}^{N+1}$ onto $\mathbb{R}^N \times 0$. We know that M can be arbitrarily closely approximated (algorithmically) in an appropriately adjusted Whitney - like C^k topology by an isomorphic manifold represented by a rectilinearly triangulated set $\hat{M}$, and that $\hat{V} := P(\hat{M})$ can also be considered to have a rectilinear simplicial structure. Therefore, for purposes of accurate approximations (which is the best we can do algorithmically), we may assume to begin with that $M = \hat{M}$ and $V = \hat{V}$, as shown in Fig. 2. Observe that the rectilinear simplicial structure of M guarantees that both M and V in such a projective setting are both objects with Whitney-regular stratifications; hence, they are objects in CDT categories.

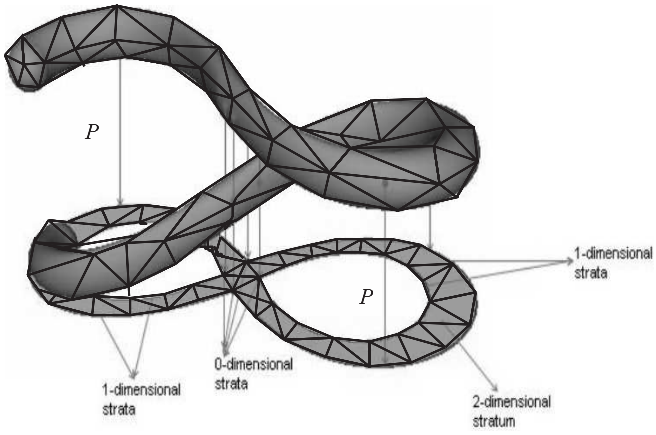


FIGURE 2. Sweep-like projective subvariety

Remark 4.1. Inasmuch as the identification of sweep-like projective varieties was inspired by swept volume research, it is appropriate that we briefly describe this connection before continuing with our discussion of effective procedures for applying singularity theory. We confine ourselves to C^k rigid sweeps, noting only that the description can be easily extended to piecewise smooth, deformed sweeps (see e.g. [2], [21], [26], and [153]). By a C^k, rigid *sweep* we mean a C^k mapping $\sigma : \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ of the form

$$\sigma(\mathbf{x}, t) := \sigma_t(\mathbf{x}) := \xi(t) + A(t)\mathbf{x},$$

where t is confined to the unit interval $[0,1]$ without loss of generality, ξ and A are, respectively, C^k vector-valued and matrix-valued functions defined on the t-interval $[0,1]$ such that $\xi(0) = 0$, $A(0) =$ the identity matrix I, and $A(t) \in \mathrm{SO}(N)$ - the Lie group of orthogonal $N \times N$ matrices having determinants $= 1$. Note that $A(t)A^{\mathsf{T}}(t) = I$ and $\det A(t) = 1$ for all $0 \leq t \leq 1$, where the superscript $^{\mathsf{T}}$ denotes the transpose operation for matrices.

The *swept volume* of a C^k manifold M (possibly with nonempty boundary) embedded in $\mathbb{R}^N$ generated by the sweep σ is defined as

$$S_\sigma(M) := \{\sigma_t(\mathbf{x}) : \mathbf{x} \in M, \, 0 \leq t \leq 1\}.$$

Each manifold

$$M(t) := \sigma_t(M) := \{\sigma_t(\mathbf{x}) : \mathbf{x} \in M\}$$

is called the *t-section* of M under the sweep σ. It is easy to show that the orbits (trajectories) $O_\sigma(\mathbf{x}) := \{\sigma_t(\mathbf{x}) : 0 \le t \le 1\}$ correspond to the solutions of the *sweep differential equation*

$$\dot{\mathbf{x}} = X_\sigma(\mathbf{x},t) := \dot{\xi}(t) + \dot{A}(t)A^{\mathsf{T}}(t)\left(\mathbf{x} - \xi(t)\right),$$

where the dot over a variable denotes d/dt, and X_σ is called the *sweep vector field* associated to σ(cf. [26]).

There are natural analogs of the above concepts for spacetime. The sweep σ in $\mathbb{R}^N$ has an associated *extended sweep* $\sigma^* : \mathbb{R}^N \times [0,1] \to \mathbb{R}^{N+1}$ in spacetime defined as

$$\sigma^*(\mathbf{x},t) := \sigma_t^*(\mathbf{x}) := (\sigma_t(\mathbf{x}), t).$$

Standard maps relating space $\mathbb{R}^N$ and spacetime $\mathbb{R}^{N+1}$ are the embedding $i : \mathbb{R}^N \to \mathbb{R}^{N+1}$ and the projection $P : \mathbb{R}^{N+1} \to \mathbb{R}^N$ defined, respectively as $i(\mathbf{x}) := (\mathbf{x}, 0)$ and $P(\mathbf{x},t) := \mathbf{x}$. Note that the composition $P \circ i = id_{\mathbb{R}^N}$, the identity map on $\mathbb{R}^N$. We shall, as is usually the case, identify M with $i(M) = M \times 0 = \{(\mathbf{x},0) : \mathbf{x} \in M\}$ in spacetime $\mathbb{R}^{N+1}$. The *extended swept volume* of M generated by σ is defined as

$$S_\sigma^*(M) := \{\sigma_t^*(\mathbf{x}) : \mathbf{x} \in M,\ 0 \le t \le 1\} = \{(\sigma_t(\mathbf{x}), t) : (\mathbf{x},t) \in M \times [0,1]\}.$$

It is easy to show that the orbits (trajectories) $O_\sigma^*(\mathbf{x}) := \{\sigma_t^*(\mathbf{x}) : 0 \le t \le 1\}$ correspond to the solutions of the *extended sweep differential equation*

$$\mathbf{u}' = (\mathbf{x}', t') = X_\sigma^*(\mathbf{x},t) := \left(X_\sigma(\mathbf{x},t), 1\right),$$

where the prime denotes d/ds.

A direct consequence of the definition, or elementary properties of differential equations, is that the extended swept volume $S_\sigma^*(M)$ is a submanifold-with-boundary of $\mathbb{R}^{N+1}$ (see [23]). On the other hand, the swept volume $S_\sigma(M)$ can have self-intersections, so it may not be a submanifold; the most one can say is that the swept volume is a subvariety of $\mathbb{R}^N$ (cf. [26]). Nevertheless, it is easy to see that the swept and extended swept volumes enjoy the following simple relationship $S_\sigma(M) = P\left(S_\sigma^*(M)\right)$, wherein we use our identification of $\mathbb{R}^N$ with $\mathbb{R}^N \times 0 = \left\{(\mathbf{x},0) : \mathbf{x} \in \mathbb{R}^N\right\}$.

Returning again to the above sweep-like projective variety context, it is clear that the topological structure of V can be considered to be inherited from the 'covering' manifold M. For example, self-intersections of V (connected with the trimming problem for swept volume representations [24]) can be identified by counting the number of components in the fibers $P^{-1}(\mathbf{x})$ for $\mathbf{x} \in V$, and the boundary elements of V correspond to singularities of the map $\Phi : M \to V$, where Φ is the restriction of P to M. We emphasize again that the singularities correspond to points $\mathbf{z} \in M$ where either the derivative of Φ exists and is rank deficient or the derivative fails to exist at 'corners' of the triangulation of M.

Finding rank deficient points is a simple matter, since it involves only solving systems of linear equations under constraints imposed by the boundary of the compact manifold M. The other singular points, where differentiability breaks down, are likewise easy to identify by investigating the simplices that are faces of the triangulation $\mathcal{T}_M$ of M. One can then project this triangulation onto V, and employ the usual strategies to obtain a triangulation of V containing the projection $\Phi(\mathcal{T}_M)$ of the triangulation of M.

Even though our description of the topological connections between V and the projection $\Phi : M \to V$ has been cursory, it is clear that given the homeomorphism type of M, there should be an effective procedure for determining the topology of V. In fact, we can boldly extrapolate this intuitive observation by surmising the validity of the following assertion.

Conjecture 4.2. *Let M be a connected, compact, m-dimensional covering manifold of a sweep-like projective variety V be CDT^0_{N+1}-decidable via a stable algorithm $\mathcal{A}_M$ with computational cost $CC(\mathcal{A}_M) = O(n^s)$, where s is a positive integer such that $1/s$ is a lower bound for the diameters of the simplices in a given triangulation of M. Then there exists a stable algorithm $\mathcal{A}_V$ for determining the isomorphism type of V in CDT^0_N, and the computational complexity of the algorithm satisfies $CC(\mathcal{A}_V) = O(n^{ms})$.*

Remark 4.3. The usual over and under representation of knots in a plane may be viewed as a sweep-like projection of a circle embedded in $\mathbb{R}^3$. The crossings of the knot in the plane represent self intersections of a 1-dimensional subvariety, which can be characterized in terms of how the components of the fibers of the projection vary as the knot is traversed in $\mathbb{R}^3$. This resonates with the knot type deciding algorithm in [74], and suggests the following query.

Problem 4.4. *How does the suggested singularity based procedure for deciding knot type compare with that in* [74]*?*

If M is in the projective role above and embedded in $\mathbb{R}^{N+1}$, one can embed it in a higher dimensional Euclidean space in which it can be deformed into the simplest representative in its isomorphism class. Thus, it may be possible to develop a more a efficient way of determining the isomorphism type of the projective variety V by increasing the dimension of the ambient space of its covering manifold. Accordingly we offer the following extension of the notion of a sweep-like projective variety.

Definition 4.5. *Let V be an object in CDT^k_N. Suppose there is a submanifold M embedded in $\mathbb{R}^{N+m}$, where m is a positive integer, such that the projection*

$$P_m : \mathbb{R}^{N+m} \to \mathbb{R}^N$$

*ignoring the last m coordinates satisfies $P_m(M) = V$. Then we say that V is a **sweep-like projective variety of codimension-m**, which makes objects fitting our original definition, sweep-like projective varieties of codimension-1.*

Let m_* be the smallest positive integer such that TOP-decidability of M is equivalent to CDT^0_{N+m}-decidability, which exists in virtue of well-known embedding theorems (cf. [133]), where M covers a sweep-like projective variety V of codimension-m in $\mathbb{R}^N$. The following result appears plausible.

Conjecture 4.6. *If $\mathcal{A}_{\#}$ is a stable algorithm for deciding the homeomorphism type of M in $\mathbb{R}^{N+m_*}$ such that $CC(\mathcal{A}_{\#}) = O(n^r)$, then there is a stable algorithm $\mathcal{A}_{\natural}$ for determining the isomorphism type of the sweep-like projective variety of codimension-m, V, in CDT_N^0, satisfying $CC(\mathcal{A}_{\natural}) = O(n^{r+m_*})$.*

Although our plan was only to sketch the elements of an effectively computable singularity theory for piecewise linear object in the CDT category, we cannot end our brief discussion without at least mentioning Thom-Boardman stratifications (cf. [15], [66], [106], and [146]). If $\varphi : M \to V$ is a piecewise linear map between piecewise linear objects in CDT categories, it is natural to collect all points of M where φ is differentiable, but has deficient φ-rank, along with points where the derivative does not exist. All together, these sets comprise a piecewise linear analog of a Thom-Boardman stratification of M. This stratification should in some sense refine the natural Whitney-regular stratification with which M is equipped. It should not be too difficult to compute such singularity-based stratifications, in aid of solving the following problem.

Problem 4.7. *Describe how this piecewise linear Thom-Boardman stratification 'refines' the given stratification of M, and determine how this may help in characterizing the topology of V.*

5. Towards a Computable Obstruction Theory

Here we shall provide only a glimpse of what appears to be the kernel of an effectively computable obstruction theory for piecewise linear objects in our CDT categories. We hope to fashion this into a cohesive theory, complemented with several meaningful applications, in a series of papers to be written in the near future.

Our first inspiration comes from a long look at Fig. 2. Observe how the 'covering' manifold M resembles a regularized lifting of its projective variety V over the 'fibration' P. If we consider a sequence of higher codimension projections

$$\cdots \to M_3 \overset{\Phi_3}{\to} M_2 \overset{\Phi_2}{\to} M_1 \overset{\Phi_1}{\to} V$$

as defined in the preceding section, we are reminded of the kind of Moore-Postnikov factorizations that can serve as the foundation for an obstruction theory (cf. [140]).

Classical obstruction theory relies on the iterative calculation of successive obstructions lying in cohomology groups with homotopy group coefficients. Given the triangulated structure of the geometric objects involved, it is likely that the usual obstructions can be computed recursively, but just how this is to be accomplished in general is certainly not obvious.

Fortunately, there are some contexts in which classical obstruction theory can be implemented in a relatively simple way, such as in [111], and we believe that the rich structure available to us in our CDT setting will enable us to simplify the whole obstruction theory process. Our basic idea - which may in a certain sense be regarded as the dual of the singularity theory approach delineated in the preceding section - is to treat the requisite obstruction theory from an essentially combinatorial viewpoint, which we illustrate for the objects in Fig. 2.

We look closely at Fig. 2 to see how the covering manifold and its projection are related, topologically speaking. Our basic question is, what operations must be performed on V to lift it to M, or dually, how can one get from M to V via a simple topological/geometric process? Looking at the triangles comprising the central self-intersection set of the projected variety, we see that by excising them, locally we obtain a set with four (rather than one or two) components. Thus this central cell is an obstruction to lifting V to M over the projection P. Dually, a thickened version of this cell needs to be attached as a bridge between two different portions of M to obtain this self-intersection of V. But there is also an overlapping cell where the projected portions of the ends of M meet. Hence, this cell is another obstruction to lifting V to M, and a thickened version of this cell must be attached as a bridge between the ends of M in order to produce the correct topology in the corresponding portion of the projection.

The simple approach outlined above has, we believe, the potential for extension to much more general settings involving piecewise linear objects in the CDT categories. Therefore, we pose the following problem for ourselves and others who might enjoy the challenge.

Problem 5.1. *Develop the basic ideas sketched in this section into a rigorous obstruction theory for piecewise linear objects in the CDT categories, and explore possible applications of this theory.*

6. Topology of Intersections

From our discussions in the preceding sections, the importance of intersections in determining the differential topological and topological nature of geometric objects is manifest. It is our intention here to focus on a relatively narrow, but increasingly important facet of intersection theory - the algorithmic aspects of intersection analysis pertaining to computer generated geometric objects - that can justifiably be claimed to belong to computational geometry, computer aided geometric design, or the relatively new field of CT. Even in this rather narrow range of the intersection theory landscape, there has been extensive research that has produced an extensive body of important and useful results. And still the research continues apace, which underscores the fact that in spite of all the progress that has been made, there remains a vast collection of important open problems. We intend to make some progress in addressing these open problems using approaches that are firmly rooted in the field of CT in general, and CDT in particular.

Examples of the type of computational geometry or computationally oriented intersection theory results we are addressing include the following, among many others: Abdel-Malek & Yeh [1] use the intrinsic differential geometry of parametric surfaces - especially critical point techniques - to detect points of intersection. The use of novel strategies employing boundary representations, as in Bajaj *et al.* [17] have also figured in numerous investigations. Heo *et al.* [76] use special properties of ringed surfaces to find local and global properties of their intersections. Algorithms adapted to the special geometric features of the surfaces considered have been developed by Johnstone [84] to describe the differential geometric features of certain types of surface intersections. Sederberg and his collaborators ([137], [138]) use methods from algebraic geometry and differential geometry to describe (or make conjectures about) important aspects of the global structure of intersections of algebraic surfaces, as do Ma & Lee [100]. The aforementioned and other researchers have developed useful algorithmic techniques for describing intersections of smooth swept surfaces with general smooth surfaces in ways that have features in common with the intersection detection techniques that we shall introduce in this paper; and Ye & Mackawa [158] bring classical elements of differential geometry to bear on the intersection description problem. A sample of intersection research in a more CT vein includes the work of Farouki *et al.* [60] on developing algorithms for piecewise linearly, topologically consistent representations of self-intersections, the use of index theories for vector fields to characterize intersections in Kreizis *et al* [94], and the development by Peters *et al.* (see e.g. [114]) of error bounds on differential geometric surface characteristics that insure the correct ambient isotopy type of their intersection sets.

The overall intersection problem for geometric objects may be conveniently described as comprised of two parts: *intersection detection* and *intersection analysis* or *interrogation.* Intersection detection is concerned with the determination of the occurrence of intersections among groups of objects, or self intersections of a single object. Intersection analysis or interrogation involves local and global characterization of intersection sets, once it has been established that an intersection actually occurs. Both of these aspects of the intersection problem play important roles in a host of practical applications in such fields as computer-aided design (CAD), computer-aided manufacturing (CAM), robotics, and virtual reality; examples of which are provided in [1], [17], and [76].

We first concentrate on a particular class of intersection detection problems involving certain types of objects that we call swept manifolds. More precisely, we devise a method for detecting intersections that is based on locating zeros of a special type of signed distance function. Then we develop some extensions and generalizations of results in [26] related to local intersection detection and interrogation using homology theory. In particular, we extend the local homological approach introduced in [26] for transverse intersections of hypersurfaces to general (nontransverse) intersections of submanifolds and varieties (with and without boundaries) having a wide range of codimensions. Moreover, we demonstrate that both our intersection detection and homological characterization methods are well suited to computer implementation.

Our organization of this section of the paper is as follows: In Subsection 6.1, we briefly summarize the topological and geometric notions that are employed to a significant extent in the sequel, define the notion of a swept manifold, and provide references that give more detailed accounts of the standard mathematical nomenclature that we use. Then, in Subsection 6.2, we develop a new method for detecting intersections of swept manifolds, and illustrate its implementation with an example. This is followed by a brief description of how homological techniques can be used to detect and analyze intersections in Subsection 6.3. In Subsection 6.4, we show how homology can be used to detect and interrogate intersections of submanifolds and subvarieties of Euclidean space, and then illustrate the effectiveness of this approach in dealing with tangential (nontransverse) intersections of geometric objects. Then we discuss the algorithmic implementation of our homological methods and their integration into widely used data structures in Subsection 6.5. Then to conclude this section, in Subsection 6.6, we summarize the contributions our results, discuss their significance, and indicate some possible future related research directions.

6.1. **Geometric preliminaries.** As discussed in the preceding sections, the objects considered in CT are usually subsets - endowed with additional structure - of the ambient Euclidean n-space $\mathbb{R}^n$. Typically, the additional structure on the subsets is at least enough to render them submanifolds or subvarieties. In most cases, the objects of interest actually have some smoothness, which induces the submanifold or subvariety structure.

We complete our brief introduction to some of the basic topological and geometric employed in the sequel with a rather general formulation of the notion of a swept manifold and a swept variety (cf. [84]).

Definition 6.1. *We say that a subset S of $\mathbb{R}^n$ is an $(m+1)$-dimensional,* ***C^kswept submanifold*** *or* ***subvariety****, respectively, if it is a C^k, $(m+1)$-dimensional submanifold (possibly with boundary) or subvariety (possibly with boundary) such that there is a C^k, m-dimensional submanifold M (possibly with boundary) or subvariety M (possibly with boundary), a C^k vector field X on $\mathbb{R}^n$, which is nonvanishing on a neighborhood of S and generates a flow φ_t, and a real interval J such that $S = \varphi_J(M) := \{\varphi_t(x) : (x,t) \in M \times J\}$. The submanifold (subvariety) M is called an* ***initial submanifold (initial subvariety)*** *of S, and each of the m-dimensional sets $\varphi_t(M) := \{\varphi_t(x) : x \in M\}$ is called a* ***t-section*** *or* ***trace*** *of S.*

Note that the vector field in the above definition is much more general than the Lie algebra vector field associated to Euclidean, affine, or any of the other Lie groups of transformations used to define swept volumes (cf. [6], [21]-[26], [153]). The above definition of submanifolds in terms of flows generated by vector fields enables us to bring the powerful tools of dynamical systems (see e.g. [69], [72], [88], and [142]) to bear on problems concerning swept manifolds and varieties. One should also observe that every C^r submanifold is locally a C^r swept manifold.

Example 6.2. Consider the 2-sphere in $\mathbb{R}^3$ defined in the usual way as

$$S^2 := \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}.$$

This is a C^{ω} swept surface in $\mathbb{R}^3$ (modulo the vanishing of the vector field at the poles) that may be generated in several ways. For example, we can take the initial submanifold to be the unit circle $S^1 := \{(x,0,z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}$, and the generating vector field as $X = (y, -x, 0)$. The flow generated by X is $\varphi_t(x,y,z) = (x\cos t - y\sin t, x\sin t + y\cos t, z)$, and is is easy to see that in this case that $S^2 = \varphi_{[0,\pi]}(S^1)$.

On the other hand, we have the related case.

Example 6.3. Let h denote the restriction of the height (z) function to the family of spheres in a thin shell containing S^2, so that

$$\operatorname{grad} h = \nabla h = \left(x^2 + y^2 + z^2\right)^{-1} \left(-xz, -yz, x^2 + y^2\right).$$

Denoting the flow generated by ∇h as ψ_t, and taking the generating submanifold to be the equator $E := \{(x,y,0) : x^2 + y^2 = 1\}$, we observe that if a is a large positive number, then $\psi_{[-a,a]}(E)$ is all of S^2 except for small polar caps at the north $(z = 1)$ and south $(z = -1)$ poles. Moreover, these caps can be made as small as one wants by taking a sufficiently large. Hence, E under the action of the flow nearly sweeps out all of S^2.

This is a rather interesting observation that we intend to investigate in much more detail in a forthcoming paper.

6.2. **Intersection detection.** Here we develop a rather effective analytical method for detecting intersections of swept submanifolds, either for a pair of such manifolds, or self-intersections of a single swept submanifold with itself. Our approach has some elements in common with the work of others, but it is at its core quite different. We shall concentrate on the detection problem, and not be concerned with the topology or geometry of the intersection set as in such investigations as Bajaj *et al.*[17], Farouki *et al.* [60], [68] , Heo *et al.*[76], Johnstone [84], Kreizis *et al.*[94], Ma & Le [100], Mow *et al.* [114], Peters *et al.* [124], Sederberg *et al.* [137, 138], and Ye & Mackawa [158]. Hence in this regard at least, the work in this section compares rather closely with that of Abdel-Malek & Yeh [1] on finding useful starting points of intersection sets.

Now let M and N be C^k $(k \geq 2)$ submanifolds of $\mathbb{R}^n$ with $\dim M = p$ and $\dim N = q$. If $\dim M + \dim N = p + q < \dim \mathbb{R}^n = n$. It follows from transversality theory (see e.g. [66], [78], and [142]) that any points of intersection can be eliminated by an arbitrarily small C^k perturbation of either or both of M and N. Consequently, as computer generated representations of geometric objects always entail small some small errors, one must assume that

$(\mathcal{A}_I)$ $\dim M + \dim N = p + q \geq n = \dim \mathbb{R}^n$.

in order to have any real hope of finding an effective algorithmic method for detecting intersections, which is suitable for computer implementation. We note here that for self-intersections, we can take M and N to be different portions of the whole object S, which is typically a subvariety of $\mathbb{R}^n$.

If U is an open subset of $\mathbb{R}^n$ in which M and N have implicit representations of the form

$$M \cap U = F^{-1}(0), \quad N \cap U = G^{-1}(0),$$

where $F : U \to \mathbb{R}^{n-p}$ and $G : U \to \mathbb{R}^{n-q}$ are C^k functions, then finding points of $(M \cap N) \cap U$ boils down to solving the system of equations

$$F(x) = 0, \quad G(x) = 0, \tag{6.1}$$

simultaneously in U. This can be a daunting task, even when only approximate zeros are being computed. Now if one can find one solution, say x_*, of (6.15), then there are results such as the implicit function theorem, various elements of singularity theory, and certain index theorems for vector fields that enable one to obtain a reasonably good characterization of $M \cap N$ locally, and even globally to a lesser extent, but first one must find a point in the intersection.

Taking d to be the usual Euclidean metric, namely $d(x, y) := |x - y|$, intersection points also can be associated with the following variational problem: Minimize

$$d^2(x, y) := (d(x, y))^2,$$

subject to the constraint

$$(x, y) \in M \times N,$$

and establish that this minimum is zero. Unfortunately, this formulation of the intersection problem is not particularly useful in a computational setting, since exact zeros are rarely computable. However, it is helpful in simplifying the problem, inasmuch as we can immediately eliminate from further consideration those portions of the two submanifolds for which $d^2(x, y)$ is greater than any convenient preassigned positive constant. In particular, suppose we identify all points $(x, y) \in M \times N$ such that $d^2(x, y) > \epsilon^2 > 0$, where ϵ is say an order of magnitude larger than the accuracy of the numerical scheme being employed. Then we can safely eliminate all such points from the list of possible intersections, and concentrate on the remaining points. So, for example, if $d^2(x, y) > \epsilon^2$ for all $(x, y) \in M \times N$, we know that there are no points of intersection at all.

In light of the above analysis, we may at this juncture assume that there exist points satisfying $d^2(x, y) \le \epsilon^2$, where ϵ is an appropriately chosen small positive number, which means that there may exist some points of intersection. We shall further assume that M and N are compact, C^k submanifolds of $\mathbb{R}^n$. In addition, we suppose that M is a swept submanifold obtained by sweeping a $(p-1)$-dimensional, C^k submanifold M_0 with the flow φ_t generated by a nonvanishing, C^k vector field $X : U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$ that is large enough to contain both M and N. We assume without loss of generality that $p = \dim M \le q = \dim N$. Moreover, we assume to begin with that N is a embedded submanifold of codimension-1, i.e. $q = n - 1$. We also assume that it is oriented, so that one can designate a positive and negative side of N (see [78], [105], [112], [115] and [140]). The references just cited, also serve as excellent sources for some of the basic mathematical concepts that we use in what follows.

By choosing ϵ sufficiently small, we guarantee that the set

$$\tau_\epsilon(N) := \{x \in \mathbb{R}^n : d(x, N) \leq \epsilon\}$$

is a C^k *tubular neighborhood* of N having the structure of a 1-disk bundle over N, where $d(x, N)$ is the distance between the point x and the submanifold N. For each $y \in N$, the fiber of $\tau_\epsilon(N)$ over y may be viewed as the interval $\mathcal{F}_y$ of length 2ϵ, bisected by y, along the normal to N at y. Each such interval (fiber) may be viewed as comprised of a positive normal vector of length ϵ pointing up from the positive side of N, and a negative normal vector of length ϵ at y pointing in the opposite (negative) direction. Whence we obtain a partition of $\tau_\epsilon(N)$ into positive and negative normal vectors of length ϵ, which we denote by $\tau_\epsilon(N)_+$ and $\tau_\epsilon(N)_-$, respectively. Consequently, $\tau_\epsilon(N)$ can be written as the disjoint union $\tau_\epsilon(N)_+ \cup N \cup \tau_\epsilon(N)_-$, and there is an associated vector field of unit normals on each side of N, where $\mathbf{n}_+(x)$ is the unit normal pointing away from N on the fiber through $x \in \tau_\epsilon(N)_+$, and $\mathbf{n}_-(x)$ is the unit normal pointing away from N on the fiber through $x \in \tau_\epsilon(N)_-$. We are now in a position to prove the following result.

Theorem 6.4. *Let M and N be C^k $(k \geq 2)$ submanifolds of $\mathbb{R}^n$ satisfying assumption $\mathcal{A}_I$, where N is of codimension-1. Moreover, suppose that the vector field X, flow φ_t, normal bundle $\tau_\epsilon(N)$, etc., are as described above, and let M_0 be the initial submanifold of the swept submanifold M. If there exist a point $x \in M_0$ and $0 < t_1 < t_2$ such that both $\varphi_{t_1}(x)$ and $\varphi_{t_2}(x)$ belong to $\tau_\epsilon(N)$, $\langle X(\varphi_{t_1}(x)), \mathbf{n}(x)\rangle$ and $\langle X(\varphi_{t_2}(x)), \mathbf{n}(x)\rangle$ have opposite signs, but $\langle X(\varphi_t(x)), \mathbf{n}(x)\rangle$ does not vanish on the interval $[t_1, t_2]$, then $\varphi_{t_*}(x) \in M \cap N$ for some $t_* \in (t_1, t_2)$.*

Proof. The desired result follows directly from the definitions and the connectedness of the orbits of a C^k vector field. The hypotheses imply that $\varphi_{t_1}(x)$ and $\varphi_{t_2}(x)$ are in opposite halves of the normal bundle $\tau_\epsilon(N)$. Accordingly as the path segment $\{\varphi_t(x) : t_1 \leq t \leq t_2\}$ is connected, and we have the partition $\tau_\epsilon(N) = \tau_\epsilon(N)_+ \cup N \cup \tau_\epsilon(N)_-$, it follows that this path segment, which is contained in M, must intersect N. Thus we have a point of intersection of the two submanifolds, and the proof is complete. □

It is easy to construct a rather effective computational scheme for implementing the above theorem having the following steps:

Step 1. Choose a small positive ϵ, and construct the normal bundle together with the normal vector field $\mathbf{n}$ on each side of $\tau_\epsilon(N)$.

Step 2. Triangulate M_0 with a mesh diameter of size δ, and collect all the vertices in a set $V = \{x_{(1)}, ..., x_{(m)}\}$..

Step 3. Use a Runge-Kutta scheme with step size $h \leq \delta$ to find the approximate positive semi-orbits starting at all points of V, and compute the distance from each of the points on the discrete approximate trajectories to N.

Step 4. Discard all approximate orbit points outside of the normal bundle $\tau_\epsilon(N)$, and group the remaining points in order of increasing t. Then a typical ordered collection of remaining points takes the form $\tilde{O}_{(j)} = \{x_{(j)}^{(1)}, x_{(j)}^{(2)}, ..., x_{(j)}^{(m_j)}\}$.

Step 5. Test the points on the approximate orbits in Step 6 to see if

$$\left\langle X\left(x_{(j)}^{(l)}\right), \mathbf{n}(x_{(j)}^{(l)})\right\rangle$$

changes sign at successive points.

Step 6. If a sign change described in Step 6 occurs, use the bisection method on the corresponding time interval to test the conditions of Theorem 6.4, thereby obtaining an intersection point whose accuracy can be improved by reducing the sizes of δ, and increasing the number of bisection iterations.

Note that the above can readily be generalized to piecewise linear manifolds.

Example 6.5. Our computational scheme can be effectively illustrated with the following example in $\mathbb{R}^4$. Let M be the 2-torus in $\mathbb{R}^4$ defined parametrically as

$$M := \{(x_1, x_2, x_3, 0) : x_1 = (2 + \cos v)\cos u,$$
$$x_2 = (2 + \cos v)\sin u, x_3 = \sin v,\ 0 \leq u, v < 2\pi\},$$

and let N be the hyperplane characterized by the Cartesian equation $x_2 - x_1 = 2$. It is easy to solve exactly for the intersection set of these two submanifolds, but we want to show how our method can be applied. Observe that M is a swept submanifold generated by the circle M_0 defined by the equations $(x_1-2)^2+x_3^2 = 1$, $x_2 = x_4 = 0$ and the vector field $X := (-x_2, x_1, 0, 0)$, which induces the flow

$$\varphi_t(x) = (x_1\cos t - x_2\sin t, x_1\sin t + x_2\cos t, x_3, x_4).$$

We choose the positive normal direction for N to coincide with the vector $(-1, 1, 0, 0)$, and the negative direction to be parallel to $(1, -1, 0, 0)$. A uniform partition of M_0 defined by points separated by the distance $\delta_n := 1/n$ can be chosen to obtain the desired accuracy. Theorem 6.4 is checked by computing $\langle X(\varphi_t(x)), \mathbf{n}(\varphi_t(x))\rangle = \pm[-x_1(\cos t + \sin t) + x_2(\sin t - \cos t)]$, depending on which side of N the trajectory lies.

6.3. **Homological intersection criteria.** We now show how homology can be used to distinguish points in the intersection of two objects. We assume that the reader is familiar with the basics of algebraic topology - more specifically, homology theory - such as can be found in texts such as [105], [116], [140], and [160]. As usual, we shall assume integral homology unless otherwise indicated. Before plunging into details of the general case, let us take a look at an example. Consider two surfaces in space that intersect transversally as in Fig. 3. Let us denote them by M and N. It is easy to see that if $x \in M \cap N$, then a neighborhood of x is homeomorphic to a union of two intersecting planes. If $x \notin M \cap N$, a neighborhood of x is homeomorphic to just one such plane (Fig.3).

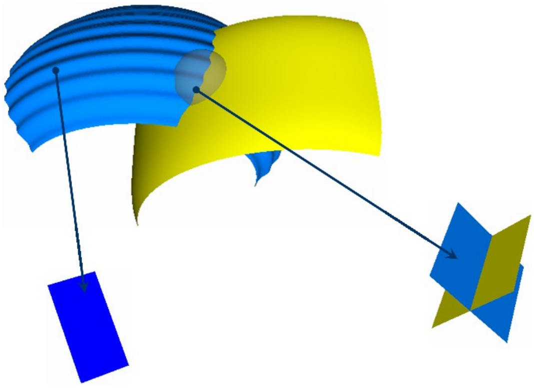


FIGURE 3. Transverse intersection of two surfaces

Unfortunately, checking if two spaces are homeomorphic is not an easy problem. In many cases, it can be reduced to comparing Betti numbers or homology groups of the spaces (cf. [42], [105], [140], [157], [160], and [161]) . But it turns out that to discern intersection points in the above example it is enough to compute the local homology groups of $M \cup N$ at a point of interest. We can do it fairly easily, obtaining the following: If $x \in M \cap N$, then the only non-trivial homology group is $H_2(M \cup N, M \cup N \setminus \{x\}) = \mathbb{Z}^3$. If $x \notin M \cap N$, then the non-trivial homology group is $H_2(M \cup N, M \cup N \setminus \{x\}) = \mathbb{Z}$. Similar homological characterization holds for transverse intersections of smooth, codimension-1 submanifolds of $\mathbb{R}^n$ for any n (cf. [140]). Things become more complicated for nontransverse intersections, especially in higher-dimensional spaces. Still, as we shall see, a description in terms of homology remains quite elegant and straightforward.

We now proceed to our analysis of the general case. Usually, objects under consideration are assumed to be smooth, compact submanifolds of $\mathbb{R}^n$ without a boundary. But since homology is homotopy invariant, we start by considering *topological* submanifolds of $\mathbb{R}^n$. To simplify our analysis, we impose some restrictions on the intersection set – we assume that it is an s-subvariety of $\mathbb{R}^n$ (cf. Blackmore *et al.* [26]), which, for example, is always the case if the intersecting manifolds are analytic, piecewise linear, or elements of the CDT categories defined in the preceding sections.

Theorem 6.6. *Let M and N be two topological submanifolds of $\mathbb{R}^n$ without boundaries, and let $I = M \cap N$ be an s-subvariety. Denote by p, q and r dimensions of M, N and I, respectively, and let $n > p \geq q > 0$.*

(1) *If $x \in (M \cup N) \setminus I$ then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if} \quad k = p, x \in M, \text{ or } k = q, x \in N \\ 0, & \text{otherwise} \end{cases}$$

(2) *If $x \in I$ the following hold:*
  (i) *if $p > q > r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if} \quad k = p, q \\ H_{k-1}(I, I \setminus \{x\}), & \text{if} \quad k = r+1 \\ 0, & \text{otherwise} \end{cases}$$

  (ii) *if $p > q = r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if} \quad k = p \\ \mathbb{Z} \oplus H_{k-1}(I, I \setminus \{x\}), & \text{if} \quad k = q = r+1 \\ 0, & \text{otherwise} \end{cases}$$

  (iii) *if $p = q > r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}^2, & \text{if} \quad k = p = q \\ H_{k-1}(I, I \setminus \{x\}), & \text{if} \quad k = r+1 \\ 0, & \text{otherwise} \end{cases}$$

  (iv) *if $p = q = r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}^2 \oplus H_{k-1}(I, I \setminus \{x\}), & \text{if} \quad k = p = q = r+1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let $x \in M \setminus I$, and consider $H_k(M \cup N, M \cup N \setminus \{x\})$. By excision, $H_k(M \cup N, M \cup N \setminus \{x\}) = H_k(U, U \setminus \{x\})$, where $U \subset M$ is a neighborhood of x in M, and from the long exact sequence for the pair $(U, U \setminus \{x\})$ we infer that $H_k(U, U \setminus \{x\}) = \widetilde{H}_{k-1}(U \setminus \{x\})$. Since M is a manifold of dimension p, the set $U \setminus \{x\}$ is homotopic to a $(p-1)$-sphere, S^{p-1}. Therefore, $\widetilde{H}_{p-1}(U \setminus \{x\}) = \mathbb{Z}$ and $H_{k-1}(U \setminus \{x\}) = 0, k \neq p$. The case $x \in N \setminus I$ is proved in the same way.

Now let $x \in I$. Again, by excision we get $H_k(M \cup N, M \cup N \setminus \{x\}) = H_k(U, U \setminus \{x\})$, where $U \subset M \cup N$ is a neighborhood of x in $M \cup N$, and the long sequence for the pair $(U, U \setminus \{x\})$ implies that $H_k(U, U \setminus \{x\}) = \widetilde{H}_{k-1}(U \setminus \{x\})$. Denoting $X = U \setminus \{x\}$, We can write $X = A \cup B$, where $A = U \cap M \setminus \{x\}$ and $B = U \cap N \setminus \{x\}$. Notice that from excision and the corresponding long exact sequence, we get $H_k(I, I \setminus \{x\}) = \widetilde{H}_{k-1}(A \cap B)$. Consider the reduced Mayer-Vietoris sequence for A, B and X:

$$\begin{aligned} \cdots \to \widetilde{H}_k(A \cap B) &\xrightarrow{\Phi} \widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \xrightarrow{\Psi} \widetilde{H}_k(X) \xrightarrow{\partial} \\ &\to \widetilde{H}_{k-1}(A \cap B) \to \cdots \to \widetilde{H}_0(X) \to 0, \end{aligned}$$

Since M and N are manifolds of dimensions p and q, A and B are, respectively, homotopic to S^{p-1} and S^{q-1}. So, the only non-trivial reduced homology groups of A and B are $\widetilde{H}_{p-1}$ and $\widetilde{H}_{q-1}$, respectively. Both of them are $\mathbb{Z}$. By assumption, $M \cap N$ is an s-subvariety of $\mathbb{R}^n$ of dimension r, which implies that U can be chosen such that $M \cap N \cap U$ is homeomorphic to a finite union of r-dimensional balls B_j^r each of which contains x. Therefore, $M \cap N \cap U \setminus \{x\} = A \cap B$ is homotopic to a connected union of $(r-1)$-dimensional spheres, so the only non-trivial reduced homology group of $A \cap B$ is $\widetilde{H}_{r-1}$.

Returning to the Mayer-Vietoris sequence, we see that it gives rise to one, two or three short exact sequences, as follows.

Case 1: $p > q > r+1$. We get three short exact sequences:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_{p-1}(X) \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_{q-1}(X) \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{H}_r(X) \longrightarrow \widetilde{H}_{r-1}(A \cap B) \longrightarrow 0,$$

from which the result follows.

Case 2: $p > q = r+1$. We obtain two short exact sequences:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_{p-1}(X) \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_r(X) \longrightarrow \widetilde{H}_{r-1}(A \cap B) \longrightarrow 0.$$

The first sequence implies that $H_{p-1}(X) \approx \mathbb{Z}$. In the second case, using the fact that all the groups involved are free and finitely generated, we infer that the sequence is a split exact sequence. Hence, $H_r(X) \approx \mathbb{Z} \oplus H_{r-1}(A \cap B)$.

Case 3: $p = q > r+1$. We have two short exact sequences:

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \widetilde{H}_{p-1}(X) \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{H}_r(X) \longrightarrow \widetilde{H}_{r-1}(A \cap B) \longrightarrow 0,$$

which yield the desired result.

Case 4: $p = q = r+1$. We get one short exact sequence:

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \widetilde{H}_r(X) \longrightarrow \widetilde{H}_{r-1}(A \cap B) \longrightarrow 0.$$

Again, using the fact that all the groups involved are free and finitely generated, we conclude that $H_r(X) \approx \mathbb{Z}^2 \oplus H_{r-1}(A \cap B)$

Notice that all nontrivial homology groups for A, B, or $A \cap B$ are considered in these four cases. Therefore, the short exact sequences for $\widetilde{H}_k(X)$ that are not considered above are of the form

$$0 \longrightarrow \widetilde{H}_k(X) \longrightarrow 0,$$

which implies that all such $\widetilde{H}_k(X)$ are trivial. This completes the proof. □

Remark 6.7. It can be seen from the proof that the hypothesis that the intersection set is an s-subvariety of $\mathbb{R}^n$ can probably be weakened, since we only need the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_r(X) \longrightarrow \widetilde{H}_{r-1}(A \cap B) \longrightarrow 0$$

to be a split exact sequence.

The above theorem has several important corollaries, the first of which generalizes Theorem 7.2 of [26].

Corollary 6.8. *Let M and N be $C^r, r \geq 1$, compact submanifolds of $\mathbb{R}^n$ without boundaries of dimensions $p < n$ and $q < n$, respectively, and suppose $M \pitchfork N$. Then if $x \in M \cap N$, the local (relative) homology satisfies*

$$H_k(M \cup N, (M \cup N) \setminus \{x\}) = \begin{cases} \mathbb{Z}^3, & \text{if } k = p, p = q = n-1 \\ \mathbb{Z}^2, & \text{if } k = p, p = q < n-1 \\ \mathbb{Z}, & \text{if } k = p, q, p+q-n; p \neq q \\ 0, & \text{otherwise} \end{cases},$$

and if $x \in (M \cup N) \setminus (M \cap N)$,

$$H_k(M \cup N, (M \cup N) \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } k = p \text{ and } x \in M \text{ or } k = q \text{ and } x \in N \\ 0, & \text{otherwise} \end{cases}$$

Proof. It follows from Theorem 6.6 and the fact that the transverse intersection of two C^r submanifolds of $\mathbb{R}^n$ of dimensions p and q is a C^r submanifold of dimension $p + q - n$. □

Corollary 6.9. *If a C^1 compact, connected, codimension-1 submanifold M of $\mathbb{R}^n$ has points of transverse self-intersection, then there exists an $x \in M$ such that $H_{n-1}(M, M \setminus \{x\}) = \mathbb{Z}^3$.*

If C^r submanifolds M and N intersect tangentially, then $M \cap N$ can have a quite complicated structure, e.g. it may not even be a submanifold. The following easily proved lemma describes, to some extent, the local structure of a tangential intersection of two C^r submanifolds of $\mathbb{R}^n$.

Lemma 6.10. *Let M and N be two C^r submanifolds of $\mathbb{R}^n$ without boundaries of dimensions p and q, respectively, and let $x \in M \cap N$ be a point where M and N intersect non-transversally. Then there is a neighborhood $U \subset \mathbb{R}^n$ of x and a C^r function $f : U \cap M \to \mathbb{R}^{n-q}$ such that $U \cap M \cap N = f^{-1}(0)$ and x is a critical point of f.*

A reader familiar with differential topology will notice that the statement of the lemma is equivalent to saying that $M \cap N$ is a C^r subvariety of $\mathbb{R}^n$. It may not be an s-subvariety though. Therefore, we still need the corresponding assumption for the statement of Theorem 6.6 to be true.

In many cases, the local homology groups of $M \cap N$ can be computed fairly easily, thereby yielding explicit formulas for the local homology at the intersection point. We demonstrate this in the following example.

Example 6.11. Consider the paraboloid M, given by $z = x^2 + y^2$, and the surface N, given in cylindrical coordinates by the following equations:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r^2 \sin(8\theta),$$

where $\theta \in [0, 2\pi], r \geq 0$. The neighborhood of the origin is shown in Fig. 4. These two surfaces intersect tangentially, and the intersection set, I, is an s-subvariety (and also a cdt_3^ω object) shown in Fig. 5. By excision, the local homology groups $H_k(I, I \setminus \{0\})$ are isomorphic to the corresponding reduced homology groups $\widetilde{H}_{k-1}(I \setminus \{0\})$. Since the set $I \setminus \{0\}$ consists of eight contractible components, we obtain $H_k(I, I \setminus \{0\}) = \mathbb{Z}^7$ for $k = 1$ and $H_k(I, I \setminus \{0\}) = 0$ for $k \neq 1$. Thus, Theorem 6.6 implies that $H_k(M \cup N, M \cup N \setminus \{0\}) = \mathbb{Z}^9$ for $k = 2$ and $H_k(M \cup N, M \cup N \setminus \{0\}) = 0$ for $k \neq 2$.

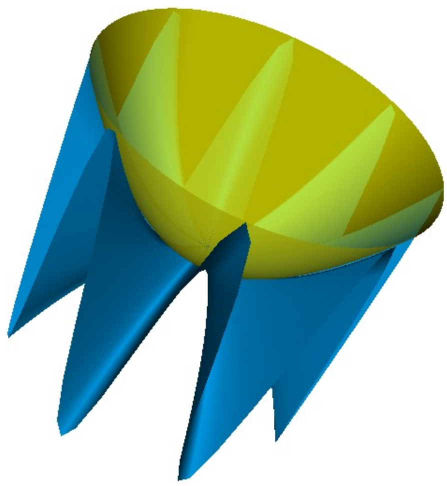


FIGURE 4. Tangentially intersecting surfaces

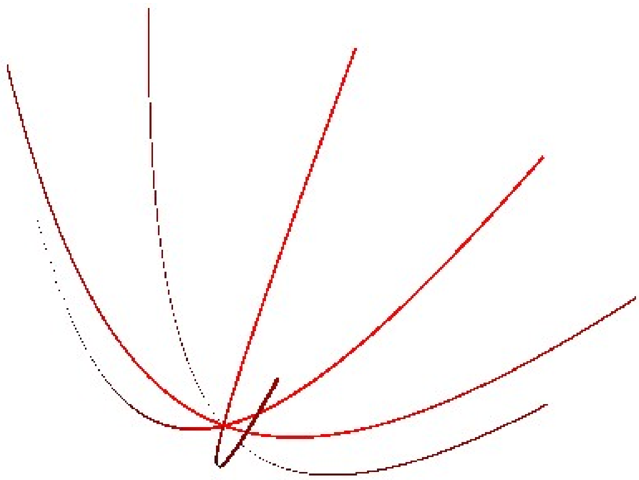


FIGURE 5. Intersection set of the surfaces

A direct application of Theorem 6.6 allows us to distinguish between intersection and non-intersection points, as well as between points of tangential and transverse intersection, but it does not provide an explicit way for determining the local structure of the intersection set, which is the primary goal in many applications. To achieve such a goal, at least partially, one would need to 'reverse' the theorem, expressing the local homology of the intersection set, $I = M \cap N$, in terms of the local homology of the union, $M \cup N$. Notice that our assumptions on the intersection set imply that the only non-trivial local homology group of I at $x \in I$ is $H_r(I, I \setminus \{x\})$, where r is the dimension of the corresponding s-subvariety. Moreover, this homology group will always be of the form $\mathbb{Z}^k$, where $k \geq 0$ corresponds to the number of homeomorphs of the r-dimensional ball meeting at x. Therefore, the local homology of the intersection set of two submanifolds of $\mathbb{R}^n$ is completely described by one integer. Looking back at the statement of Theorem 6.6, it becomes clear that such an integer can be retrieved from the homology of the union, $M \cup N$. Indeed, suppose that all the homology groups, $H_j(M \cup N, (M \cup N) \setminus \{x\})$, $0 \leq j \leq n$, are known. Theorem 6.6 implies that this collection contains no more than three nonzero groups. Having exactly three nontrivial homology groups, $H_{l_j}(M \cup N, (M \cup N) \setminus \{x\})$, $j = 1, 2, 3$, $l_1 < l_2 < l_3$, corresponds to the case $2(i)$ of the theorem, from which it follows that $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1}$, where $r_1 = \mathrm{rank} H_{l_1}(M \cup N, (M \cup N) \setminus \{x\})$. Similarly, if there are only two nonzero homology groups, $H_{l_1}(M \cup N, (M \cup N) \setminus \{x\})$ and $H_{l_2}(M \cup N, (M \cup N) \setminus \{x\})$, $l_1 < l_2$, and $r_2 = \mathrm{rank} H_{l_2}(M \cup N, (M \cup N) \setminus \{x\}) = 2$, then the case $2(iii)$ of the theorem implies that again $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1}$. If, on the other hand, $r_2 = 1$, then $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1-1}$, provided that $I \neq \{x\}$. Applying similar reasoning to the rest of the cases, we obtain the following.

Corollary 6.12. *Let M and N be two submanifolds of $\mathbb{R}^n$ without boundaries, and let $I = M \cap N$ be an s-subvariety. Suppose also that $n > \mathrm{dim} M \geq \mathrm{dim} N > \mathrm{dim} I > 0$. If $x \in M \cup N$ and $H_j = H_j(M \cup N, (M \cup N) \setminus \{x\})$, $0 < j < n$, then one of the following holds.*

1. *$H_{l_i} \neq 0$, $i = 1, 2, 3$, $l_1 < l_2 < l_3$, and $H_j = 0$, $j \neq l_1, l_2, l_3$. In this case $x \in I$, $\mathrm{dim} I = l_1 - 1$, $\mathrm{dim} N = l_2$, $\mathrm{dim} M = l_3$, and $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1}$, where $r_1 = \mathrm{rank} H_{l_1}$.*
2. *$H_{l_i} \neq 0$, $i = 1, 2$, $l_1 < l_2$, and $H_j = 0$, $j \neq l_1, l_2$. Then there are three possibilities.*
   - (i) *$H_{l_1} = \mathbb{Z}$ and $H_{l_2} = \mathbb{Z}$. In this case $x \notin I$, $\mathrm{dim} N = l_1$, $\mathrm{dim} M = l_2$.*
   - (ii) *$H_{l_2} = \mathbb{Z}^2$. In this case $x \in I$, $\mathrm{dim} I = l_1 - 1$, $\mathrm{dim} N = \mathrm{dim} M = l_2$, and $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1}$, where $r_1 = \mathrm{rank} H_{l_1}$.*
   - (iii) *$H_{l_1} \neq \mathbb{Z}$ and $H_{l_2} = \mathbb{Z}$. In this case $x \in I$, $\mathrm{dim} I = l_1 - 1$, $\mathrm{dim} N = l_1$, $\mathrm{dim} M = l_2$, and $H_{l_1-1}(I, I \setminus \{x\}) = \mathbb{Z}^{r_1-1}$, where $r_1 = \mathrm{rank} H_{l_1}$.*
3. *$H_l \neq 0$ for some $0 < l < n$, and $H_j = 0$, $j \neq l$. In this case $x \in I$, $\mathrm{dim} I = l - 1$, $\mathrm{dim} N = \mathrm{dim} M = l$, and $H_{l-1}(I, I \setminus \{x\}) = \mathbb{Z}^r$, where $r = \mathrm{rank} H_l$.*

6.4. **Manifolds with boundary.** Theorem 6.6 can be easily generalized to the case of topological submanifolds with boundaries. The proof remains virtually unchanged: we just have to consider cases when a point of interest belongs to the boundary of each of the submanifolds. If $x \in \partial M \cap N$ or $x \in M \cap \partial N$, then the set A or, respectively, B used in the proof is contractible. Therefore, the corresponding elements in the Mayer-Vietoris sequence are zeros. Following the proof, we can conclude that if $x \in \partial M \cap (N \setminus \partial N)$ or $x \in (M \setminus \partial M) \cap \partial N$, then the local homology will be as in the items $(i) - (iv)$ of Theorem 6.6 with $\mathbb{Z}$ factored out at the corresponding places. If $x \in \partial M \cap \partial N$, then we should factor out $\mathbb{Z}^2$. Thus, we obtain the following theorem.

Theorem 6.13. *Let M and N be two topological submanifolds of $\mathbb{R}^n$ with boundaries, and let $I = M \cap N$ be an s-subvariety (with a boundary). Denote by p, q and r dimensions of M, N and I, respectively, and let $n > p \geq q > 0$. If $x \in I \cap (\partial M \cup \partial N)$ the following hold:*

(i) *if $p > q > r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } \ k = p, x \notin \partial M \ \text{ or} \\ & \quad\ k = q, x \notin \partial N \\ H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = r+1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) *if $p > q = r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } \ k = p, x \notin \partial M \\ \mathbb{Z} \oplus H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = q = r+1, \\ & \quad\ x \notin \partial N \\ H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = q = r+1, \\ & \quad\ x \in \partial N \\ 0, & \text{otherwise} \end{cases}$$

(iii) *if $p = q > r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } \ k = p = q, x \notin \partial M \cap \partial N \\ H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = r+1 \\ 0, & \text{otherwise} \end{cases}$$

(iv) *if $p = q = r+1$, then*

$$H_k(M \cup N, M \cup N \setminus \{x\}) = \begin{cases} \mathbb{Z} \oplus H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = p = q = r+1, \\ & \quad\ x \notin \partial M \cap \partial N \\ H_{k-1}(I, I \setminus \{x\}), & \text{if } \ k = p = q = r+1, \\ & \quad\ x \in \partial M \cap \partial N \\ 0, & \text{otherwise} \end{cases}$$

The corresponding generalization of Corollary 6.8 is:

Corollary 6.14. *Let M and N be $C^r, r \geq 1$, compact submanifolds of $\mathbb{R}^n$ with boundaries of dimensions $p < n$ and $q < n$, respectively, and suppose $M \pitchfork N$. Then if $x \in \partial M \cap (N \setminus \partial N)$ or $x \in (M \setminus \partial M) \cap \partial N$, the local (relative) homology* *satisfies*

$$H_k(M \cup N, (M \cup N) \setminus \{x\}) = \begin{cases} \mathbb{Z}^2, & \text{if } k = p, p = q = n - 1 \\ \mathbb{Z}, & \text{if } k = p, p = q < n - 1 \\ \mathbb{Z}, & \text{if } k = p + q - n; p \neq q \\ \mathbb{Z}, & \text{if } k = p, x \in (M \setminus \partial M) \cap \partial N; p \neq q \\ \mathbb{Z}, & \text{if } k = q, x \in \partial M \cap (N \setminus \partial N); p \neq q \\ 0, & \text{otherwise} \end{cases},$$

and if $x \in \partial M \cap \partial N$, then

$$H_k(M \cup N, (M \cup N) \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } k = p, p = q = n - 1 \\ \mathbb{Z}, & \text{if } k = p + q - n; p \neq q \\ 0, & \text{otherwise} \end{cases}$$

These results show that the local homology allows us to distinguish not only intersection points from non-intersection points, but also boundary points from non-boundary points in the intersection set of two transverse, codimension-1 submanifolds of $\mathbb{R}^n$.

6.5. **Applications of intersection techniques.** Throughout the years, models in CAGD have evolved from simple surfaces to complicated, nonmanifold, higher-dimensional spaces. The way these models are represented can be quite different, but here we will focus on models that are simplicial complexes (cf. [115], [116], and [133]). Such complexes can be very general and may easily model rather intricate topological spaces, which makes them highly attractive to those who deal with nonmanifolds. Nonmanifoldness itself can assume various guises, but the type that most frequently occurs in geometric models results from the intersection of several (usually two) manifolds. The intersections can be both transverse and tangential, and their number can be quite large. When one performs some operations on such nonmanifold models (e.g. smoothing), topological artifacts may appear in a neighborhood of a nonmanifold point. Also, some topological defects at (or around) intersections in a model are often produced during its construction. Detecting such flaws is problematical, since some of them may not be visible to a human eye. There are several procedures for reducing such deficiencies in a geometric model, but most of them are restricted to two- or three-dimensional cases. Results obtained in the previous subsection allow us to design a general framework for analyzing and improving geometric nonmanifold models in any finite-dimensional space. The basic idea is rather naive – simply compute the local homology groups at a point of interest. Such a point is usually a nonmanifold point, and the local homology is usually known to a user, since a designer knows the number of manifolds and in what way they should intersect at each point of the model. Therefore, as follows from Theorem 6.6 and its corollaries, comparing the computed local homology with the desired one should determine whether or not there is a defect in the model at this point.

Besides detecting flaws, the foregoing results can often be used to determine whether a given model is a manifold. Indeed, the local homology at a manifold point has only one nontrivial homology group, which is homeomorphic to $\mathbb{Z}$. In most models, the nontrivial homology groups at a nonmanifold point are different from $\mathbb{Z}$. In fact, many models turn out to be codimension-1, s-subvarieties. To detect nonmanifoldness in such a model, we can browse through all vertices of the model and check if the local homology group of the corresponding dimension is $\mathbb{Z}$. This may not be very efficient, and we explain later.

We now provide more details regarding actual implementation of the above ideas. Obviously, any algorithm that does such computations depends in a fundamental way on the data structure used to describe the model. There are several nonmanifold structures, e.g. Radial Edge Data Structure (cf. [155]), Tri-Cyclic Data Structure (cf. [71]), NeMeSi Data Structure (cf. [131]). Though there are some significant differences among them, they all have an important common feature - they provide means for local analysis of the topology of the model. For example, it is always possible to compute the link of a point quite efficiently. This allows us to describe the basic steps of the intersection detection algorithm without getting involved in tedious details. Suppose that a model is represented by a simplicial complex S (described by one of the data structures). Then the primary steps are the following:

Step 1. Choose (using some method) a vertex $v \in S$ and compute its link $L(v)$.
Step 2. Compute the homology groups of $L(v)$. By excision, these are the local homology groups of S at v.
Step 3. Using Theorem 6.6, try to determine what kind of intersection, if any, occurs at the point v. If needed, go to Step 1.

These steps are very general, and many intermediate steps that were skipped require much work. For example, computation of homology groups is a important question that has been addressed in quite a few papers and books related to CT (cf [18], [40], [42], [55], [85], [86], [117], [157], [160], and [161]). Fortunately, the link of a vertex in a simplicial complex does not usually contain a large number of simplices. Therefore, even the most basic algorithms, say the reduction algorithm, can be used without any negative repercussions. Also, it is worth noting that Theorem 6.6 may not provide an answer in some exotic cases. But, as we have already mentioned, this does not usually happen in practice. Another important question is how to choose a point in Step 1. Though existing data structures provide excellent means for analyzing local topological structure of a simplicial complex, they do not incorporate any information that would facilitate global analysis. As a result, it is impossible, for example, to figure out *a priori* in what direction it is best to move if we want to determine whether there is a nonmanifold point in the model. Therefore, an exhaustive search has to be performed in such cases, and this is highly inefficient.

We also should point out that computing local homology groups (as indicated in Step 2) may be an overkill in some cases. In fact, if one looks for a transverse intersection in a simplicial complex that represents a codimension-1, s-subvariety, the following lemma [26] is much easier to apply than the homological criteria developed above.

Lemma 6.15. *Let M and N be two codimension-1 submanifolds of $\mathbb{R}^n$ without boundaries, and suppose $\mathcal{T}$ is a triangulation of $M \cup N$. Then each $(n-2)$-simplex of $\mathcal{T}$ that represents a part of the transverse intersection of M and N is a face of exactly four $(n-1)$-simplices.*

Thus, identifying transverse intersections is much less computationally expensive than locating nontransverse intersections, since adjacency information is readily available in any nonmanifold data structure. It is also possible to obtain similar results for tangential intersections; but only on a case-by-case basis that would not provide the kind of uniform classification that is available through the homology approach.

7. Some Applications

The application areas that we shall briefly describe in this section are: (1) virtual sculpting and surgery; (2) modeling of heterogeneous biomaterials such as bones; and (3) high-speed, topologically consistent, scientific visualization for high performance computer architectures. The descriptions that follow shall be from the perspective of CDT.

7.1. **Virtual sculpting and surgery.** The main idea for both virtual sculpting and surgery is to perform the process, in as realistic an environment as possible, without actually crafting anything or performing any invasive procedures. Both of these virtual activities have a great deal in common; a fairly simple geometric object (either a sculpting tool of some sort or possibly a human hand, or a particular surgical instrument in the case of surgery) is moved (in a virtual reality environment) by the user so that it interacts with another geometric object, which is usually a block of modeling clay in the case of sculpting, or some part of the human anatomy in the case of surgery. Thus the basic elements in both processes involve the swept volume of a geometric object and the Boolean subtraction of this swept volume from another object in order to obtain the end result. As virtual reality applications require essentially real-time responses for the system, algorithms for performing the geometric operations in these process must be extremely fast (see e.g. [26], [48], and [148]).

One nice feature of the swept volumes generated for these applications is that they are automatically sweep-like projective varieties. Working with M.C. Leu and others, we are now in the process of using the singularity based methods described in Section 5 to create topologically consistent algorithms for creating representations of the objects and modeling the operations involved in virtual sculpting and surgery in ways that are an order of magnitude more efficient than our previous algorithms, such as those in [26], [98], and [101]. In fact, we are now a testing such an algorithm $\mathcal{A}_s$ for objects in $\mathbb{R}^3$, which is stable, TOP-decidable, and has a computational cost of $CC(\mathcal{A}_s) = O(n^2)$, where n represents both the number of triangles in a triangulation of the initial object and 1/(time step) in the sweeping process. Moreover, it appears that by more effective use of available data structures, we may be able to reduce this to

$CC(\mathcal{A}_s) = O(n \log n)$. Either of these (low) costs is more than sufficient for rendering complicated objects and intricate sweeping motions in real-time within the virtual reality environment that we have created to test various computing strategies.

7.2. **Modeling heterogeneous materials.** There are many important applications that require the modeling, analysis and graphical representation of heterogeneous materials; especially those in biomedical engineering that call for computer assisted methods for tissue engineering (see e.g. [26], [144], and [145]). In joint research with W. Regli and W. Sun, reported on briefly in [26], we have been studying the viability of employing methods of computational differential topology - primarily in the guise of swept volume based techniques. To further explain the applicability of the some of the results from this paper to tissue analysis and modeling, we shall merely describe a few new developments in this direction.

If one considers the complex canal-like structure of a bone, it appears that the geometry can be very effectively modeled using sweeps of the openings in the structure of a typical cross-section at a particular z value along the main axis (z-axis) of the bone (cf. [26]). Thus, the requisite structure can be obtained from flows of a system of differential equations of the form $d\mathbf{x}/dz = \mathbf{X}(\mathbf{x}, z)$. One new wrinkle that we have introduced in our recent modeling attempts, is the use of systems that generate chaotic flow regimes. Such systems have the nice feature of creating bone models that have a random generic structure that may be especially well suited to the fabrication of bone tissue replacements that are acceptable regardless of the specific characteristics of an individual bone.

When it comes to using our new dynamical system models for generating bone structure, one has to be able to deal with the branching of canals that is a standard fixture in bone tissue configurations. One can view the corresponding topological bifurcations as being analogs of handlebody decompositions of manifolds, and the flows along the canal boundaries as simulations of gradient flows associated to Morse functions that determine handlebody decompositions and their associated Reeb graphs. As mentioned in Section 3, we have been working on effectively computable, sweep based methods for describing handlebody and Reeb graph structure for manifolds - and generalizations of these approaches for varieties. It is our intention to employ these methods in an effort to create efficient methods for accurately modeling bone tissue structure in a way that satisfies certain prescribed topological constraints. As one can see, such problems and applications lie squarely in the realm of CDT.

7.3. **Topologically consistent scientific visualizations.** We have recently embarked on an interdisciplinary research project with R. Kopperman and T.J. Peters aimed at creating and analyzing high-speed scientific visualizations via algorithmic means that guarantee that the evolving scenes have the correct isomorphism type in at least one of the categories discussed above. An important goal of this interdisciplinary research effort is to devise methods and create software for use in the visualization of complex, evolving configurations of the type encountered in life science applications such as those connected with the high performance computer architectures currently being used by IBM Deep Computing scientists. As indicated in such investigations as [128] and [132], visualization is an area in which although much progress has been made, there are many nagging intrinsic problems that require novel approaches for their resolution.

Our research in this area has already produced some minor revelations. One of which is the potential use of results in T_0-topology (such as in [92] and [93]) in characterizing preservation of isomorphism type, and another is how well suited visualization is to modeling using sweep techniques, which give rise to geometric constructs that suggest the application of cobordism theory as presented in [109] and [151]. To illustrate how this works, we have developed the following framework for modeling the type of dynamic processes that we want to visualize. We take our ambient space to be $\mathbb{R}^n$, observing that $n = 2$ or 3 in most applications, and let Ω be an open n-cube in which all the action in the process unfolds. Although we take the time period to be the unit interval, this incurs no loss in generality, through appropriate scaling.

Definition 7.1. *Let $M_1, ..., M_N$ be smooth submanifolds, and let $\Phi_k : \Omega \times [0,1] \to \Omega$, $1 \le k \le N$, be diffeotopies. We call the pair $\{\Phi, M\}$ comprised of the map*

$$\Phi := \Phi_1 \times \Phi_2 \times \cdots \times \Phi_N : \Omega^N \times [0,1] \to \Omega^N,$$

and the collection of submanifolds $M := \{M_1, ..., M_N\} (= M_1 \times \cdots \times M_N)$ in Ω, a dynamic sweep process, or more precisely, the dynamic Φ-sweep of the (initial) ensemble M, and say that

$$F_t = F_t(\Phi, \mathcal{M}) := \phi_t(\mathcal{M}) := (\varphi_{1,t}(M_1), ..., \varphi_{N,t}(M_N))$$

is the t-frame of the sweep process, where $\phi_t := \Phi(\cdot, t)$ and $\varphi_{k,t} := \Phi_k(\cdot, t)$, $1 \le k \le N$. The submanifolds $\varphi_{1,t}(M_1), ..., \varphi_{N,t}(M_N)$ are the sweeping elements in the dynamic sweep process, Φ is the sweep, and its coordinate functions Φ_k are the element sweeps.

A visualization of a Φ-sweep is a computer generated graphical representation of the image (changing with time) of $\varphi_{1,t}(M_1) \cup \cdots \cup \varphi_{N,t}(M_N)$ on a specified disjoint hypersurface (possibly changing with time) generated by the projection from a vantage point $v(t) \in \mathbb{R}^n \setminus (\varphi_{1,t}(M_1) \cup \cdots \cup \varphi_{N,t}(M_N))$.

The appellation sweep in the above is indeed apt: for example, in swept volume theory, $\cup\{\phi_t(\mathcal{M}) : 0 \le t \le 1\}$ is just the (deformed) swept volume of $\mathcal{M}$ generated by the sweep map Φ. Accordingly we can build upon the rich algebraic structure, the extensive body of theory (including techniques for analyzing topological consistency), and the extremely efficient algorithms for analysis and representation to devise innovative solutions to the fundamental problems envisaged in this project. Observe also that the overall sweep between the initial and terminal scene is essentially a corbordism between these two configurations. This framework is well worth studying as a means of describing the dynamic processes to be visualized, and its very nature suggests that additional concepts from differential topology might prove to be quite useful in this regard.

8. Concluding Remarks

We have made a concerted effort in this paper to help lay the foundations for the differentiable aspects of CT. In the process, we have established a plausible category to work in comprised of subvarieties embedded in Euclidean spaces having a Whitney regular stratification, and morphism of these subvarieties that respect the embedding in the given ambient space and the stratified structure.

Our main focus was on effective procedures for rendering approximate representations of the objects that incorporate subalgorithms for determining the isomorphism type of the approximations of a given geometric object. This naturally led to the identification of fundamental problems related to these algorithms. A new result on the existence of tubular-like neighborhoods for subvarieties of interest enabled us to reduce the classification in the fundamental problems to the topological category, along the lines of what we have dubbed the *self-intersection precedes knotting principle*. In aid of obtaining algorithmic methods for determining the isomorphism types of the objects in the CDT category, we extended Munkres' results on differentiable triangulations to these objects. In the course of our exposition, we provided an overview of the state-of-the-art of the classification of geometric objects - including recent developments concerning the Poincaré Conjecture - from the perspective of effective computability. We also highlighted some of the possibilities and limitations of the algorithmic approach to the classification of isomorphism type, made several significant conjectures in this regards, and identified a number of important open problems. Along the way we identified a class of subvarieties - called sweep-like projective varieties - that are particularly well suited to recursive characterization of isomorphism type in virtue of the fact that this question is, in effect, lifted to a simpler object; namely a submanifold. These special subvarieties were also shown to play a featured role in the topological analysis of subvarieties using singularity/classification theory and obstruction theory.

Among the contributions of this paper are a brief development of an effectively computable singularity theory based method for analyzing and representing certain subvarieties; for example, sweep-like projective varieties. We observed that piecewise linear approximation makes it possible to implement singularity theory in an algorithmic fashion, and we briefly described how such algorithms can be devised. Another relatively new idea - at least in terms of possibly serving as a general approach to the representation and isomorphism identification problem - was very succinctly outlined in our section on the elements of an effectively computable obstruction theory. Triangulation of the objects under consideration and associated piecewise linear approximations were also key factors in the development of a simplified, algorithmic obstruction theory.

Even in our brief sketches of effective procedures for applying singularity theory and obstruction theory, the importance of intersections - especially self-intersections in the subvarieties - is manifest. The problems of intersection detection and intersection analysis were studied using a new a analytic method and some innovative homology based techniques. We demonstrated the effectiveness of our analytic method for intersection detection, and showed how our homological approach to characterizing the topology of intersections extended some earlier results that we obtained [26]. In particular, we were able to extend our earlier homology based results to nontransverse intersections and objects that are not manifolds. Moreover, we described some of the ways in which these intersection results can be implemented, in combination with several well-known data structures, for applications in CAGD.

The main content of the paper concluded with an indication of some of the many applications of the results that were obtained. We chose to illustrate the utility of the ideas and techniques developed with some applications in the areas of virtual sculpting/surgery, the modeling of heterogeneous biomaterials such as bones, and the verification and maintenance of topological consistency during scientific visualization, with particular emphasis upon scalability for high performance computer architectures.

Many interesting problems have been discussed, and several potentially fruitful research directions in CT have been outlined in this paper. We are planning to try to solve some of the open problems, resolve a few of the conjectures, and follow up on one or more of the research leads presented here in the near future. Our hope is that we have persuaded a few readers to do the same.

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References

[1] K. Abdel-Malek and H. J. Yeh, *On the determination of starting points for parametric surface intersections*, CAD **29** (1997), 21–35.

[2] K. Abdel-Malek and H. J. Yeh, *Geometric representation of the swept volume using Jacobian rank deficiency conditions*, CAD **29** (1997), 457–468.

[3] K. Abdel-Malek, H. J. Yeh and S. Othman, *Swept volumes: void and boundary identification*, CAD **30** (1998), 1009–1018.

[4] K. Abdel-Malek, W. Seaman and H. J. Yeh, *NC verification of up to 5-axis machining processes using manifold stratification*, ASME J. Manufacturing Sci. and Engineering **122** (2000), 121–135.

[5] K. Abdel-Malek, J. Yang and D. Blackmore, *On swept volume formulations: implicit surfaces*, CAD **33** (2001), 113–121.

[6] K. Abdel-Malek, D. Blackmore and K. Joy, *Swept volumes: foundations, perspectives and applications*, Int. J. Shape Modeling (submitted).

[7] K. Abe *et al.*, *Computational topology for reconstruction of surfaces with boundary: integrating experiments and theory*, in Proceedings of the IEEE International Conference on Shape Modeling and Applications, June 15 û 17, 2005, Cambridge, MA, IEEE Computer Society, Los Alimitos, CA, 288 - 297.

[8] K. Abe *et al.*, *Computational topology for isotopic surface reconstruction*, Theoretical Computer Science, Special Issue – Spatial Representation: Discrete vs. Continuous Computational Models, Edited by R. Kopperman, P. Panangaden, M.B. Smyth, D. Spreen and J. Webster, **365** (3), 184–198, 2006.

[9] P. Agarwal, H. Edelsbrunner and Y. Wang, *Computing the writhing number of a polygonal knot*, Discrete Comput. Geom. **32** (2004), 37–53.

[10] N. Amenta, S. Choi, T. Dey and N. Leekha, *A simple algorithm for homeomorphic surface reconstruction*, in ACM Symposium on Computational Geometry, 2000, pp. 213-222.

[11] N. Amenta, S. Choi and R. Kolluri, *The power crust, union of balls and the medial axis transform*, Comput. Geom.: Theory and Applicatons **19** (2001), 127–173.

[12] N. Amenta, T. Peters and A. Russell, *Computational topology: ambient isotopic approximation of 2-manifolds* (invited paper), Theoretical Computer Sci. **305** (2003), 3–15.

[13] L.-E. Andersson, S. Dorney, T. Peters and N. Stewart, *Polyhedral perturbations that preserve topological form*, CAGD **12** (1995), 785–799.

[14] L.-E. Andersson, T. Peters and N. Stewart, *Equivalence of topological form for curvilinear geometric objects*, Int. J. Computational Geom. and Appls. **10** (6) (2000), 609–622.

[15] V. Arnold, S. Gusein-Zade and A. Varchenko, *Singularities of Differentiable Maps, Vols. I & II*, Birkhäuser, Boston, 1985.

[16] U. Axen and H. Edelsbrunner, *Auditory Morse analysis of triangulated manifolds*, in Mathematical Visualization, H.-C. Hege and K. Polthier (eds.), Springer-Verlag, Berlin, 1988, pp. 223-236.

[17] C. Bajaj, C. Hoffmann, R. Lynch and J. Hopcroft, *Tracing surface intersections*, CAGD **5** (1988), 285–307.

[18] S. Basu, *Computing the Betti numbers of arrangements via spectral sequences*, J. Computer and System Sciences **67** (2003), 244–262.

[19] M. Bern, *et al.*, *Emerging Challenges in Computational Topology*, Report of NSF Workshop on Computational Topology, June 11-12, 1999, Miami Beach, FL, `http://xxx.lanl.gov/abs/cs/9909001`.

[20] J. Bisceglio, C. Mow and T. Peters, *Boolean algebraic operands for engineering design* (preprint), `www.cse.uconn.edu/~tpeters`.

[21] D. Blackmore, M. C. Leu and F. Shih, *Analysis and modelling of deformed swept volumes*, CAD **26** (1994), 315–326.

[22] D. Blackmore, M. C. Leu and L. P. Wang, *The sweep-envelope differential equation algorithm and its application to NC machining verification*, CAD **29** (1997), 629–637.

[23] D. Blackmore, M. C. Leu, L. P. Wang and H. Jiang, *Swept volumes: a retrospective and prospective view*, Neural, Parallel and Scientific Computations **5** (1997), 81–102.

[24] D. Blackmore, R. Samulyak and M. C. Leu, *Trimming swept volumes*, CAD **31** (1999), 215–223.

[25] D. Blackmore, R. Samulyak and M. C. Leu, *A singularity theory approach to swept volumes*, Int. J. Shape Modeling **6** (2000), 105–129.

[26] D. Blackmore, D., Y. Mileyko, M.C. Leu, W. Regli, and W. Sun, *Computational topology and swept volumes*, Geometric and Algorithmic Aspects of Computer-Aided Design and Manufacturing, R. Janardan, M. Smid and D. Dutta (eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. **67**, AMS, Providence, RI, 2005, pp. 53–78.

[27] J.-D. Boissonnat, D. Cohen-Steiner and G. Vegter, *Meshing implicit surfaces with certified topology*, RR-4930 `http://www.inria.fr/rrrt/rr-4930.html`

[28] P.-T. Bremer, V. Pascucci, H. Edelsbrunner and B. Hamann, *A topological hierarchy for functions on triangulated surfaces*, IEEE Trans. Vis. Comput. Graphics **10** (2004), 385–396.

[29] H.-D. Cao and X.-P. Zhu, *A complete proof of the PoincarÚ and geometrization conjectures - an application of the Hamilton-Perelman Theory of the Ricci flow*, Asian J. Math. **10** (2006), 165–492.

[30] E. Chazal and D. Cohen-Steiner, *A condition for isotopic approximation*, Proc. ACM. Symp. Solid Modeling and Applications, 2004.

[31] B. Chazelle, H. Edelsbrunner, L. Guibas and M. Sharir, *A singly exponential stratification scheme for real semi-algebraic varieties and its application*, Proc. 16th Int. Colloq. Automata, Languages and Programming, Lect. Notes in Comp. Sci., No. 372, Springer-Verlag, Berlin, 1989, pp. 179–192.

[32] H. L. Cheng, H. Edelsbrunner and P. Fu, *Shape space from deformation*, Proc. 6th Pacific Conf. Comput. Graphics Appl., Singapore, 1998, pp. 104–113.

[33] K. Cole-McLaughlin, H. Edelsbrunner, J. Harer, V. Natarajan and V. Pascucci, *Loops in Reeb graphs of 2-manifolds*, Discrete Comput. Geom. **32** (2004), 231–244.

[34] N. Cornish and J. Weeks, *Measuring the shape of the universe*, Not. AMS **45** (1998), 1463–1471.

[35] F. Crimins and D. Souvaine, *Constructing differentiable homeomorphisms between isomorphic triangulations*, The 14th Canadian Conference of Computational Geometry, Lethbridge, Alberta, Canada, 2002, pp.170 – 173.

[36] T. Culver, J. Keyser and D. Manocha, *Exact computation of a medial axis of a polyhedron*, CAGD **2**1 (1) (2004), 65–98.

[37] J. Damon, *On the smoothness and geometry of boundaries associated to skeletal structures, I: sufficient conditions for smoothness*, Ann. Inst. Fourier **53** (2003), 1941–1985.

[38] J. Damon, *Determining the geometry of boundaries of objects from medial data*, Int. J. Comp. Vision **63** (2005), 45–64.

[39] F. Dachille IX, H. Qin and A. Kaufman, *A novel haptics-based interface and sculpting system for physics based geometric design*, CAD **33** (2001), 403–420.

[40] C. Delfinado and H. Edelsbrunner, *An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere*, CAGD **12** (1995), 771–784.

[41] P. Dmitrov, J. Damon and K. Siddiqi, *Flow invariants for shape*, CVPR (2003).

[42] T. Dey and S. Guha, *Computing homology groups of simplicial complexes in* $\mathbb{R}^3$, J. ACM **45** (1998), 266–287.

[43] T. Dey, H. Edelsbrunner and S. Guha, *Computational topology, in Advances in Discrete and Computational Geometry*, Contemporary Mathematics **223**, American Mathematical Society (1999), 109–143.

[44] T. Dey, H. Edelsbrunner, S. Guha and D. Nekhayev, *Topology preserving edge contraction*, Publ. Inst. Math. (*Beograd*) (*N.S.*) **66** (1999), 23–45.

[45] T. Dey and S. Goswami, *Tight Cocone: a water-tight surface reconstructor*, Proc. Eighth ACM Sympos. on Solid Modeling and Applications (2003), 127–134.

[46] T. Dey, H. Woo and W. Zhao, *Approximate medial axis for CAD models*, Proc. Eighth ACM Symposium on Solid Modeling and Applications (2003), 280–285.

[47] S. Donaldson and P. Kronheimer, *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, England, 1990.

[48] J. Dorman and A. Rockwood, *Surface design using hand motion with smoothing*, CAD **33** (2001), 389–402.

[49] A. Edalat and A. Lieutier, *Foundations of a computable solid modeling*, Proc. Fifth ACM Symposium on Solid Modeling and Applications, Ann Arbor, MI (1999), 278–284.

[50] H. Edelsbrunner, *Geometry and Topology for Mesh Generation*, Cambridge University Press, Cambridge, England, 2001.

[51] H. Edelsbrunner, J. Harer, A. Mascarenhas and V. Pascucci, *Time-varying Reeb graphs for continuous space-time data*, Proc. 20th Ann. Sympos. Comput. Geom. (2004), 366–372.

[52] H. Edelsbrunner and J. Harer, *Jacobi sets of multiple Morse functions*, in Foundations of Computational Mathematics, F. Cucker, R. DeVore, P. Olver and E. Sueli (eds.), Cambridge University Press, Cambridge, England, (2002), 37–57.

[53] H. Edelsbrunner, J. Harer, V. Natarajan and V. Pascucci, *Morse-Smale complexes for piecewise linear 3-manifolds*, Proc. 19th Ann. Sympos. Comput. Geom. (2003), 361–370.

[54] H. Edelsbrunner, J. Harer and A. Zomorodian, *Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds*, Discrete Comput. Geom. **30** (2003), 87–107.

[55] H. Edelsbrunner, D. Letscher and A. Zomorodian, *Topological persistence and simplification*, Discrete Comput. Geom. **28** (2002), 511–533.

[56] H. Edelsbrunner and N. Shah, N.R., *Triangulating topological spaces*, Intl. J. of Computational Geometry and Applications **7** (4) (1997), 365–378.

[57] H. Edelsbrunner and A. Zomorodian, *Computing linking numbers of a filtration*, in Proc. 1st Intl. Workshop Alg. BioInformatics (2001), 112–127.

[58] G. Farin, *Curves and Surfaces for Computer Aided Design: A Practical Guide*, Academic, Boston, 1988.

[59] R. Farouki, *Closing the gap between CAD model and downstream application*, SIAM News **32** (5), June 1999.

[60] R. Farouki, C-Y. Han, J. Hass and T. Sederberg, *Topologically consistent trimmed surface approximations based on triangular patches*, CAGD **21** (2004), 459–478.

[61] R. Forman, *Morse theory for cell complexes*, Adv. Math. (1998), 90–145.

[62] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.

[63] M. Freedman and F. Quinn, *Topology of 4-Manifolds*, Princeton University Press, Princeton, NJ, 1990.

[64] A. Gain and A. Dodgson, *Preventing self-intersection under free-form deformation*, IEEE Trans. on Visualization and Computer Graphics **7** (4) (2001), 289 – 298.

[65] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove and D. Scott, *Continuous Lattices and Domains*, Encyclopedia of Mathematics, Vol.9, Cambridge University Press, Cambridge, 2003.

[66] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer-Verlag, New York, 1973.

[67] M. Gopi, *On sampling and reconstructing surfaces with boundaries*, Proc. Canadian Conf. on Computational Geometry, Lethbridge, Canada (2002), 49–53.

[68] T. Grandine, T. A. and F. W. Klein, IV, *A new approach to the surface intersection problem*, CAGD **14** (1997), 111–134.

[69] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[70] L. Guibas, *Computational geometry and visualization: problems at the interface*, in Scientific Visualization of Physical Phenomena, N. Patrakalikas (ed.), Springer-Verlag, New York (1991), 45–59.

[71] E. Gursoz, Y. Choi and F. Prinz, *Vertex-based representation of non-manifold boudaries*, in Geometric Modeling for Product Engineering, J. Turner, M. Wozny, and K. Preiss (eds.), Elsevier Science, North-Holland (1990), 107–130.

[72] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.

[73] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[74] J. Hass, J. Lagarias and N. Pippenger, *The computational complexity of knot and link problems*, J. ACM **46** (1999), 185–211.

[75] G. Hemion, *The Classification of Knots and 3-Dimensional Space*, Oxford University Press, Oxford, 1992.

[76] H-S. Heo, S. J. Hong, J-K. Seong, M-S. Kim and G. Elber, *The intersection of two ringed surfaces and some related problems*, Graph. Models **63** (2001), 228–244.

[77] G. Herman, *Geometry of Digital Spaces*, Birkhäuser, Boston, 1998.

[78] M. W. Hirsch, *Differential Topology*, Springer-Verlag, New York, 1976.

[79] C. Hoffmann, *The problems of accuracy and robustness in geometric computations*, Computer **22** (1989), 31–41.

[80] C. Hoffmann, *Geometric and Solid Modeling - An Introduction*, Morgan Kaufmann, San Mateo, CA, 1989, out of print, but available at `http://www.cs.purdue.edu/homes/cmh/MyHome.html`

[81] H. Hoppe, T. DeRose, T. Duchamp, J. McDonald and W. Stuetzle, *Surface reconstruction from unorganized points*, Proc. ACM SIGGRAPH '92 (1992), 71–78.

[82] D. Husemoller, *Fibre Bundles*, Springer-Verlag, New York, 1975.

[83] F. Jaeger, D. Vertigan and D. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Proc. Cambridge Philos. Soc. **108** (1990), 35–53.

[84] J. Johnstone, *A new intersection algorithm for cyclides and swept surfaces using circle decomposition*, CAGD **10** (1993), 1–24.

[85] T. Kacyzynski, M. Mrowzek and M. Slusarek, *Homology computation by reduction of chain complexes*, Computers & Math. Appl. **35** (1998), 59–70.

[86] T. Kacyzynski, K. Mischaikow and M. Mrowzek, *Computational Homology*, Springer-Verlag, New York, 2004.

[87] K. Kase, A. Makinouchi, T. Nakagawa, H. Suzuki and F. Kimura, *Shape error evaluation method of free-form surfaces*, CAD **31** (1999), 495–505.

[88] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.

[89] M. Kervaire and J. Milnor, *Groups of homotopy spheres: I*, Annals of Math. **77** (1963), 504–537.

[90] H. King, K. Knudson and N. Mramor, *Generating discrete Morse functions from point data*, Experimental Math. **14** (2005), 435–444.

[91] Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vols. I & II*, Interscience, New York, 1963 & 1969.

[92] R. Kopperman, V. Tkachuk and R. Wilson, *The approximation of compacta by finite T_0-spaces*, Quaestiones Math. **26**(3) (2003), 155–170.

[93] R. Kopperman, S. Matthews and H. Pajoohesh, *Universal partial metrizability*, Applied General Topology **5** (1) (2004), 115—127.

[94] G. Kreizis, N. Patrikalakis and F. Wolter, *Topological and differential-equation methods for surface intersections*, CAD **24** (1992), 41–55.

[95] B. Kvasov, *Algorithms for shape preserving local approximation with automatic selection of tension parameters*, CAGD **17** (2000), 17–37.

[96] H. Lawson, *The Qualitative Theory of Foliations*, Regional Conf. Series in Math., No. 27, AMS, Providence, 1975.

[97] M. C. Leu, D. Blackmore and B. Maiteh, *Deformed swept volume analysis of NC machining with cutter deflection*, in Machining Impossible Shapes, B. K. Choi and R. Jerard (eds.), Kluwer, Boston (1999), 158–166.

[98] M. C. Leu, B. Maiteh, D. Blackmore and L. Fu, *Creation of freeform solid models in virtual reality*, Annals of CIRP **50** (2000), 73–76.

[99] Y.C. Lu, *Singularity Theory and an Introduction to Catastrophe Theory*, Springer-Verlag, New York, 1976.

[100] Y. Ma and Y.-S. Lee, *Detection of loops and singularities of surface intersections*, CAD **30** (1998), 1959–1976.

[101] B. Maiteh, M. C. Leu, D. Blackmore and L. Abdel-Malek, *Swept volume computation for machining simulation and virtual reality application*, J. Materials Processing & Manufacturing Science **7** (1999), 380–390.

[102] R. Malgouyres and A. Lenoir, *Topology preservation within digital surfaces*, Graphical Models **62** (2000), 71–84.

[103] M. Mäntyalä, *Computational topology: a study on topological manipulations and interrogations in computer graphics and geometric modeling*, Acta Polytechnica Scandinavica, Vol. 37, Mathematics and Computer Science Series. Finnish Academy of Technical Sciences, Helsinki, 1983.

[104] A. Mäntylä, *An Introduction to Solid Modeling*, Computer Science Press, Rockville, 1988.

[105] W. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, New York, 1967.

[106] J. Mather, *Stratifications and mappings*, in Dynamical Systems, M. Peixoto (ed.), Academic Press, New York, 1973.

[107] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Annals of Math. **64** (1956), 399–405.

[108] J. Milnor, *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962–972.

[109] J. Milnor, *Lectures on the h-Cobordism Theorem*, Princeton University Press, Princeton, NJ, 1965.

[110] J. Milnor and R. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, NJ, 1974.

[111] J. Mitchell, *Topological obstructions to blending algorithms*, CAGD **17** (2000), 673–694.

[112] E. Moise, *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, New York, 1977.

[113] J. Morgan, *Recent progress on the Poincaré Conjecture and the classification of 3-manifolds*, Bull. AMS **42** (2004), 57–78.

[114] C. Mow, T. Peters and N. Stewart, *Specifying useful error bounds for geometry tools: an intersector exemplar*, CAGD **20** (2003), 247–251.

[115] J. Munkres, *Elementary Differential Topology*, Annals of Math. Studies 54, Princeton University Press, Princeton, NJ, 1966.

[116] J. Munkres, *Elements of Algebraic Topology*, Perseus Publ., Cambridge, MA, 1984.

[117] P. Niyogi, S. Smale and S. Weinberger, *Finding the homology of submanifolds with high confidence from random samples*, preprint.

[118] N. Patrikalakis, T. Sakkalis and G. Shen, *Boundary representation models: Validity and rectification*, Proc. 9th Conf. on Mathematics of Surfaces, Cambridge University, Sept. 2000,R. Cipolla and R. Martin (eds.), Springer-Verlag, New York (2000), 389–40.

[119] N. Patrikalakis and T. Maekawa, *Shape Interrogation for Computer Aided Design and Manufacturing*, (Mathematics and Visualization), Springer-Verlag, New York, 2002.

[120] G. Perelman, *Spaces with curvature bounded below*, Proc. of ICM'94, Birkhäuser (1995), 517–525.

[121] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv.math.DG/0211159, 2002.

[122] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv.math.DG/0303109, 2003.

[123] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv.math.DG/0307245, 2003.

[124] T. Peters, *et al.*, *I-TANGO: Intersections — Topology, Accuracy and Numerics for Geometric Objects (in Computer Aided Design)*, NSF Award Abstract - #0138098 `http://www.nsf.gov/awardsearch/showAward.do?AwardNumber=0138098`

[125] T. Peters, J. Bisceglio, D. Ferguson, C. Hoffmann, T. Maekawa, N. Patrikalakis, T. Sakkalis and N. Stewart, *Computational topology for regular closed sets (within the I-TANGO project)*, invited article, Topology Atlas **9** (1) (2004), 12 pp. `http://at.yorku.ca/t/a/i/c/50.htm`.

[126] T. Peters, D. Rosen and S. Dorney, S. M., *The diversity of topological applications within computer aided geometric design*, Annals of the New York Academy of Science **728** (1994), 198–209.

[127] T. Peters and N. Stewart, *Computational topology for CFD: theorems, criteria and issues*, Proc. First MIT Conf. on Computational Fluid and Solid Mechanics, Elsevier (2001), 131–134.

[128] S. Plantinga and G. Vegter, *Contour generators of evolving implicit surfaces*, Proc. Eighth ACM Sympos. on Solid Modeling and Applications, June, 2003, pp. 23–32.

[129] G. M. Reed, A. W. Roscoe and R. F. Wachter, *Topology and Category Theory in Computer Science*, Oxford University Press, Oxford, England, 1991.

[130] A. Requicha, *R.F. Representations of solid objects - theory, methods and systems*, ACM Computing Surveys **12** (1980), 437–464.

[131] J. Rossignac and M. O'Connor, *R.F.A dimension-independent model for point sets with internal structures and incomplete boundaries*, in Geometric Modeling for Product Engineering, J. Turner, M. Wozny, and K. Preiss (eds.), Elsevier Science, North Holland (1990), 145–180.

[132] J. Rossignac and J. Kim, *R.F. Computing and visualizing pose interpolating 3-D motions*, CAD **33** (2001), 279–291.

[133] T. Rushing, *Topological Embeddings*, Academic Press, New York, 1973.

[134] Point Clouds in Imaging Science, SIAM News **37**, Sept. 2004.

[135] T. Sakkalis and T. Peters, *Ambient isotopic approximations for surface reconstruction and interval solids*, Proc. Eighth ACM Sympos. on Solid Modeling and Applications, June, 2003, pp. 176–184.

[136] T. Sakkalis, T. Peters and J. Bisceglio, *Application of ambient isotopy to surface approximation and interval solids*, (invited paper), CAD **36** (11) (2004), 1089–1100.

[137] T. Sederberg, H. Christiansen and S. Kaz, *An improved test for closed loops in surface intersections*, CAD **21** (1989), 505–508.

[138] T. Sederberg, J. Zheng and X. Song, *A conjecture on tangent intersections of surface patches*, CAGD **21** (2004), 1–2.

[139] V. Shapiro, *Maintenance of geometric representations through space decompositions*, Int. J. Comp. Geom. and Appl. **7** (1997), 383–393.

[140] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[141] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Annals of Math. **74** (1961), 391–406.

[142] S. Smale, *The Mathematics of Time*, Springer-Verlag, New York, 1980.

[143] J. Stallings, *Polyhedral homotopy spheres*, Bull. AMS **66** (1962), 485–488.

[144] W. Sun and P. Lal, *Recent developments in computer-aided tissue engineering - a review*, J. Computer Methods and Programs in Biomedicine **67** (2002), 85–103.

[145] W. Sun, A. Darling, B. Starly and J. Nam, *Computer-aided tissue engineering: overview, scope and challenges*, Biotechnology & Applied Biochemistry **39** (2004), 29–47.

[146] R. Thom, *Ensemble et morphisms stratifiés*, Bull. AMS (1969), 240–284.

[147] A. Thompson, *Thin position and the recognition problem for S^3*, Math. Res. Lett. **1** (1994), 613–630.

[148] T. Thompson, D. Johnson and E. Cohen, *Direct haptic rendering of sculpted models*, Computer Graphics (1997 Sympos. on Interactive 3D Graphics) (1997), 167–176.

[149] W. Thurston, *Hyperbolic structures on 3-manifolds*, Annals of Math. **124** (1986), 203–246.

[150] G. Vegter, *Computational topology*, in Handbook of Discrete and Computational Geometry, J. Goodman and J. O'Rourke (eds.), CRC Press, Boca Raton (1997), 517–536.

[151] C. Wall and Ranicki, Surgery on Compact Manifolds, 2nd ed., AMS, Providence, RI, 1999.

[152] A. Wallace, *Modifications and cobounding manifolds, II*, J. Math. Mech. **10** (1961), 773–809.

[153] G. Wang, J. Sun and X. Hua, *The sweep-envelope differential equation algorithm for general deformed swept volumes*, CAGD **17** (2000), 399–418.

[154] J. Weeks, *Reconstructing the global topology of the universe from the cosmic microwave background*, Classical Quantum Gravity **15** (1998), 2599–2604.

[155] K. Weiler, *The radial edge data structure: A topological representation for non-manifold geometric boundary modeling*, in Geometric Modeling for CAD Applications, M. Wozny, H. McLaughlin, and H. Encarnacao (eds.), Elsevier Science, North Holland (1988), 3–36.

[156] H. Whitney, *Tangents to an analytic variety*, Annals of Math. **81** (1965), 496–540.

[157] A. Yao, *Decision tree complexity and Betti numbers*, Proc. 25th Sympos. on Theory of Computing, ACM (1994), 615–624.

[158] X. Ye and T. Mackawa, *Differential geometry of intersection curves of two surfaces*, CAGD **16** (1999), 767–788.

[159] E. Zeeman, *The Poincaré conjecture for* $n \geq 5$, in Topology of 3-Manifolds and Related Topics, Prentice Hall, Englewood Cliffs, NJ, 1962, p. 240.

[160] A. Zomorodian, *Topology for Computing*, Cambridge University Press, New York, NY, 2005.

[161] A. Zomorodian and G. Carlsson, *Computing persistent homology*, Discrete Comput. Geom. **33** (2) (2005), 249–274.

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