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Sandwich-type characterization of completely regular spaces

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Abstract. All the higher separation axioms in topology, except for complete regularity, are known to have sandwich-type characterizations. This note provides a characterization of complete regularity in terms of inserting a continuous real-valued function. The known fact that each continuous real valued function on a compact subset of a Tychonoff space has a continuous extension to the whole space is obtained as a corollary.

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1. Introduction

All the higher separation axioms in general topology, except for complete regularity, are known to have sandwich-type (= insertion-type) characterizations. A canonical example is provided by the Katětov-Tong-Hahn insertion theorem for normal spaces (see [6], [10], and [2]). A topological space is *normal* if, given two disjoint closed sets A and B, there exist two disjoint open sets U and V containing A and B respectively. Also recall that, given a topological space X, a function $f : X \to \mathbb{R}$ is lower [upper] semicontinuous if $f^{-1}(t, \infty)$ $[f^{-1}(-\infty, t)]$ is open for each $t \in \mathbb{R}$.

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Theorem (Katětov-Tong-Hahn). *Let X be a topological space. Then the following are equivalent*:

(1) *X is normal.*
(2) *If $g, h : X \to \mathbb{R}$, g is upper semicontinuous, h is lower semicontinuous, and $g \leq h$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $g \leq f \leq h$.*

More examples can be seen in [8]. In this note we give an insertion-type characterization of completely regular spaces. We note that insertion theorems usually have Urysohn-type lemmas and Tietze-type extension theorems as corollaries, and so does the insertion theorem of this note.

2. Sandwich-type characterization of completely regular spaces

We need some notation. Let $C(X, \mathbb{I})$ [$USC(X, \mathbb{I})$] be the set of all continuous [upper semicontinuous] functions from a topological space X to $\mathbb{I} = [0, 1]$. We recall that X is *completely regular* (no lower separation axiom assumed) if, whenever $K \subset X$ is closed and $x \in X \setminus K$, there exists an $f \in C(X, \mathbb{I})$ such that $f(x) = 1$ and $f(K) = \{0\}$. Equivalently, X is completely regular if and only if, given an open set $U \subset X$ and $x \in U$, there is a continuous $f : X \to \mathbb{I}$ such that $1_{\{x\}} \leq f \leq 1_U$. Here and elsewhere 1_A denotes the characteristic function of a subset $A \subset X$. By using some ideas of fuzzy topology (cf. [4]) or point-free topology (cf. [5]), one has a yet more convenient formulation.

Statement *A topological space X is completely regular if and only if, whenever $U \subset X$ is open, there exists an open cover $\mathcal{V}$ of U with the property that for every $V \in \mathcal{V}$ there is an $f_V \in C(X, \mathbb{I})$ such that $1_V \leq f_V \leq 1_U$.*

Proof. The *only if* part: by complete regularity, we have $1_{\{x\}} \leq g_x \leq 1_U$ for each $x \in U$, where $g_x \in C(X, \mathbb{I})$. Let $V_x = g_x^{-1}(\frac{1}{2}, 1]$ and $f_x = \min(1, 2g_x)$. Then $\mathcal{V} = \{V_x\}_{x \in U}$ is an open cover of U and $1_{V_x} \leq f_x \leq 1_U$. The *if* part is evident. □

We recall that two disjoint subsets A and B of a topological space X are *completely separated* if there exists an $f \in C(X, \mathbb{I})$ such that $f = \mathbf{1}$ on A and $f = \mathbf{0}$ on B. Equivalently, if $1_A \leq f \leq 1_{X \setminus B}$.

To state our insertion theorem, we need a "general" property of a function $f : X \to \mathbb{I}$ holding, in particular, for $1_{\{x\}}$. The right choice is to require each $[f \geq t]$ to be compact for all $t > 0$. This can actually be taken as a definition, but we prefer to distinguish a class of maps for which this property becomes a characterization. In what follows, $\mathbf{t}$ stands for the constant map on X taking the value $t \in \mathbb{I}$. All the infs and the sups of families of functions are pointwise. In particular, $(\inf \mathcal{K})(x) = \inf\{k(x) : k \in \mathcal{K}\}$.

Definition Given a topological space X, an $f : X \to \mathbb{I}$ is called *compact-like* if, given a $t \in \mathbb{I} \setminus \{0\}$ and $\mathcal{K} \subset USC(X, \mathbb{I})$ with $\min(f, \inf \mathcal{K}) < \mathbf{t}$, there exists a finite $\mathcal{K}_0 \subset \mathcal{K}$ such that $\min(f, \inf \mathcal{K}_0) < \mathbf{t}$.

Properties *Let X be a topological space. The following hold*:

(1) $f : X \to \mathbb{I}$ *is compact-like iff* $[f \geq t]$ *is compact for all* $t \in \mathbb{I} \setminus \{0\}$.
(2) $A \subset X$ *is compact iff* 1_A *is compact-like.*
(3) *If* X *is compact, then* $USC(X, \mathbb{I})$ *consists of compact-like functions.*
(4) *If* X *is Hausdorff and* $f : X \to \mathbb{I}$ *is compact-like, then* $f \in USC(X, \mathbb{I})$.

Proof. For (1): let $\mathcal{U}$ be an open cover of $[f \geq t]$ with $t > 0$, then $\min(f, \inf\{1_{X \setminus U} : U \in \mathcal{U}\}) < \mathbf{t}$. But then the finite subfamily $\mathcal{U}_0 \subset \mathcal{U}$ for which $\min(f, \inf\{1_{X\setminus U} : U \in \mathcal{U}_0\}) < \mathbf{t}$ yields $[f \geq t] \subset \bigcup \mathcal{U}_0$. Conversely, let $\min(f, \inf \mathcal{K}) < \mathbf{t}$ with $t > 0$. Then $\varnothing = [\min(f, \inf \mathcal{K}) \geq t] = [f \geq t] \cap \bigcap_{k \in \mathcal{K}} [k \geq t]$. By the finite intersection property, there exists a finite $\mathcal{K}_0 \subset \mathcal{K}$ with $\varnothing = [f \geq t] \cap \bigcap_{k \in \mathcal{K}_0} [k \geq t] = [\min(f, \inf \mathcal{K}_0) \geq t]$. This translates into $\min(f, \inf \mathcal{K}_0) < \mathbf{t}$ and proves (1). Finally, (2) follows from (1), while (3) and (4) are obvious. □

The concept of a compact-like function shows that sandwich-type characterizations of higher separation axioms, viz.: *perfect normality* [9], *complete normality* [7], *normality* ([6] and [10]), continue to hold for the case of *complete regularity.* We shall need the following general insertion theorem.

Theorem 1 (Blair [1], Lane [8]). *For X a topological space and two arbitrary functions $g, h : X \to \mathbb{I}$, the following statements are equivalent*:

(1) *There exists a continuous function $f : X \to \mathbb{I}$ such that $g \leq f \leq h$.*
(2) *If $s < t$ in $\mathbb{I}$, then $[g \geq t]$ and $[h \leq s]$ are completely separated.*

The equivalence (1) $\Leftrightarrow$ (2), in the theorem which follows, is well known (see 3.11(c) in [3]). We provide a short proof for completeness.

Theorem 2 *For X a topological space, the following are equivalent*:

(1) *X is completely regular.*
(2) [Urysohn-type lemma] *Every two disjoint subsets of X, one of which is compact and the other is closed, are completely separated.*
(3) [Insertion] *If $g, h : X \to \mathbb{I}$, g is compact-like, h is lower semicontinuous, and $g \leq h$, then there exists a continuous function $f : X \to \mathbb{I}$ such that $g \leq f \leq h$.*

Proof. (1) $\Rightarrow$ (2): Let $A \cap B = \varnothing$, A being compact, B being closed. By complete regularity, there exist an open cover $\mathcal{U}$ of $X \setminus B$ and a family $\{f_U\}_{U \in \mathcal{U}} \subset C(X, \mathbb{I})$ such that $1_U \leq f_U \leq 1_{X \setminus B}$. Since A is compact, $A \subset \bigcup \mathcal{U}_0$ for a finite $\mathcal{U}_0$ of $\mathcal{U}$. Then $1_A \leq g = \sup\{f_U : U \in \mathcal{U}_0\} \leq 1_{X \setminus B}$. The continuous g completely separates A and B.

(2) $\Rightarrow$ (3): Let $g \leq h$ be as in (3). For any $s, t \in \mathbb{I}$ with $s < t$ one has $[g \geq t] \cap [h \leq s] = \varnothing$ where $[g \geq t]$ is compact and $[h \leq s]$ is closed. By Theorem 1, there is a continuous $f \in C(X, \mathbb{I})$ such that $g \leq f \leq h$.

(3) $\Rightarrow$ (1): This is obvious, for if $x \in U$ with U open, then $1_{\{x\}} \leq 1_U$ where $1_{\{x\}}$ is compact-like and 1_U is lower semicontinuous. □

It is a heuristic principle that a sandwich-type theorem provides an extension theorem. This is the case of our sandwich theorem. In order to avoid speaking about compact-closed sets in a completely regular space, we shall assume X to be Tychonoff (completely regular + Hausdorff).

Corollary ([3], 3.11(c)). *Let X be a Tychonoff space, let $A \subset X$ be compact, and let $f : A \to \mathbb{R}$ be continuous. Then there exists a continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof. Given a compact subset $A \subset X$ and a continuous function $f : A \to \mathbb{R}$, the set $f(A)$ is bounded and we can assume that $f(A) \subset \mathbb{I}$. Now, define $g, h : X \to \mathbb{I}$ as follows: $g = f = h$ on A, $g = \mathbf{0}$, and $h = \mathbf{1}$ on $X \setminus A$. Since A is closed, h is lower semicontinuous. Also, if $t > 0$, then $[g \geq t] = [f \geq t]$ is closed in A, hence compact in X. By Theorem 2, there exists a continuous $F : X \to \mathbb{I}$ with $g \leq F \leq h$. Clearly, F extends f to the whole of X. □

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