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# On the topology of generalized quotients

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Abstract. Generalized quotients are defined as equivalence classes of pairs $(x, f)$, where $x$ is an element of a nonempty set $X$ and $f$ is an element of a commutative semigroup $G$ acting on $X$. Topologies on $X$ and $G$ induce a natural topology on $\mathcal{B}(X, G)$, the space of generalized quotients. Separation properties of this topology are investigated.

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## 1. Preliminaries

Let $X$ be a nonempty set and let $S$ be a commutative semigroup acting on $X$ injectively. For $(x, \varphi), (y, \psi) \in X \times S$ we write

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \ \ \psi x = \varphi y.$$

This is an equivalence relation in $X \times S$. Finally, we define

$$\mathcal{B}(X, S) = (X \times S)/\!\sim,$$

the set of generalized quotients. The equivalence class of $(x, \varphi)$ will be denoted by $\frac{x}{\varphi}$.

Elements of $X$ can be identified with elements of $\mathcal{B}(X, S)$ via the embedding $\iota : X \to \mathcal{B}(X, S)$ defined by $\iota(x) = \frac{\varphi x}{\varphi}$, where $\varphi$ is an arbitrary element of $S$. The action of $G$ can be extended to $\mathcal{B}(X, S)$ via $\varphi \frac{x}{\psi} = \frac{\varphi x}{\psi}$. If $\varphi \frac{x}{\psi} = \iota(y)$, for some $y \in X$, we will write $\varphi \frac{x}{\psi} \in X$ and $\varphi \frac{x}{\psi} = y$. For instance, we have $\varphi \frac{x}{\varphi} = x$.

Other properties of generalized quotients and several examples can be found in [2] and [4].

If $X$ is a topological space and $G$ is a commutative semigroup of continuous maps acting on $X$, equipped with its own topology, then we can define the product topology on $X \times G$ and then the quotient topology on $\mathcal{B}(X, S) = (X \times G)/_\sim$.

It is easy to show that the embedding $\iota : X \to \mathcal{B}(X,S)$ is continuous. Moreover, the map $\frac{x}{\psi} \mapsto \frac{\varphi x}{\psi}$ is continuous for every $\varphi \in G$. These and other topological properties of generalized quotients can be found in [1].

In this note we will always assume that the topology on $G$ is discrete. In most examples, it is a natural assumption.

Let $Y$ be a topological space and let $\sim$ be an equivalence relation. If $y \in Y$, then by $[y]$ we denote the equivalence class of $y$, that is, $[y] = \{w \in Y : w \sim y\}$. The map $q : Y \to Y/_\sim$, defined by $q(y) = [y]$, is called the quotient map. A subset $U \subset Y$ is called saturated if $y \in U$ implies $[y] \subset U$. In other words, $U$ is saturated if $U = q^{-1}(q(U))$. Let $Z = Y/_\sim$. A set $V \subset Z$ is open (in the quotient topology) if and only if $V = q(U)$ for some open saturated $U \subset Y$.

Whenever convenient, we use convergence arguments. The sequential convergence defined by the topology of $\mathcal{B}(X,G)$ is not easily characterized. The following theorem is often useful.

**Theorem 1.1.** *Let $\frac{x_n}{\varphi_n} \in \mathcal{B}(X,G)$, $n \in \mathbb{N}$. If there exist a $\psi \in G$ and a $y \in X$ such that $\frac{x_n}{\varphi_n} = \frac{y_n}{\psi}$, for all $n \in \mathbb{N}$, and $y_n \to y$ in the topology of $X$, then $\frac{x_n}{\varphi_n} \to \frac{y}{\psi}$ in the topology of $\mathcal{B}(X,G)$.*

*Proof.* If $U$ is an open neighborhood of $\frac{y}{\psi}$ in $\mathcal{B}(X,G)$, then $(y,\psi) \in q^{-1}(U)$. Since $q^{-1}(U)$ is open in $X \times G$, there exists an open $V \subset X$ such that $(y,\psi) \in V \times \{\psi\} \subset q^{-1}(U)$. But then $y_n \in V$ for almost all $n \in \mathbb{N}$, because $y_n \to y$ in the topology of $X$. Hence, $(y_n,\psi) \in q^{-1}(U)$ for almost all $n \in \mathbb{N}$ or, equivalently $\frac{y_n}{\psi} = \frac{x_n}{\varphi_n} \in U$ for almost all $n \in \mathbb{N}$. □

In this note we investigate some separation properties of the topology of $\mathcal{B}(X,G)$.

## 2. General separation properties

We are interested in the general question whether a separation property of $X$ is inherited by $\mathcal{B}(X,G)$. First we consider $T_1$.

**Theorem 2.1.** *If $X$ is $T_1$ and the topology of $G$ is discrete, then $\mathcal{B}(X,G)$ is $T_1$.*

*Proof.* If $\frac{x}{\varphi} \in \mathcal{B}(X,G)$, then $(X \times G) \setminus q^{-1}\left(\frac{x}{\varphi}\right)$ is an open saturated subset of $X \times G$. □

Now we give an example of a Banach space $X$ and a semigroup $G$ of continuous injections on $X$ for which $\mathcal{B}(X,G)$ is not Hausdorff.

If $f, g : \mathbb{R} \to \mathbb{R}$ and the set $\{t \in \mathbb{R} : f(t) \neq g(t)\}$ is meager in the usual topology of $\mathbb{R}$, then we will write $f \simeq g$. Let $B(\mathbb{R})$ be the space of all bounded real-valued functions on $\mathbb{R}$ and let $X = B(\mathbb{R})/_\simeq$. With respect to the norm

$$\|[f]\| = \inf\{\|g\|_\infty : g \simeq f\}$$

$X$ is a Banach space. Let

$$G = \{[f] \in X : \{t \in \mathbb{R} : f(t) = 0\} \text{ is a meager set in } \mathbb{R}\}.$$

Then $G$ is a semigroup of injections acting on $X$ by pointwise multiplication. Note that $\mathcal{B}(X,G)$ can be identified with $\mathbb{R}^{\mathbb{R}}/_{\simeq}$. To show that the topology of $\mathcal{B}(X,G)$ is not Hausdorff we need two simple lemmas. In what follows, we will not distinguish between functions and equivalence classes of functions. The indicator function of a set $A$ will be denoted by $I_A$.

**Lemma 2.2.** *If $(A_n)$ is a sequence of subsets of $\mathbb{R}$ such that $A_n \subset A_{n+1}$, for each $n \in \mathbb{N}$, and $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n$ is meager, then for each $f \in \mathcal{C}(\mathbb{R})$ the sequence $f_n = fI_{A_n}$ is convergent to $f$ in $\mathcal{B}(X,G)$.*

*Proof.* Define a function $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(t) = \begin{cases} 1 & \text{if } t \in A_1, \\ \frac{1}{n} & \text{if } t \in A_n \setminus A_{n-1}. \end{cases}$$

It is easy to see that $f_n g \to fg$ in $X$. Consequently $f_n \to f$ in $\mathcal{B}(X,G)$. □

**Corollary 2.3.** *If a set $U \subset \mathcal{B}(X,G)$ is sequentially open and $\frac{f}{g} \in U$, then for each $r \in \mathbb{R}$ there exists a open neighborhood $V \subset \mathbb{R}$ of $r$ such that $\frac{fI_{\mathbb{R}\setminus V}}{g} \in U$.*

**Lemma 2.4.** *If $(A_n)$ is a sequence of subsets of $\mathbb{R}$ such that $A_{n+1} \subset A_n$, for each $n \in \mathbb{N}$, and the set $\bigcap_{n=1}^{\infty} A_n$ is meager, then for each $f \in X$ the sequence $(f_n)$, where $f_n = fI_{A_n}$, is convergent to $0$ in $\mathcal{B}(X,G)$.*

*Proof.* Use

$$g(t) = \begin{cases} 1 & \text{if } t \notin A_1, \\ \frac{1}{n} & \text{if } t \in A_n \setminus A_{n+1}. \end{cases}$$

□

**Theorem 2.5.** *If $U$ is a nonempty sequentially open subset of $\mathcal{B}(X,G)$, then $U$ is sequentially dense in $\mathcal{B}(X,G)$.*

*Proof.* It is enough to prove that there exists a sequence $F_n \in U$ such that $F_n \to 0$ in $\mathcal{B}(X,G)$. Consider an arbitrary element $f/g \in U$ and assume that $(r_n)$ is a sequence of all rational numbers. Then, by Corollary 2.3, there exits a neighborhood $V_1$ of $r_1$ such, that

$$F_1 = \frac{fI_{\mathbb{R}\setminus V_1}}{g} \in U.$$

Next we find a neighborhood $V_2$ of $r_2$ such, that

$$F_2 = \frac{fI_{\mathbb{R}\setminus (V_1\cup V_2)}}{g} \in U.$$

By induction, we construct a sequence $V_n \subset \mathbb{R}$ such that $V_n$ is a neighborhood of $r_n$ and

$$F_n = \frac{fI_{\mathbb{R}\setminus (V_1\cup \ldots \cup V_n)}}{g} \in U.$$

The set $\bigcup_{n=1}^{\infty} V_n$ is open and dense in $\mathbb{R}$. Hence, the complement of $\bigcup_{n=1}^{\infty} V_n$ is a meager set. By Lemma 2.4, $fI_{\mathbb{R}\setminus (V_1\cup \ldots \cup V_n)} \to 0$ in $\mathcal{B}(X,G)$, and consequently $F_n \to 0$ in $\mathcal{B}(X,G)$. □

Since $X$ in this example is a Banach space, no separation property of $X$ above $T_1$ will be inherited by the topology of $\mathcal{B}(X,G)$ without additional assumptions. In the remaining part of this note we give examples of theorems that discribe special situations in which the topology of $\mathcal{B}(X,G)$ is Hausdorff.

## 3. Hausdorff property in special cases

First we introduce some notation and make some useful observations. If $U \subset X \times G$, then $U = \bigcup_{\varphi \in G} U_\varphi \times \{\varphi\}$, where $U_\varphi \subset X$. For every $\psi \in G$ let $\Pi_\psi : X \times G \to X$ be the projection defined by

$$\Pi_\psi \left( \bigcup_{\varphi \in G} U_\varphi \times \{\varphi\} \right) = U_\psi.$$

If $A \subset X \times G$, then the smallest saturated set containing $A$ will be denoted by $\Sigma A$. We have the following straightforward characterization on $\Sigma A$.

**Proposition 3.1.** *If $A \subset X \times G$, then*

$$\Sigma A = \bigcup_{\varphi, \psi \in G} \varphi^{-1} \psi \Pi_\varphi A \times \{\psi\}.$$

*In other words, for every $\psi \in G$, we have*

$$\Pi_\psi \Sigma A = \bigcup_{\varphi \in G} \varphi^{-1} \psi \Pi_\varphi A.$$

**Corollary 3.2.** *A set $A \subset X \times G$ is saturated if and only if*

$$\varphi^{-1} \psi \Pi_\varphi A \subset \Pi_\psi A$$

*for every $\varphi, \psi \in G$.*

**Theorem 3.3.** *If $X$ is Hausdorff and every $\varphi \in G$ is an open map, then $\mathcal{B}(X,G)$ is Hausdorff.*

*Proof.* Let $\frac{x}{\varphi}$ and $\frac{y}{\psi}$ be two distinct elements of $\mathcal{B}(X,G)$. It suffices to find open and saturated subsets of $X \times G$ that separate $(x,\varphi)$ and $(y,\psi)$. Since $\psi x \neq \varphi y$ and $X$ is Hausdorff, there exist open and disjoint sets $U, V \subset X$ such that $\psi x \in U$ and $\varphi y \in V$. Define

$$A = \psi^{-1} U \times \{\varphi\} \quad \text{and} \quad B = \varphi^{-1} V \times \{\psi\}.$$

Consider the sets $\Sigma A$ and $\Sigma B$. By Proposition 3.1, $\Sigma A$ and $\Sigma B$ are open sets. If $(z,\gamma) \in \Pi_\gamma \Sigma A$, then $z \in \varphi^{-1}\gamma\psi^{-1}U$, again by Proposition 3.1. This means that $\varphi z = \gamma \psi^{-1} u$ for some $u \in U$. Hence, $(z,\gamma) \sim (\psi^{-1}u, \varphi)$. Similarly, if $(z,\gamma) \in \Pi_\gamma \Sigma B$, there exists a $v \in V$ such that $(z,\gamma) \sim (\varphi^{-1}v, \psi)$. Therefore, $(\psi^{-1}u,\varphi) \sim (\varphi^{-1}v,\psi)$, which implies $u = v$, contradicting $U \cap V = \varnothing$. □

For topological spaces $X$ and $Y$, by $\mathcal{C}(X,Y)$ we denote the space of continuous maps from $X$ to $Y$. For a continuous $\varphi : X \to X$, by $\varphi^* : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$ we denote the adjoint map, that is, $(\varphi^* f)x = f(\varphi x)$ where $f \in \mathcal{C}(X,Y)$.

**Theorem 3.4.** *Let $X$ be a topological space, $G$ a commutative semigroup of continuous injections from $X$ into $X$, equipped with the discrete topology, such that $\varphi(X)$ is dense in $X$ for all $\varphi \in G$. Let $Y$ be a Hausdorff space and let $\mathcal{F} \subset \mathcal{C}(X,Y)$ be such that $\mathcal{F}$ separates points in $X$ and for every $\varphi \in G$ we have $\mathcal{F} \subset \varphi^*(\mathcal{F})$. Then the topology of $\mathcal{B}(X,G)$ is Hausdorff.*

*Proof.* First note that, since $\varphi(X)$ is dense in $X$, $\varphi^*$ is a injection. For $f \in \mathcal{F}$ and $\varphi \in G$ define $f_\varphi$ to be the unique function in $\mathcal{F}$ such that $\varphi^* f_\varphi = f$. Then, for any $\varphi, \psi \in G$, we have $\psi^* f_\psi = f = (\varphi\psi)^* f_{\varphi\psi}$ and hence $\psi^* f_\psi = \psi^* \varphi^* f_{\varphi\psi}$. Since $\psi^*$ is injective, we have $f_\psi = \varphi^* f_{\varphi\psi}$. Thus, $f_\psi(x) = \varphi^* f_{\varphi\psi}(x) = f_{\varphi\psi}(\varphi x)$ for any $x \in X$.

Consider two distinct elements $F_1$ and $F_2$ of $\mathcal{B}$. Without loss of generality, we can assume that $F_1 = \frac{x_1}{\varphi}$ and $F_2 = \frac{x_2}{\varphi}$, for some $x_1 \neq x_2$. There exists an $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$. Let $\Omega_1, \Omega_2 \subset Y$ be open disjoint neighborhoods of $f(x_1)$ and $f(x_2)$, respectively. For every $\psi \in G$ let

$$U_\psi = \varphi^{-1}\left(f_\psi^{-1}(\Omega_1)\right) \quad \text{and} \quad V_\psi = \varphi^{-1}\left(f_\psi^{-1}(\Omega_2)\right).$$

We will show that

$$U = \bigcup_{\psi \in G} U_\psi \times \{\psi\} \quad \text{and} \quad V = \bigcup_{\psi \in G} V_\psi \times \{\psi\}$$

are disjoint saturated open sets that separate $(x_1, \varphi)$ and $(x_2, \varphi)$. It suffices to prove that the sets are saturated. Since the sets are defined the same way, we will only prove it for $U$. Suppose $x \in U_\psi$ and $(x,\psi) \sim (y,\gamma)$. Then $\gamma x = \psi y$ and

$$f_\gamma(\varphi y) = f_{\psi\gamma}(\varphi\psi y) = f_{\psi\gamma}(\varphi\gamma x) = f_\psi(\varphi x) \in \Omega_1.$$

Thus $y \in U_\gamma$. □

**Example 3.5.** Let $X = \{x \in \mathcal{C}(\mathbb{R}) : x(0) = 0\}$, with the topology of uniform convergence on compact sets, and let $G = \{\Lambda^n : n \in \mathbb{N}_0\}$, where $\Lambda x(t) = \int_0^t x(s)\,ds$ and $\mathbb{N}_0$ denotes the set of all nonnegative integers. To show that the topology of $\mathcal{B}(X,G)$ is Hausdorff we use Theorem 3.4 with $Y = \mathbb{R}$ and $\mathcal{F} = \{f \in \mathcal{D}(\mathbb{R}) : f \neq 0\}$, where $\mathcal{D}(\mathbb{R})$ is the space of smooth functions with compact support. If $f \in \mathcal{F}$ and $x \in X$, then we define $f(x) = \int_{-\infty}^{\infty} f(t)x(t)\,dt$.

Clearly, $\Lambda^n$ is injective and $\Lambda^n(X)$ is dense in $X$ for every $n \in \mathbb{N}$. Moreover, $\mathcal{F}$ separates points in $X$. If $f \in \mathcal{F}$ and $n \in \mathbb{N}$, then there exists a $g \in \mathcal{F}$ such that $f(x) = g(\Lambda^n x)$ for every $x \in X$, namely $g = (-1)^n f^{(n)}$. Thus all the assumptions of the theorem are met.

The assumption that $x(0) = 0$, in the definition of $X$, may seem artificial. It is made for convenience and it does not affect the final result. Note that for any $x \in \mathcal{C}(\mathbb{R})$ we have $\frac{x}{\Lambda^n} = \frac{\Lambda x}{\Lambda^{n+1}}$ and $\Lambda x \in X$. One can prove that, in general, $\mathcal{B}(X,G) = \mathcal{B}(gX,G)$ for any $g \in G$ (see [1]).

In the next theorem we assume that $G$ is generated by a single function, that is, $G = \{\varphi^n : n \in \mathbb{N}_0\}$.

**Proposition 3.6.** *Let $G = \{\varphi^n : n \in \mathbb{N}_0\}$ and $A \subset X \times G$. $A$ is saturated if and only if, for all $i, j \in \mathbb{N}_0$,*

$$z \in \Pi_i A \quad \text{if and only if} \quad \varphi^j z \in \Pi_{i+j} A, \tag{3.1}$$

*where $\Pi_k = \Pi_{\varphi_k}$.*

*Proof.* Assume that (3.1) holds for some $A \subset X \times G$, $x \in \Pi_n A$, and $\varphi^n y = \varphi^m x$ for some $y \in X$ and $m \in \mathbb{N}_0$. If $n \leq m$, then $y = \varphi^{m-n}x$. Hence, if we take $j = m - n$, $i = n$, and $z = x$, we obtain $y = \varphi^{m-n}x \in \Pi_m A$, by (3.1). If $n > m$, then $x = \varphi^{n-m}y$, and thus, $\varphi^{n-m}x \in \Pi_n A$. Hence $y \in \Pi_m$, by (3.1). Therefore $A$ is saturated.

Assume now that $A \subset X \times G$ is saturated. Then, by Corollary 3.2, we have $\varphi^j \Pi_i A \subset \Pi_{i+j} A$. Hence, if $z \in \Pi_i$, then $\varphi^j z \in \Pi_{i+j} A$. Now, conversely, if $\varphi^j z \in \Pi_{i+j} A$, then $z \in \Pi_i$ since $(z, \varphi^i) \sim (\varphi^j z, \varphi^{i+j})$ and $A$ is saturated. □

**Corollary 3.7.** *If $G = \{\varphi^n : n \in \mathbb{N}_0\}$, then $A \subset X \times G$ is saturated if and only if*

$$\Pi_{j-1} A = \varphi^{-1} \Pi_j A$$

*for every $j \in \mathbb{N}$.*

**Theorem 3.8.** *If $X$ is a normal space, $\varphi : X \to X$ is a closed and continuous injection, and $G = \{\varphi^n : n \in \mathbb{N}_0\}$, then $\mathcal{B}(X, G)$ is a Hausdorff space.*

*Proof.* Consider two distinct points in $\mathcal{B}(X, G)$. Without loss of generality, we can assume that they are represented by $\frac{x}{\varphi^n}$ and $\frac{y}{\varphi^n}$ for some $x, y \in X$ and $n \in \mathbb{N}$. Then $x \neq y$ and there exist open sets $U_n, V_n \subset X$ such that $x \in U_n$, $y \in V_n$, and $\overline{U_n} \cap \overline{V_n} = \varnothing$. Since $\varphi$ is a closed injective map, $\varphi(\overline{U_n})$ and $\varphi(\overline{V_n})$ are disjoint closed sets. Whereas $X$ is normal, there exist open sets $U_{n+1}, V_{n+1} \subset X$ such that

$$\varphi(\overline{U_n}) \subset U_{n+1},\ \varphi(\overline{V_n}) \subset V_{n+1}, \quad \text{and} \quad \overline{U_{n+1}} \cap \overline{V_{n+1}} = \varnothing.$$

Similarly, by induction, we can construct open sets $U_{n+k}, V_{n+k} \subset X$ such that

$$\varphi(\overline{U_{n+k}}) \subset U_{n+k+1},\ \varphi(\overline{V_{n+k}}) \subset V_{n+k+1}, \quad \text{and} \quad \overline{U_{n+k+1}} \cap \overline{V_{n+k+1}} = \varnothing,$$

for all $k = 1, 2, \ldots$. Now, for $m = n, n+1, n+2, \ldots$, we define open subsets of $X \times G$:

$$U'_m = \bigcup_{j=0}^{m} \left(\varphi^{j-m} U_m\right) \times \{\varphi^j\} \quad \text{and} \quad V'_m = \bigcup_{j=0}^{m} \left(\varphi^{j-m} V_m\right) \times \{\varphi^j\}.$$

Note that $U'_n \subset U'_{n+1} \subset \ldots$, $V'_n \subset V'_{n+1} \subset \ldots$, and $U'_m \cap V'_m = \varnothing$ for all $m \geq n$. Finally, let

$$U = \bigcup_{m=n}^{\infty} U'_m \quad \text{and} \quad V = \bigcup_{m=n}^{\infty} V'_m.$$

Clearly, $U$ and $V$ are disjoint open subsets of $X \times G$ such that $(x, \varphi^n) \in U$ and $(y, \varphi^n) \in V$. Since $U$ and $V$ are defined the same way, it suffices to show that

$U$ is saturated. Note that

$$\Pi_j U = \bigcup_{m=n}^{\infty} \varphi^{j-m} U_m$$

if $j = 0, \ldots, n$, and

$$\Pi_j U = \bigcup_{m=j}^{\infty} \varphi^{j-m} U_m$$

if $j > n$. Since $\Pi_{j-1} U = \varphi^{-1} \Pi_j U$ for every $j \in \mathbb{N}$, it follows that $U$ is saturated by Corollary 3.7. $\square$

**Corollary 3.9.** *If $X$ is a compact Hausdorff space and $G$ is generated by a continuous injection, then $\mathcal{B}(X, G)$ is a Hausdorff space.*

Now we consider the case when $X$ has an algebraic structure, namely $X$ is a topological semigroup. A nonempty set $X$ with an associative operation $(x, y) \to xy$ from $X \times X$ into $X$ is called a semigroup. If the topology of $X$ is Hausdorff and the semigroup operation is continuous (with respect to the product topology on $X \times X$), then $X$ is called a topological semigroup. Our main result follows from a theorem of Lawson and Madison (see Theorem 1.56 in [3]).

**Theorem 3.10** (Lawson and Madison)**.** *Let $S$ be a locally compact $\sigma$-compact semigroup and let $R$ be a closed congruence on $S$. Then $S/R$ is a topological semigroup.*

An equivalence $\sim$ in a semigroup $A$ is called a congruence if

$$a \sim b \quad \text{implies} \quad ca \sim cb \quad \text{for all } c \in A.$$

If $(X, \cdot)$ is a semigroup and $G$ is a commutative semigroup of injective homomorphisms on $X$, then $X \times G$ is a semigroup with respect to the binary operation $*$ defined by

$$(x, \varphi) * (y, \psi) = ((\psi x) \cdot (\varphi y), \varphi\psi),$$

where $x, y \in X$ and $\varphi, \psi \in G$.

**Lemma 3.11.** *The equivalence $\sim$ in $X \times G$ defined by*

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \quad \psi x = \varphi y$$

*is a congruence with respect to $*$.*

*Proof.* Let $(x, \varphi), (y, \psi), (z, \gamma) \in X \times G$ and $(x, \varphi) \sim (y, \psi)$. Then

$$(x, \varphi) * (z, \gamma) = ((\gamma x) \cdot (\varphi z), \varphi\gamma) \quad \text{and} \quad (y, \psi) * (z, \gamma) = ((\gamma y) \cdot (\psi z), \psi\gamma).$$

Since $\psi x = \varphi y$ and $G$ is commutative, we have

$$\psi\gamma((\gamma x) \cdot (\varphi z)) = (\psi\gamma\gamma x) \cdot (\psi\gamma\varphi z) = (\varphi\gamma\gamma y) \cdot (\varphi\gamma\psi z) = \varphi\gamma((\gamma y) \cdot (\psi z)),$$

which means $(x, \varphi) * (z, \gamma) \sim (y, \psi) * (z, \gamma)$. $\square$

A relation $\sim$ in a topological space $Y$ is called closed if $\{(a,b) \in Y \times Y : a \sim b\}$ is a closed subset of $Y \times Y$ with respect to the product topology.

**Lemma 3.12.** *If $X$ is Hausdorff, then $\sim$ is a closed relation in $X \times G$.*

*Proof.* We have to show that the set

$$\mathcal{R} = \{((x,\varphi),(y,\psi)) : (x,\varphi),(y,\psi) \in X \times G \text{ and } (x,\varphi) \sim (y,\psi)\}$$

is closed in $(X \times G) \times (X \times G)$. Consider $((x,\varphi),(y,\psi)) \notin \mathcal{R}$. Then $(x,\varphi) \not\sim (y,\psi)$ and hence $\psi x \neq \varphi y$. Since $X$ is Hausdorff, there are open and disjoint $U, V \subset X$ such that $\psi x \in U$ and $\varphi y \in V$. Then

$$(x,\varphi) \times (y,\psi) \in (\psi^{-1}(U) \times \{\varphi\}) \times (\varphi^{-1}(V) \times \{\psi\}).$$

Clearly, $(\psi^{-1}(U) \times \{\varphi\}) \times (\varphi^{-1}(V) \times \{\psi\})$ is open and disjoint with $\mathcal{R}$. □

In view of the above lemmas, the theorem of Lawson and Madison gives us the following result.

**Theorem 3.13.** *If $X$ is a Hausdorff semigroup and $(X \times G)$ is locally compact $\sigma$-compact, then $\mathcal{B}(X,G)$ is Hausdorff.*

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