Applied General Topology
© Universidad Politécnica de Valencia
Volume 9, No. 2, 2008
pp. 177-184

On pseudo-k-spaces

Annamaria Miranda

Abstract. In this note a new class of topological spaces generalizing k-spaces, the pseudo-k-spaces, is introduced and investigated. Particular attention is given to the study of products of such spaces, in analogy to what is already known about k-spaces and quasi-k-spaces.

2000 AMS Classification: 54D50, 54D99, 54B10, 54B15

Keywords: Quotient map, product space, locally compact space, (locally) pseudocompact space, pseudo-k-space.

1. Introduction

The first example of two k-spaces whose cartesian product is not a k-space was given by Dowker (see [2]). So a natural question is when a k-space satisfies that its product with every k-spaces is also a k-space. In 1948 J.H.C. Whitehead proved that if X is a locally compact Hausdorff space then the cartesian product $i_X \times g$, where i_X stands for the identity map on X, is a quotient map for every quotient map g. Using this result D.E. Cohen proved that if X is locally compact Hausdorff then $X \times Y$ is a k-space for every k-space Y (see Theorem 3.2 in [1]). Later the question was solved by Michael who showed that a k-space has this property iff it is a locally compact space (see [5]).

A similar question, related to quasi-k-spaces, was answered by Sanchis (see [8]). Quasi-k-spaces were investigated by Nagata (see [7]) who showed that "a space X is a quasi-k-space (resp. a k-space) if and only if X is a quotient space of a regular (resp. paracompact) M-space (see [6]).

The study of quasi-k-spaces suggests to define a larger class of spaces simply replacing countable compactness with pseudocompactness in the definition.

This note begins with the study of general properties about pseudo-k-spaces which leads on results about products of pseudo-k-spaces, in analogy with those known about k-spaces and more generally about quasi-k-spaces.

For terminology and notations not explicitly given we refer to [3].

2. Pseudo-k-spaces

We consider pseudocompact spaces which are not necessarily Tychonoff. Recall that

Definition 2.1. *A topological space X is called pseudocompact if every continuous real-valued function defined on X is bounded.*

Definition 2.2. *A topological space X is called locally compact (resp. locally countably compact) if each point of X has a compact (resp. countably compact) neighborhood.*

In analogy with the definitions of locally compact (resp. locally countably compact) space we have the following

Definition 2.3. *A topological space X is called locally pseudocompact if each point of X has a pseudocompact neighborhood.*

Clearly a locally compact space is locally pseudocompact and we have

Proposition 2.4. *The cartesian product of a locally pseudocompact space X and a locally compact space Y is locally pseudocompact.*

Proof. It suffices to observe that Corollary 3.10.27 in [3] holds even if the pseudocompact factor is not necessarily Tychonoff. □

Proposition 2.5. *If all spaces X_s are pseudocompact then the sum $\oplus_{s\in S}X_s$, where $X_s \neq \varnothing$ for $s \in S$, is locally pseudocompact.*

Now we are going to define a new class of spaces which is larger than the class of k-spaces.

Definition 2.6. *A topological space X is called a pseudo-k-space if X is a Hausdorff space and X is the image of a locally pseudocompact Hausdorff space under a quotient mapping.*

In other words, pseudo-k-spaces are Hausdorff spaces that can be represented as quotient spaces of locally pseudocompact Hausdorff spaces. Clearly every locally pseudocompact Hausdorff space is a pseudo-k-space.

We can compare this kind of spaces with the one of quasi-k-spaces. To this aim recall that

Definition 2.7. *A Hausdorff space X is a quasi-k-space if, and only if, a subset $A \subset X$ is closed in X whenever the intersection of A with any countably compact subset Z of X is closed in Z.*

Condition (2) in Theorem 2.11 yields

Proposition 2.8. *Every quasi-k-space is a pseudo-k-space.*

The following example will show that the class of quasi-k-spaces is strictly contained in the class of pseudo-k-spaces.

Definition 2.9. *A Hausdorff space X is called H-closed if X is a closed subspace of every Hausdorff space in which it is contained.*

For a Hausdorff space X, this definition is equivalent to say that every open cover $\{U_s\}_{s\in S}$ of X contains a finite subfamily $\{U_{s_1}, U_{s_2}, ..., U_{s_k}\}$ such that $\overline{U}_{s_1} \cup \overline{U}_{s_1} \cup ... \cup \overline{U}_{s_1} = X$.

Example 2.10. A H-closed space which is not a quasi-k-space.

Let $\Im$ be the family of all free ultrafilters on $\mathbb{N}$, let $k\mathbb{N} = \mathbb{N} \cup \Im$ be the Katětov extension of $\mathbb{N}$. We have that

(1) $k\mathbb{N}$ is a H-closed space;
(2) $k\mathbb{N}$ is not a quasi-k-space.

It is enough to show that all countably compact subsets of $k\mathbb{N}$ have finite cardinality. Let $Y \subset X = k\mathbb{N}$ be countably compact. $\Im$ is closed and discrete in X so $Y \cap \Im$ is closed and discrete in Y, therefore $Y \cap \Im = \{p_1, \dots, p_n\}$. Hence $Y = S \cup \{p_1, \dots, p_n\}$, where $S \subset \mathbb{N}$.
Assume that S is infinite. Since $p_1, \dots, p_n$ are distinct ultrafilters, there exists $S_1 \subset S$ such that $|S_1| = \omega$, $S_1 \in p_1$ and $S_1 \notin p_i$ for every $i \neq 1$. In fact let $H_i \in p_1$ such that $H_i \notin p_i$ for every $i \neq 1$, then $S_1 = \bigcap_{i=1}^{n} H_i \in p_1$ and $S_1 \notin p_i$ for every $i \neq 1$, otherwise $S_1 \in p_i$ and $S_1 \subset H_i$ implies $H_i \in p_i$. Moreover S_1 is infinite. Indeed, if p is an ultrafilter, $A = \{x_1, \dots, x_n\}$ and $A \in p$, then $\{x_i\} \notin p$ implies that $\mathbb{N} \backslash \{x_i\} \in p$, for every i, so $\bigcap_{i=1}^{n} \mathbb{N} \backslash \{x_i\} = \mathbb{N} \backslash A \in p$, a contradiction.
Now, let $G \subset S_1$ such that $|G| = \omega$ and $|S_1 \backslash G| = \omega$. Then $G \in p_1$ or $\mathbb{N} \backslash G \in p_1$. Since $S_1 \in p_1$ it follows that $G \in p_1$ or $S_1 \backslash G \in p_1$. Let us suppose that $S_1 \backslash G \in p_1$. Then $G \notin p_1$. Therefore $G \notin p_i$ for every i.
Since $G \notin p_i \quad \forall\ i \in \{1, \dots, n\}$, it follows that for every i there exists $A_i \in p_i$ such that $G \cap A_i = \varnothing$, so $V_i = A_i \cup \{p_i\}$ is an open neighborhood of p_i such that $V_i \cap G = \varnothing$, therefore $p_i \notin \overline{G}$ for every i, hence G is closed in Y and, since $G \subset \mathbb{N}$, G is also discrete. So G is an infinite closed discrete subspace of the countably compact space Y, a contradiction. Hence S is finite.

In conclusion, since any H-closed space is a pseudocompact space, $k\mathbb{N}$ is a pseudo-k-space which is not a quasi-k-space.

Now we give two useful characterizations of pseudo-k-spaces.

Theorem 2.11. *Let X be a Hausdorff space. The following conditions are equivalent:*

1. *X is a pseudo-k-space.*
2. *For each $A \subset X$, the set A is closed provided that the intersection of A with any pseudocompact subspace Z of X is closed in Z.*
3. *X is a quotient space of a topological sum of pseudocompact spaces.*

Proof. (1)⇒(2) Let X be a pseudo-k-space and let $f : Y \to X$ be a quotient mapping of a locally pseudocompact Hausdorff space Y onto X. Suppose that the intersection of a set A with any pseudocompact subspace P of X is closed in P. Take a point $y \in \overline{f^{-1}(A)}$ and a neighborhood $U \subset Y$ of the point y such that U is pseudocompact. Since the space $f(U)$ is pseudocompact (see Theorem 3.10.24 [3] which holds even if the range space Y is not Tychonoff), the set $A \cap f(U)$ is closed in $f(U)$.

Now, if $y \notin f^{-1}(A)$ then $f(y) \notin A \cap f(U)$ so there exists an open set T in X containing $f(y)$ such that $T \cap (A \cap f(U)) = \varnothing$.
It follows that $f^{-1}(T) \cap f^{-1}(A) \cap U = \varnothing$ where the set $f^{-1}(T) \cap U$ represents a neighborhood of y disjoint from $f^{-1}(A)$. This is a contradiction. Then $y \in f^{-1}(A)$.

(2)⇒(3) Now consider a Hausdorff space X and denote by $\mathcal{P}(X)$ the family of non-empty pseudocompact subspaces of X. Let $\tilde{X} = \oplus\{P : P \in \mathcal{P}(X)\}$. The surjective mapping $f : \nabla_{P \in \mathcal{P}(X)}, i_P : \tilde{X} \to X$, where i_P is the embedding of the subspace P in the space X, is continuous (see Proposition 2.1.11 [3]).
Suppose now that A is closed in $\tilde{X}$, this means $A \cap P$ closed in P, for every pseudocompact subset P of X. Then, by (2), A is closed in X. It follows that f is a quotient map.

(3)⇒(1) If X is a quotient space of a topological sum of pseudocompact spaces then X is a pseudo-k-space, by Proposition 2.5. □

Corollary 2.12. *A Hausdorff space X is a pseudo-k-space if, and only if, a subset $A \subset X$ is open in X whenever the intersection of A with any pseudocompact subset P of X is open in P.*

Regarding the continuity of a mapping whose domain is a pseudo-k-space we have the following

Theorem 2.13. *A mapping f of a pseudo-k-space X to a topological space Y is continuous if and only if for every pseudocompact subspace $P \subset X$ the restriction $f|_P : P \to Y$ is continuous.*

From the definition of a pseudo-k-space we obtain

Theorem 2.14. *If there exists a quotient mapping $f : X \to Y$ of a pseudo-k-space X onto a Hausdorff space Y, then Y is a pseudo-k-space.*

Theorem 2.11 yields

Theorem 2.15. *The sum $\oplus_{s\in S} X_s$ is a pseudo-k-space if and only if all spaces are pseudo-k-spaces.*

3. On products of pseudo-k-spaces

The cartesian product of two pseudo-k-spaces need not be a pseudo-k-space. So, when a pseudo-k-space satisfies that its product with every pseudo-k-space is also a pseudo-k-space?
Proposition 2.4 states that the cartesian product of a locally compact space and a locally pseudocompact Hausdorff space is a locally pseudocompact space. This result, together with Definition 2.6, yields

Theorem 3.1. *The cartesian product $X \times Y$ of a locally compact Hausdorff space X and a pseudo-k-space Y is a pseudo-k-space.*

Proof. Let $g : Z \to Y$ be a quotient mapping of a locally pseudocompact Hausdorff space Z onto a pseudo-k-space Y. The cartesian product $f : id_X \times g : X \times Z \to X \times Y$ is a quotient mapping, by virtue of the Whitehead Theorem (see Lemma 4 in [9], or Theorem 3.3.17 in [3]). Now, since, by Proposition 2.4, $X \times Z$ is a locally pseudocompact Hausdorff space, it follows that $X \times Y$ is a pseudo-k-space. □

The previous Theorem gives a sufficient condition to obtain that the cartesian product of two pseudo-k-spaces is a pseudo-k-space. This condition, for regular spaces, is also necessary, as we will see in Theorem 3.4.

Now, starting from a regular space X which is not locally compact, we define, following a construction introduced by Michael in [5], a normal pseudo-k-space $Y(X)$ such that the product $X \times Y(X)$ is not a pseudo-k-space. This enable us not only to give examples of two pseudo-k-spaces whose product is not a pseudo-k-space, but also to show Theorem 3.4.

Suppose that X is a regular space which is not locally compact at some $x_0 \in X$. Let $\{U_\alpha\}_{\alpha\in A}$ be a local base of non-compact closed sets at x_0. For every $\alpha \in A$ let $\lambda(\alpha)$ be a limit ordinal and $\{F_\lambda\}_{\lambda<\lambda(\alpha)}$ be a well-ordered family of non-empty closed subsets of U_α whose intersection is empty.
Each $\lambda(\alpha)+1$, equipped with the order topology, is a compact Hausdorff space. Therefore $\lambda(\alpha) + 1$ is a normal pseudo-k-space.
Then, by Theorem 2.15 jointly with Theorem 2.27 in [3], the topological sum $\Lambda = \oplus\{\lambda(\alpha) + 1 : \alpha \in A\}$ is a normal pseudo-k-space.
Now, let us denote by $Y(X)$ the quotient space obtained by identifying all the final points $\lambda(\alpha) \in \lambda(\alpha) + 1$ to a single points y_0.

We have the following

Theorem 3.2. *The space $Y(X)$ is a normal pseudo-k-space. Moreover, if P is a pseudocompact subset of $Y(X)$, then $|\{\alpha \in A : P \cap \lambda(\alpha) \neq \varnothing\}| < \omega$.*

Proof. Let us denote by $g : \Lambda \longrightarrow Y(X)$ the canonical projection defining $Y(X)$. It is easy to verify that g is a closed mapping. So, since the normality preserves under closed mappings, it follows that $Y(X)$ is normal. Moreover, since g is a continuous surjective closed map, then g is a quotient mapping. Then, by Theorem 2.14, the space $Y(X)$ is a pseudo-k-space.

Now, suppose that there exists $B \subset A$, $|B| \geq \omega$, such that a pseudocompact subset P of $Y(X)$ meets each element of the family $\{\lambda(\alpha) : \alpha \in B\}$. Observe that for every $\alpha \in A$, since $\lambda(\alpha)$ is open in $Y(X)$, the set $\lambda(\alpha) \cap P$ is open in P. Then the set $\{\lambda(\alpha) \cap P : \alpha \in B\}$ is a locally finite family of non-empty open subsets of P. Since P is a Tychonoff space, this is equivalent to say that P is not pseudocompact (see Theorem 3.10.22 in [3]), a contradiction. □

Theorem 3.3. *Let X be a regular space which is not locally compact at a point x_0. The cartesian product $X \times Y(X)$ is not a pseudo-k-space.*

Proof. Let X be a regular space which is not locally compact at a point x_0. Let us show that the cartesian product $X \times Y(X)$ is not a pseudo-k-space. It suffices to find a subset H of $X \times Y(X)$, which is not closed even if the intersection of H with any pseudocompact subspace P of the space $X \times Y(X)$ is closed in P.

Recall that, in the definition of $Y(X)$, the set A denotes an index set and to each $\alpha \in A$ is associated a limit ordinal $\lambda(\alpha)$ such that $\bigcap_{\lambda<\lambda(\alpha)} F_\lambda$ is empty.

Now fix $\alpha \in A$ and $\lambda \in \lambda(\alpha)+1$ and define $E_\lambda = \bigcap_{\mu<\lambda} F_\mu$. Then $E_{\lambda(\alpha)} = \varnothing$. Moreover the set $S_\alpha = \cup\{E_\lambda \times \{\lambda\} \quad \lambda \in \lambda(\alpha)+1\}$ is closed in $X \times (\lambda(\alpha)+1)$, which implies that it is closed in $X \times \Lambda$.

Denote by g the canonical projection $g : \Lambda \longrightarrow Y(X)$ and by h the function $id_X \times g$, and define the set

$$H = \bigcup_{\alpha \in A} h(S_\alpha) \subset X \times Y(X).$$

We shall show that H is the set we are searching for.

First let us prove that the intersection of H with any pseudocompact subset P of $X \times Y(X)$ is closed in P. The projection $p_y(P)$ is a pseudocompact subset in $Y(X)$ so, by virtue of Theorem 3.2, we have

$$|\{\alpha \in A : p_y(P) \cap \lambda(\alpha) \neq \varnothing\}| < \omega$$

Then P meets finitely many $X \times g(\lambda(\alpha)+1) = X \times (\lambda(\alpha) \cup \{y_0\}) \supset h(S_\alpha)$. Now, since $h(S_\alpha)$ is closed in $X \times Y(X)$ for each $\alpha \in A$, it follows that the set $H \cap P = \bigcup_{\alpha \in A} (h(S_\alpha) \cap P)$ is closed in P.

Now let us show that H is not closed in $X \times Y(X)$. The point $(x_0, y_0) \in X \times Y(X)$ belongs to $\overline{H}$ but does not belong to H. Take a neighborhood $U \times V$ of $(x_0, y_0), U$ open in X, V open in $Y(X)$, and let U_β a closed non-compact neighborhood $U_\beta \subset U$, for some $\beta \in A$. Now, consider the canonical projection $g : \Lambda \to Y(X)$, and fix $\lambda \in g^{-1}(V) \cap \lambda(\beta)$. The set $h(E_\lambda \times \{\lambda\}) \neq \varnothing$ is contained in $(U \times V) \cap H$. Therefore $(x_0, y_0) \in \overline{H}$.
Suppose that $(x_0, y_0) \in H$, then $(x_0, y_0) \in h(S_\alpha)$ for some $\alpha \in A$. This is a contradiction.

□

Theorems 3.1 and 3.3 provide the following characterization for locally compact spaces.

Theorem 3.4. *Let X be a regular space. The following conditions are equivalent:*

(1) *X is locally compact.*
(2) *$X \times Y$ is a pseudo-k-space, for each pseudo-k-space Y.*

Proof. (1)⇒(2) It follows from Theorem 3.1.
(2)⇒(1) Let X be a regular space which is not locally compact at a point x_0. Then, by virtue of Theorems 3.2 and 3.3, the space $Y(X)$ is a pseudo-k-space such that $X \times Y(X)$ is not a pseudo-k-space. □

In terms of products of mappings we have

Theorem 3.5. *Let X be a regular space. The following conditions are equivalent:*

(1) *X is locally compact.*
(2) *$id_X \times g$ is a quotient map with domain a locally pseudocompact Hausdorff space, for every quotient map g with domain a locally pseudocompact Hausdorff space Y.*

Proof. (1)⇒(2) It comes directly from Whitehead Theorem (see Theorem 3.3.17 in [3]) and Proposition 2.4.
(2)⇒(1) If X is not locally compact then we can consider $Y(X)$, defined as before, and the projection map $g : \Lambda \to Y(X)$, which is a quotient map with domain the locally pseudocompact Hausdorff space Λ. It is easy to show that $h = id_X \times g$ is not a quotient map with domain a locally pseudocompact Hausdorff space. Indeed if h was a quotient map with domain a locally pseudocompact Hausdorff space then $X \times Y(X)$ should be a pseudo-k-space, but $X \times Y(X)$ is not a pseudo-k-space by virtue of Theorem 3.3. □

References

[1] D. E. Cohen, *Spaces with weak topology*, Quart. J. Math., Oxford **5** (1954), 77–80.

[2] C. H. Dowker, *Topology of metric complexes*, Amer. J. Math. **74** (1952), 555–577.

[3] R. Engelking, *General Topology*, Sigma Ser. Pure Math. 6 (Heldermann, Berlin, 1989).

[4] T. Jech, *Set Theory*, Academic Press, 1978.

[5] E. Michael, *Local compactness and cartesian product of quotient maps and k-spaces*, Ann. Inst. Fourier **18** (1968), 281–286.

[6] K. Morita, *Product of normal spaces with metric spaces*, Math. Ann. **154** (1964), 365–382.

[7] J. Nagata, *Quotient and bi-quotient spaces of M-spaces*, Proc. Japan Acad. **45** (1969), 25–29.

[8] M. Sanchis, *A note on quasi-k-spaces*, Rend. Ist. Mat. Univ. Trieste Suppl. **XXX** (1999), 173–179.

[9] J. H. C. Whitehead, *A note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. **54** (1948), 1125–1132.

Received February 2007

Accepted October 2007

A. Miranda (amiranda@unisa.it)
Dip. di Matematica e Informatica, Università di Salerno, Via Ponte Don Melillo, 84084 Fisciano (Salerno), Italy