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Thin subsets of balleans

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Abstract. A ballean is a set endowed with some family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. We characterize the ideal generated by the family of all thin subsets in an ordinal ballean, and apply this characterization to metric spaces and groups.

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Let G be a group with the identity e. A subset $A \subseteq G$ is called *thin* if $|gA \cap A| < \aleph_0$ for every $g \in G$, $g \neq e$. For thin subsets, its modifications, applications and references see [4]. We denote by $\mathcal{T}_G$ the family of all thin subsets of G. Then the smallest ideal $\mathcal{T}_G^*$ (in the Boolean algebra of all subsets of G) containing $\mathcal{T}_G$ is the family of all finite unions of thin subsets. Thus, to characterize $\mathcal{T}_G^*$, we need some test which, for given $A \subseteq G$ and $m \in \mathbb{N}$, detect whether A can be represented as a union of $\leqslant m$ thin subsets.

Let (X, d) be a metric space. We say that a subset $A \subseteq X$ is *thin* if, for every $r \in \mathbb{R}^+$, there exists a bounded subset $Y \subseteq X$ such that $A \cap B(x, r) = \{x\}$ for every $x \in A \setminus Y$, where $B_d(x, r) = \{x \in Y : d(x, y) \leqslant r\}$. As in the group case, to characterize the ideal $\mathcal{T}^*(X, d)$ generated by the family $\mathcal{T}(X, d)$ of all thin subsets of (X, d), we ask for a test recognizing if a subset $A \subseteq X$ is a union of $\leqslant m$ thin subsets.

It is easy to see that a subset $A \subseteq G$ is thin if and only if, for every finite subset F of G containing e, there exists a finite subset Y of G such that $A \cap Fg = \{g\}$ for every $x \in A \setminus Y$. Following [1], we say that Fg is a ball of radius F around g.

From this point of view, the definitions of the thin subsets in groups and metric spaces are very similar syntactically. To formalize this similarity we use the ballean approach from [5]. A ballean is a set endowed with some family of its subsets which are called the balls. The property of the family of ball are postulated in such a way that the balleans can be considered as the counterparts of the uniform topological spaces (see Section 1 for precise definition).

In Section 1 we define the thin subsets of a ballean and, for every ordinal ballean, characterize the ideal generated by the thin subsets.

The group and metric spaces have the natural ballean structures. In Section 2 we apply the result from Section 1 to justify the following two tests.

A subset A of a metric space X can be partitioned in $\leqslant m$ thin subsets if and only if, for every $r \in \mathbb{R}^+$, there exists a bounded subset $Y \subseteq X$ such that $|A \cap B(x,r)| \leqslant m$ for every $x \in A \setminus Y$.

A subset A of a countable group G can be partitioned in $\leqslant m$ thin subsets if and only if, for every finite subset F of G, there exists a finite subset Y of G such that $|A \cap Fx| \leqslant m$ for every $x \in A \setminus Y$. **We do not know whether this test is effective for an uncountable group.**

1. Ballean context

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are not-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(X, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the support of $\mathcal{B}$, P is called the *set of radii*.

Given any $x \in X$, $A \subseteq X$, we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\}, \; B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha).$$

A ball structures $\mathcal{B}$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
$$B(x,\alpha) \subseteq B^*(x,\alpha'), \; B^*(x,\beta) \subseteq B(x,\beta');$$
- for any $\alpha, \beta \in P$, there exist $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

We note that a ballean can also be defined in terms of entourages of diagonal in $X \times X$. In this case it is called a coarse structures [7].

A ballean $\mathcal{B}$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, y)$. All balleans under considerations are suppose to be connected. Replacing each ball $B(x,\alpha)$ to $B(x,\alpha) \cap B^*(x,\alpha)$, we may suppose that $B^*(x,\alpha) = B(x,\alpha)$ for all $x \in X$, $\alpha \in P$. A subset $Y \subseteq X$ is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $Y \subseteq B(x,\alpha)$.

We use a preordering $\leqslant$ on the support X of $\mathcal{B}$ defined by the rule: $\alpha \leqslant \beta$ if and only if $B(x,\alpha) \leqslant B(x,\beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha \leqslant \alpha'$. A ballean $\mathcal{B}$ is called *ordinal* if there exists a cofinal subset $P' \subseteq P$ well ordered by $\leqslant$.

Let $\mathcal{B} = (X, P, B)$ be a ballean, $m \in \mathbb{N}$. We say that a subset $A \subseteq X$ is *m-thin* if, for every $\alpha \in P$, there exists a bounded subset $Y_\alpha \subseteq X$ such that $|B(x,\alpha) \cap A| \leqslant m$ for every $x \in A \setminus Y_\alpha$. A 1-thin subset is called *thin*. Thus,

A is thin if, for every $\alpha \in P$, there exists a bounded subset Y_α of X such that $B(x, \alpha) \cap A = \{x\}$ for every $x \in A \setminus Y_\alpha$. In the terminology of [5], the thin subsets are called pseudodiscrete. For pseudodiscreteness see also [2], [6].

We use the following notation:

$\mathcal{T}(\mathcal{B})$ is the family of all thin subsets of X;
$\mathcal{T}_m(\mathcal{B})$ is the family of all m-thin subsets of X;
$\bigcup_m \mathcal{T}(\mathcal{B})$ is the family of all unions of $\leqslant m$ thin subsets of X;
$\mathcal{T}^*(\mathcal{B})$ is the ideal generated by $\mathcal{T}(\mathcal{B})$.

Clearly, $\mathcal{T}^*(\mathcal{B}) = \bigcup_{m \in \mathbb{N}}(\bigcup_m \mathcal{T}(\mathcal{B}))$.

Lemma 1.1. *For every ballean $\mathcal{B}$, we have $\bigcup_m \mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}_m(\mathcal{B})$.*

Proof. Let $A_1, \dots, A_n$ be thin subsets of X. For every $\alpha \in P$, we pick $\gamma(\alpha) \in P$ such that $B(B(x, \alpha), \alpha) = B(x, \gamma(\alpha))$. For all $\alpha \in P$ and $i \in \{1, \dots, m\}$, we choose a bounded subset $Y_\alpha(i)$ such that $B(x, \alpha) \cap A_i = \{x\}$ for every $x \in A_i \setminus Y_\alpha(i)$, and put $Y_\alpha = Y_\alpha(1) \cup \ldots \cup Y_\alpha(m)$. We take an arbitrary element $a \in (A_1 \cup \ldots \cup A_m) \setminus Y_\alpha$ and suppose that $|B(a, \alpha) \cap (A_1 \cup \ldots \cup A_m)| > m$. Then there exists $j \in \{1, \dots, m\}$ such that $|A_j \cap B(a, \alpha)| > 2$. Let $b, c \in A_i \cap B(a, \alpha)$, $b \neq c$. Then $c \in B(b, \gamma(\alpha))$ contradicting the choice of $Y_\alpha(j)$. □

The following theorem gives a characterization of $\mathcal{T}^*(\mathcal{B})$ in the case of an ordinal ballean $\mathcal{B}$.

Theorem 1.2. *For every ordinal ballean $\mathcal{B}$ and $m \in \mathbb{N}$, we have $\mathcal{T}_m(\mathcal{B}) = \bigcup_m \mathcal{T}(\mathcal{B})$.*

Proof. In view of Lemma 1.1, it suffices to show that $\mathcal{T}_m(\mathcal{B}) \subseteq \bigcup_m \mathcal{T}(\mathcal{B})$. Let $A \in \mathcal{T}_m(\mathcal{B})$. We may suppose that P is will ordered by $\leqslant$. We construct inductively a family $\{Y_\alpha : \alpha \in P\}$ of bounded subsets of X such that $|B(x, \alpha) \cap A|$ for every $x \in A \setminus Y_\alpha$ and $Y_\alpha \subseteq Y_\beta$ for all $\alpha \leqslant \beta$. Then we consider a graph Γ with the set of vertices A and the set of edges E defined as follows: $\{x, y\} \in E$ if and only if $x \neq y$ and there exists $\alpha \in P$ such that $x, y \in A \setminus Y_\alpha$ and $y \in B(x, \alpha)$. We show that $deg(x) \leqslant m - 1$ for every $x \in A$, where $deg(x) = |\{y \in A : \{x, y\} \in E\}|$. We suppose the contrary and choose $x \in A$ and distinct vertices $y_1, \dots, y_m$ such that $\{x, y_i\} \in E$ for every $i \in \{1, \dots, m\}$. By the definition of E, for every $i \in \{1, \dots, m\}$, there exists $\alpha_i \in P$ and a bounded subset Y_{α_i} of X such that $y_i \in B(x, \alpha_i)$ and $x, y_i \in A \setminus Y_{\alpha_i}$. Let $\alpha = max\{\alpha_1, \dots, \alpha_m\}$ and $\alpha = \alpha_j$. Then $y_1, \dots, y_m \in B(x, \alpha_j)$ and $y_1, \dots, y_m \in A \setminus Y_{\alpha_j}$ because $Y_{\alpha_i} \subseteq Y_{\alpha_j}$ for all $i \in \{1, \dots, m\}$, so we get a contradiction with the choice of Y_{α_j} because $|B(x, \alpha_j) \cap A| \leqslant m$.

By [3, Corollary 12.2], the chromatic number of Γ does not exceed m. Hence A can be partitioned $A = A_1 \cup \ldots \cup A_k$, $\kappa \leqslant m$ so that, for every $i \in \{1, \dots, k\}$ and $x, y \in A_i$, we have $\{x, y\} \notin E$.

We show that each subset A_i is thin. For every $\alpha \in P$, we put $Z_\alpha = B(Y_\alpha, \alpha)$. Suppose that there exists $x \in A_i \setminus Z_\alpha$ such that $|B(x, \alpha) \cap A_i| > 1$. Let $y \in B(x, \alpha) \cap A_i$ and $y \neq x$. Since $x \notin Z_\alpha$ then $y \notin Y_\alpha$. Thus $x, y \in A \setminus Y_{\alpha_i}$ and $y \in B(x, \alpha)$, so $\{x, y\} \in E$ contradicting the choice of A_i. □

2. Applications

Theorem 2.1. *Let (X, d) be a metric space, $m \in \mathbb{N}$. A subset $A \subseteq X$ can be partitioned in $\leqslant m$ thin subsets if and only if , for every $r \in \mathbb{R}^+$, there exists a bounded subset Y of X such that $|B(x, r) \cap A| \leqslant m$ for every $x \in A \setminus Y$.*

Proof. We consider (X, d) as the ballean $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$. Clearly, $\mathcal{B}(X, d)$ is ordinal so we can apply Theorem 1.2. □

Let G be a group, κ be an infinite cardinal, $\mathcal{F}_\kappa(G) = \{F \subseteq G : |F| < \kappa, e \in F\}$. We consider the ballean

$$\mathcal{B}_\kappa(G) = (G, \mathcal{F}_\kappa(G), B),$$

where $B(g, F) = Fg$ for all $g \in G$, $F \in \mathcal{F}_\kappa(G)$. If $\kappa > |G|$, $\mathcal{B}_\kappa(G)$ is bounded. For $\kappa = |G|$, $\mathcal{B}_\kappa(G)$ is ordinal. Indeed, let $g_0 = e$, $\{g_\alpha : \alpha < \kappa\}$ be a numeration of G, $F_\alpha = \{g_\beta : \beta \leqslant \alpha\}$. Then the well ordered by $\subseteq$ family $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$ is cofinal in $\mathcal{F}$.

We say that a subset $A \subset G$ is κ-thin if $|gA \cap A| < \kappa$ for every $g \in G$, $g \neq e$. In the case $\kappa = \aleph_0$, we get the thin subsets defined in the very beginning of the paper.

Lemma 2.2. *Let A be a subset of a group G. If A is thin in the ballean $\mathcal{B}_\kappa(G)$ then A is κ-thin. If A is κ-thin and κ is regular then A is thin in the ballean $\mathcal{B}_\kappa(G)$.*

Proof. Let A be thin in $\mathcal{B}_\kappa(G)$. For every $g \in G$, $g \neq e$, we put $F_g = \{e, g\}$ and choose a bounded subset Y_g in $\mathcal{B}_\kappa(G)$ such that $B(x, F) \cap A = \{x\}$ for every $x \in A \setminus Y_g$. Then $gx \notin A$ for every $x \in A \setminus Y_g$ so $gA \cap A \subseteq Y_g$. Since Y_g is bounded in $\mathcal{B}_\kappa(G)$ then $|Y_g| < \kappa$ and A is κ-thin.

Let A be κ-thin, $F \in \mathcal{F}_\kappa(G)$. We put $Y = \bigcup\{gA \cap A : g \in F \setminus \{e\}\}$. Since $|gA \cap A| < \kappa$, $|F| < \kappa$ and κ is regular, $|Y| < \kappa$ so Y is bounded in $\mathcal{B}_\kappa(G)$. For every $x \in A \setminus Y$, we have $Fx \cap A = \{x\}$ hence A is thin in $\mathcal{B}_\kappa(G)$. □

Theorem 2.3. *Let G be a group of regular cardinality κ, $m \in \mathbb{N}$. A subset $A \subseteq G$ can be partitioned in $\leqslant m$ κ-thin subsets if and only if, for every $F \subset G$, $|F| < \kappa$, there exists a subset $Y \subseteq G$ such that $|Y| < \kappa$ and $|Fx \cap A| \leqslant m$ for every $x \in A \setminus Y$.*

Proof. Since the ballean $\mathcal{B}_\kappa(G)$ is ordinal, in view of Lemma 2.2, we can apply Theorem 1.2. □

Remark 2.4. A subset A of a group G is called almost thin if the set $\Delta(A) = \{g \in G : gA \cap A \ is\ infinite\}$ is finite. By [4, Theorem 3.1], every almost thin subset of a group G can be partitioned in $3^{|\Delta(A)|-1}$ thin subsets, but the union of two thin subsets needs not to be almost thin [4, Theorem 3.2].

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