

The Zariski topology on the graded primary spectrum of a graded module over a graded commutative ring

SAIF SALAM  AND KHALDOUN AL-ZOUBI 

Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan (smsalam19@sci.just.edu.jo, kfzoubi@just.edu.jo)

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ABSTRACT

Let R be a G -graded ring and M be a G -graded R -module. We define the graded primary spectrum of M , denoted by $\mathcal{PS}_G(M)$, to be the set of all graded primary submodules Q of M such that $(Gr_M(Q) :_R M) = Gr((Q :_R M))$. In this paper, we define a topology on $\mathcal{PS}_G(M)$ having the Zariski topology on the graded prime spectrum $Spec_G(M)$ as a subspace topology, and investigate several topological properties of this topological space.

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1. INTRODUCTION AND PRELIMINARIES

Let G be a multiplicative group with identity e and R be a commutative ring with identity. Then R is called a G -graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g . If $r \in R$, then r can be written uniquely as $\sum_{g \in G} r_g$, where r_g is the component of r in R_g . The set of all homogeneous elements of R is denoted

by $h(R)$, i.e. $h(R) = \bigcup_{g \in G} R_g$. Let R be a G -graded ring and I be an ideal of R .

Then I is called G -graded ideal of R if $I = \bigoplus_{g \in G} (I \cap R_g)$. By $I \triangleleft_G R$, we mean that I is a G -graded ideal of R , (see [13]). The graded radical of I is the set of all $a = \sum_{g \in G} a_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $a_g^{n_g} \in I$.

By $Gr(I)$ (resp. \sqrt{I}) we mean the graded radical (resp. the radical) of I , (see [18]). The graded prime spectrum $Spec_G(R)$ of a graded ring R consists of all graded prime ideals of R . It is known that $Spec_G(R)$ is a topological space whose closed sets are $V_G^R(I) = \{p \in Spec_G(R) \mid I \subseteq p\}$ for each graded ideal I of R (see, for example, [14, 16, 18]).

Let R be a G -graded ring and M a left R -module. Then M is said to be a G -graded R -module if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where

M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g . If $x \in M$, then x can be written uniquely as $\sum_{g \in G} x_g$,

where x_g is the component of x in M_g . The set of all homogeneous elements of M is denoted by $h(M)$, i.e. $h(M) = \bigcup_{g \in G} M_g$. Let $M = \bigoplus_{g \in G} M_g$ be a G -graded

R -module. A submodule N of M is called a G -graded R -submodule of M if $N = \bigoplus_{g \in G} (N \cap M_g)$. By $N \leq_G M$ (resp. $N <_G M$) we mean that N is

a graded submodule (resp. a proper graded submodule) of M , (see [13]). If $N \leq_G M$, then $(N :_R M) = \{r \in R \mid rM \subseteq N\}$ is a graded ideal of R , (see [3, Lemma 2.1]). A proper graded submodule P of M is called a graded prime submodule of M if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $m \in P$ or $r \in (P :_R M)$. It is easily seen that, if P is a graded prime submodule of M , then $(P :_R M)$ is a graded prime ideal of R (see [3, Proposition 2.7]). The graded prime spectrum of M , denoted by $Spec_G(M)$, is the set of all graded prime submodules of M . A proper graded submodule Q of M is called a graded primary submodule of M , if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in Q$, then either $m \in Q$ or $r \in Gr((Q :_R M))$. Graded prime submodules and Graded primary submodules of graded modules have been studied by various authors (see, for example [1, 2, 3, 4, 5, 15]). The graded radical of a proper graded submodule N of M , denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N . If N is not contained in any graded prime submodule of M , then $Gr_M(N) = M$, (see [5]).

A graded R -module M is called a multiplication graded R -module if any $N \leq_G M$ has the form IM for some $I \triangleleft_G R$. If N is a graded submodule of a multiplication graded module M , then $N = (N :_R M)M$, (see [15]). A graded submodule N of a graded module M is called graded maximal submodule of M if $N \neq M$ and there is no graded submodule L of M such that $N \subset L \subset M$. A graded ring R is called graded integral domain, if whenever $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded principal ideal domain R is a

graded integral domain in which every graded ideal of R is generated by a homogeneous element. One can easily see that, if R is a graded principal ideal domain, then every non-zero graded prime ideal of R is graded maximal.

Let M be a G -graded R -module and let $\zeta^*(M) = \{V_G^*(N) \mid N \leq_G M\}$ where $V_G^*(N) = \{P \in \text{Spec}_G(M) \mid N \subseteq P\}$ for any $N \leq_G M$. Then M is called a G -top module if the set $\zeta^*(M)$ is closed under finite union. In this case, $\zeta^*(M)$ generates a topology on $\text{Spec}_G(M)$ and this topology is called the quasi Zariski topology on $\text{Spec}_G(M)$. In contrast with $\zeta^*(M)$, $\zeta(M) = \{V_G(N) \mid N \leq_G M\}$ where $V_G(N) = \{P \in \text{Spec}_G(M) \mid (N :_R M) \subseteq (P :_R M)\}$ for any $N \leq_G M$ always generates a topology on $\text{Spec}_G(M)$. Let M be a G -graded R -module. Then the map $\varphi : \text{Spec}_G(M) \rightarrow \text{Spec}_G(R/\text{Ann}(M))$ by $\varphi(P) = (P :_R M)/\text{Ann}(M)$ is called the natural map on $\text{Spec}_G(M)$. For more details concerning the topologies on $\text{Spec}_G(M)$ and the natural map on $\text{Spec}_G(M)$, one can look in [7, 14].

In this paper, we call the set of all graded primary submodules Q of a graded module M satisfying the condition $(Gr_M(Q) :_R M) = Gr((Q :_R M))$ the graded primary spectrum of M and denote it by $\mathcal{PS}_G(M)$. It is easy to see that $\text{Spec}_G(M) \subseteq \mathcal{PS}_G(M)$. The converse inclusion is not always true. For example, if F is a G -graded field and M is a G -graded F -module, then $\text{Spec}_G(M) = \mathcal{PS}_G(M) = \{N \mid N <_G M\}$. But if we take the ring of integers $R = \mathbb{Z}$, $G = \mathbb{Z}_2$ and $M = \mathbb{Z} \times \mathbb{Z}$, then R is a G -graded ring by $R_0 = R$ and $R_1 = \{0\}$. Also, M is a G -graded R -module by $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. By some computations, we can see that $N = \mathbb{Z} \times 4\mathbb{Z} \in \mathcal{PS}_G(M)$. However, $N \notin \text{Spec}_G(M)$, since $2 \in h(R)$ and $(2, 0) \in h(M)$ such that $2(0, 2) \in N$ but $(2, 0) \notin N$ and $2 \notin (N :_R M)$. For a G -graded R -module M , it is clear that $Gr_M(Q) \neq M$ for any $Q \in \mathcal{PS}_G(M)$ as $Gr((Q :_R M)) \in \text{Spec}_G(R)$. We introduce the primary G -top module which is a generalization of the G -top module. For this, we define the variety of any $N \leq_G M$ by $\nu_G^*(N) = \{Q \in \mathcal{PS}_G(M) \mid N \subseteq Gr_M(Q)\}$ and we set $\Omega^*(M) = \{\nu_G^*(N) \mid N \leq_G M\}$. Then M is called a primary G -top module if $\Omega^*(M)$ is closed under finite union. When this the case, the topology generated by $\Omega^*(M)$ is called the quasi-Zariski topology on $\mathcal{PS}_G(M)$. In particular, every primary G -top module is a G -top module. Next, we define another variety of any $N \leq_G M$ by $\nu_G(N) = \{Q \in \mathcal{PS}_G(M) \mid (N :_R M) \subseteq (Gr_M(Q) :_R M)\}$. Then the collection $\Omega(M) = \{\nu_G(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$, which is called the Zariski topology on $\mathcal{PS}_G(M)$, or simply \mathcal{PZ}_G -topology. We give some properties of these topologies. We also relate some properties of the graded primary spectrum $\mathcal{PS}_G(M)$ and $\text{Spec}_G(R/\text{Ann}(M))$ by introducing the map $\rho : \mathcal{PS}_G(M) \rightarrow \text{Spec}_G(R/\text{Ann}(M))$ given by $\rho(Q) = (Gr_M(Q) :_R M)/\text{Ann}(M)$. It should be noted that $(Gr_M(Q) :_R M) \in \text{Spec}_G(R)$, since Q is a graded primary submodule of M and $(Gr_M(Q) :_R M) = Gr((Q :_R M))$. In the last two sections, we find a base for the Zariski topology on $\mathcal{PS}_G(M)$ and we make certain observations and obtain a few results involving some conditions under which $\mathcal{PS}_G(M)$ is compact, irreducible, T_0 -space or spectral space.

Throughout this paper, G is a multiplicative group, R is a commutative G -graded ring with identity and M is a G -graded R -module. We assume that $\text{Spec}_G(M)$ and $\mathcal{PS}_G(M)$ are non-empty.

2. THE ZARISKI TOPOLOGY ON $\mathcal{PS}_G(M)$

In this section, we introduce different varieties for graded submodules of graded modules. Using the properties of these varieties, we define the quasi Zariski topology and the \mathcal{PZ}_G -topology on $\mathcal{PS}_G(M)$. We also give some relationships between $\mathcal{PS}_G(M)$, $\text{Spec}_G(R/\text{Ann}(M))$ and $\text{Spec}_G(M)$.

Theorem 2.1. *Let M be a G -graded R -module. For any G -graded submodule N of M , we define the variety of N by $\nu_G^*(N) = \{Q \in \mathcal{PS}_G(M) \mid N \subseteq \text{Gr}_M(Q)\}$. Then the following hold:*

- (1) $\nu_G^*(0) = \mathcal{PS}_G(M)$ and $\nu_G^*(M) = \emptyset$.
- (2) If $N, N' \leq_G M$ and $N \subseteq N'$, then $\nu_G^*(N') \subseteq \nu_G^*(N)$.
- (3) $\bigcap_{i \in I} \nu_G^*(N_i) = \nu_G^*(\sum_{i \in I} N_i)$ for any indexing set I and any family of graded submodules $\{N_i\}_{i \in I}$.
- (4) $\nu_G^*(N) \cup \nu_G^*(N') \subseteq \nu^*(N \cap N')$ for any $N, N' \leq_G M$.
- (5) $\nu_G^*(N) = \nu_G^*(\text{Gr}_M(N))$ for any $N \leq_G M$.

Proof. (1) and (2) are obvious.

(3) Since $N_i \subseteq \sum_{i \in I} N_i$ for all $i \in I$, then by (2) we have $\nu_G^*(\sum_{i \in I} N_i) \subseteq \nu_G^*(N_i)$ for all $i \in I$. Therefore $\nu_G^*(\sum_{i \in I} N_i) \subseteq \bigcap_{i \in I} \nu_G^*(N_i)$. Conversely, let $Q \in \bigcap_{i \in I} \nu_G^*(N_i)$. Then $N_i \subseteq \text{Gr}_M(Q)$ for all $i \in I$, which implies that $\sum_{i \in I} N_i \subseteq \text{Gr}_M(Q)$. Hence $Q \in \nu_G^*(\sum_{i \in I} N_i)$.

(4) Since $N \cap N' \subseteq N$ and $N \cap N' \subseteq N'$, then by (2) we have $\nu_G^*(N) \subseteq \nu_G^*(N \cap N')$ and $\nu_G^*(N') \subseteq \nu_G^*(N \cap N')$. Therefore $\nu_G^*(N) \cup \nu_G^*(N') \subseteq \nu_G^*(N \cap N')$.
 (5) As $N \subseteq \text{Gr}_M(N)$, we obtain $\nu_G^*(\text{Gr}_M(N)) \subseteq \nu_G^*(N)$. Conversely, let $Q \in \nu_G^*(N)$. Then $N \subseteq \text{Gr}_M(Q)$. So $\text{Gr}_M(N) \subseteq \text{Gr}_M(\text{Gr}_M(Q)) = \text{Gr}_M(Q)$. Thus $Q \in \nu_G^*(\text{Gr}_M(N))$. \square

Note that the reverse inclusion in Theorem 2.1 (4) is not always true. Take $R = \mathbb{Z}, G = \mathbb{Z}_2, M = \mathbb{Z} \times \mathbb{Z}, N = 4\mathbb{Z} \times \{0\}$ and $N' = \{0\} \times 4\mathbb{Z}$. Then R is a G -graded ring by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Also M is a G -graded R -module by $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. Moreover, $N, N' \leq_G M$. Now, $\nu_G^*(N \cap N') = \nu_G^*(\{(0, 0)\}) = \mathcal{PS}_G(M)$. Let $P = \{(0, 0)\}$. Then $P \in \text{Spec}_G(M) \subseteq \mathcal{PS}_G(M)$. It follows that $P \in \nu_G^*(N \cap N')$ and $\text{Gr}_M(P) = P$. But $N \not\subseteq P$ and $N' \not\subseteq P$. Thus $P \notin \nu_G^*(N) \cup \nu_G^*(N')$.

By Theorem 2.1 (1), (3), and (4), the collection $\Omega^*(M) = \{\nu_G^*(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$ if and only if $\Omega^*(M)$ is closed under finite union. When this is the case, we call M a primary G -top module and we call the generated topology the quasi Zariski topology on $\mathcal{PS}_G(M)$, or \mathcal{PZ}_G^q -topology for short. It is clear that, every G -graded simple

R -module is a primary G -top module. In the following theorem, we show that every multiplication graded R -module is a primary G -top module.

Theorem 2.2. *If M is a multiplication graded R -module, then M is a primary G -top module.*

Proof. Let $N, N' \leq_G M$. It is sufficient to show that $\nu_G^*(N \cap N') \subseteq \nu_G^*(N) \cup \nu_G^*(N')$. So let $Q \in \nu_G^*(N \cap N')$. Then $N \cap N' \subseteq Gr_M(Q)$. It follows that $(N :_R M) \cap (N' :_R M) = (N \cap N' :_R M) \subseteq (Gr_M(Q) :_R M) = Gr((Q :_R M)) \in Spec_G(R)$ as Q is a graded primary submodule. Therefore $(N :_R M) \subseteq (Gr_M(Q) :_R M)$ or $(N' :_R M) \subseteq (Gr_M(Q) :_R M)$. Since M is a multiplication graded R -module, then $N = (N :_R M)M \subseteq (Gr_M(Q) :_R M)M = Gr_M(Q)$ or $N' = (N' :_R M)M \subseteq (Gr_M(Q) :_R M)M = Gr_M(Q)$. Thus $Q \in \nu_G^*(N) \cup \nu_G^*(N')$. \square

Now we define another variety for a graded submodule N of a G -graded R -module M . We set $\nu_G(N) = \{Q \in PS_G(M) \mid (N :_R M) \subseteq (Gr_M(Q) :_R M)\}$. We state some properties of this variety in the following theorem to construct the Zariski topology on $\mathcal{PS}_G(M)$.

Theorem 2.3. *Let M be a G -graded R -module and let $N, N', N_i \leq_G M$ for any $i \in I$, where I is an indexing set. Then the following hold:*

- (1) $\nu_G(0) = \mathcal{PS}_G(M)$ and $\nu_G(M) = \emptyset$.
- (2) $\bigcap_{i \in I} \nu_G(N_i) = \nu_G(\sum_{i \in I} (N_i :_R M)M)$.
- (3) $\nu_G(N) \cup \nu_G(N') = \nu_G(N \cap N')$.
- (4) If $N \subseteq N'$, then $\nu_G(N') \subseteq \nu_G(N)$.

Proof. (1) and (4) are trivial.

(2) Let $Q \in \bigcap_{i \in I} \nu_G(N_i)$. Then $(N_i :_R M) \subseteq (Gr_M(Q) :_R M)$ for all $i \in I$. So $(N_i :_R M)M \subseteq (Gr_M(Q) :_R M)M$ for all $i \in I$. This implies that $\sum_{i \in I} (N_i :_R M)M \subseteq (Gr_M(Q) :_R M)M$. Therefore $(\sum_{i \in I} (N_i :_R M)M : M) \subseteq ((Gr_M(Q) :_R M)M :_R M) = (Gr_M(Q) :_R M)$. Hence $Q \in \nu_G(\sum_{i \in I} (N_i :_R M)M)$. For the reverse inclusion, let $Q \in \nu_G(\sum_{i \in I} (N_i :_R M)M)$. Then $(\sum_{i \in I} (N_i :_R M)M :_R M) \subseteq (Gr_M(Q) :_R M)$. But for any $i \in I$, we have $(N_i :_R M)M :_R M \subseteq (\sum_{i \in I} (N_i :_R M)M :_R M) \subseteq (Gr_M(Q) :_R M)$. Thus $Q \in \bigcap_{i \in I} \nu_G(N_i)$.

(3) For any $Q \in \mathcal{PS}_G(M)$, we have $Q \in \nu_G(N \cap N') \Leftrightarrow (N \cap N' :_R M) \subseteq (Gr_M(Q) :_R M) = Gr((Q :_R M)) \Leftrightarrow (N :_R M) \cap (N' :_R M) \subseteq Gr((Q :_R M)) \in Spec_G(R) \Leftrightarrow (N :_R M) \subseteq Gr((Q :_R M))$ or $(N' :_R M) \subseteq Gr((Q :_R M)) \Leftrightarrow Q \in \nu_G(N) \cup \nu_G(N')$. Hence $\nu_G(N \cap N') = \nu_G(N) \cup \nu_G(N')$. \square

In view of Theorem 2.3 (1), (2) and (3), the collection $\Omega(M) = \{\nu_G(N) \mid N \leq_G M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}_G(M)$, which is called the Zariski topology on $\mathcal{PS}_G(M)$, or \mathcal{PZ}_G -topology for short.

Now we state some relations between the varieties $V_G^*(N)$, $V_G(N)$, $\nu_G^*(N)$ and $\nu_G(N)$ for any graded submodule N of a G -graded R -module M . These relations will be used continuously throughout the rest of this paper.

Lemma 2.4. *Suppose that N and N' are graded submodules of a G -graded R -module M and that I is a G -graded ideal of R . Then the following hold:*

- (1) $V_G(N) = \nu_G(N) \cap \text{Spec}_G(M)$.
- (2) $V_G^*(N) = \nu_G^*(N) \cap \text{Spec}_G(M)$.
- (3) If $\text{Gr}((N :_R M)) = \text{Gr}((N' :_R M))$, then $\nu_G(N) = \nu_G(N')$. The converse is also true if $N, N' \in \mathcal{PS}_G(M)$.
- (4) $\nu_G(N) = \nu_G((N :_R M)M) = \nu_G^*((N :_R M)M) = \nu_G^*(\text{Gr}((N :_R M)M))$. In particular, $\nu_G^*(IM) = \nu_G(IM)$.

Proof. The proof is straightforward. □

Corollary 2.5. *Every primary G -top module is a G -top module.*

Proof. Let M be a primary G -top module and $N, N' \leq_G M$. By Lemma 2.4 (2), we have $V_G^*(N) \cup V_G^*(N') = (\text{Spec}_G(M) \cap \nu_G^*(N)) \cup (\text{Spec}_G(M) \cap \nu_G^*(N')) = \text{Spec}_G(M) \cap (\nu_G^*(N) \cup \nu_G^*(N')) = \text{Spec}_G(M) \cap \nu_G^*(J) = V_G^*(J)$ for some graded submodule J of M and hence M is a G -top module. □

By Corollary 2.5, if M is a primary G -top module, then $\zeta^*(M) = \{V_G^*(N) \mid N \leq_G M\}$ induces the quasi Zariski topology on $\text{Spec}_G(M)$ which will be, by Lemma 2.4 (2), a topological subspace of $\mathcal{PS}_G(M)$ equipped with \mathcal{PZ}_G^q -topology. Also by Lemma 2.4 (1), $\text{Spec}_G(M)$ with the Zariski topology is a topological subspace of $\mathcal{PS}_G(M)$ equipped with the \mathcal{PZ}_G -topology for any G -graded R -module M .

Consider φ and ρ as described in the introduction. Let M be a G -graded R -module. For $p \in \text{Spec}_G(R)$, we set $\mathcal{PS}_G^p(M) = \{Q \in \mathcal{PS}_G(M) \mid (\text{Gr}_M(Q) :_R M) = p\}$.

Proposition 2.6. *The following statements are equivalent for any G -graded R -module M :*

- (1) *If whenever $Q, Q' \in \mathcal{PS}_G(M)$ with $\nu_G(Q) = \nu_G(Q')$, then $Q = Q'$.*
- (2) *$|\mathcal{PS}_G^p(M)| \leq 1$ for every $p \in \text{Spec}_G(R)$.*
- (3) *ρ is injective.*

Proof. (1) \Rightarrow (2): Let $p \in \text{Spec}_G(R)$ and $Q, Q' \in \mathcal{PS}_G^p(M)$. Then $Q, Q' \in \mathcal{PS}_G(M)$ and $(\text{Gr}_M(Q) :_R M) = (\text{Gr}_M(Q') :_R M) = p$. By Lemma 2.4 (3), we have $\nu_G(Q) = \nu_G(Q')$. So, by the assumption (1), $Q = Q'$.

(2) \Rightarrow (3): Assume that $\rho(Q) = \rho(Q')$, where $Q, Q' \in \mathcal{PS}_G(M)$. Then $(\text{Gr}_M(Q) :_R M) = (\text{Gr}_M(Q') :_R M)$. Let $p = (\text{Gr}_M(Q) :_R M) \in \text{Spec}_G(R)$. Then we get $Q, Q' \in \mathcal{PS}_G^p(M)$ and by the hypothesis we obtain $Q = Q'$.

(3) \Rightarrow (1): Let $Q, Q' \in \mathcal{PS}_G(M)$ with $\nu_G(Q) = \nu_G(Q')$. Then, by Lemma 2.4 (3), we have $(\text{Gr}_M(Q) :_R M) = (\text{Gr}_M(Q') :_R M)$. So $\rho(Q) = \rho(Q')$. Since ρ is injective, then $Q = Q'$. □

Corollary 2.7. *If $|\mathcal{PS}_G^p(M)| = 1$ for every $p \in \text{Spec}_G(R)$, then ρ is bijective.*

Proof. It is clear by Proposition 2.8. □

Let M be a G -graded R -module. From now on, we will denote $R/\text{Ann}(M)$ by \bar{R} and any graded ideal $I/\text{Ann}(M)$ of \bar{R} by \bar{I} . In the following lemma, we recall some properties of the natural map φ of $\text{Spec}_G(M)$. These properties are important in the rest of this section.

Lemma 2.8 ([14, Proposition 3.13 and Proposition 3.15]). *Let M be a G -graded R -module. Then the following hold:*

- (1) φ is continuous and $\varphi^{-1}(V_G^{\bar{R}}(\bar{I})) = V_G(IM)$ for every graded ideal I of R containing $\text{Ann}(M)$.
- (2) If φ is surjective, then φ is both open and closed with $\varphi(V_G(N)) = V_G^{\bar{R}}((N :_R M))$ and $\varphi(\text{Spec}_G(M) - V_G(N)) = \text{Spec}_G(\bar{R}) - V_G^{\bar{R}}((N :_R M))$ for any $N \leq_G M$.

In the next two propositions, we give similar results for ρ .

Proposition 2.9. *Let M be a G -graded R -module. Then $\rho^{-1}(V_G^{\bar{R}}(\bar{I})) = \nu_G(IM)$, for every graded ideal I of R containing $\text{Ann}(M)$. Therefore ρ is continuous mapping.*

Proof. For any $Q \in \mathcal{PS}_G(M)$, we have $Q \in \rho^{-1}(V_G^{\bar{R}}(\bar{I})) \Leftrightarrow \rho(Q) \in V_G^{\bar{R}}(\bar{I}) \Leftrightarrow \bar{I} \subseteq (\overline{Gr_M(Q)} :_R M) \Leftrightarrow I \subseteq (Gr_M(Q) :_R M) \Leftrightarrow IM \subseteq (Gr_M(Q) :_R M)M \Leftrightarrow (IM :_R M) \subseteq ((Gr_M(Q) :_R M)M :_R M) = (Gr_M(Q) :_R M) \Leftrightarrow Q \in \nu_G(IM)$. Hence $\rho^{-1}(V_G^{\bar{R}}(\bar{I})) = \nu_G(IM)$. □

Proposition 2.10. *Let M be a G -graded R -module. If ρ is surjective, then ρ is both open and closed; more precisely, for any $N \leq_G M$, $\rho(\nu_G(N)) = V_G^{\bar{R}}((N :_R M))$ and $\rho(\mathcal{PS}_G(M) - \nu_G(N)) = \text{Spec}_G(\bar{R}) - V_G^{\bar{R}}((N :_R M))$.*

Proof. By Proposition 2.9, we have $\rho^{-1}(V_G^{\bar{R}}(\bar{I})) = \nu_G(IM)$ for every $I \triangleleft_G R$ containing $\text{Ann}(M)$. So $\rho^{-1}(V_G^{\bar{R}}((N :_R M))) = \nu_G((N :_R M)M) = \nu_G(N)$ for any $N \leq_G M$. It follows that $V_G^{\bar{R}}((N :_R M)) = \rho(\rho^{-1}(V_G^{\bar{R}}((N :_R M)))) = \rho(\nu_G(N))$ as ρ is surjective. For the second part, note that $\mathcal{PS}_G(M) - \nu_G(N) = \mathcal{PS}_G(M) - \rho^{-1}(V_G^{\bar{R}}((N :_R M))) = \rho^{-1}(\text{Spec}_G(\bar{R})) - \rho^{-1}(V_G^{\bar{R}}((N :_R M))) = \rho^{-1}(\text{Spec}_G(\bar{R}) - V_G^{\bar{R}}((N :_R M)))$. This implies that $\rho(\mathcal{PS}_G(M) - \nu_G(N)) = \rho(\rho^{-1}(\text{Spec}_G(\bar{R}) - V_G^{\bar{R}}((N :_R M)))) = \text{Spec}_G(\bar{R}) - V_G^{\bar{R}}((N :_R M))$. □

Corollary 2.11. *Let M be a G -graded R -module. Then ρ is bijective if and only if ρ is a homeomorphism.*

The following theorem is a result for Lemma 2.8, Proposition 2.9 and Proposition 2.10.

Theorem 2.12. *Let M be a G -graded R -module. Consider the following statements:*

- (1) $\text{Spec}_G(M)$ is connected.

- (2) $\mathcal{PS}_G(M)$ is connected.
- (3) $\text{Spec}_G(\overline{R})$ is connected.

- (i) If ρ is surjective, then $(1) \Rightarrow (2) \Leftrightarrow (3)$.
- (ii) If φ is surjective, then all the three statements are equivalent.

Proof. (i) $(1) \Rightarrow (2)$: Assume that $\text{Spec}_G(M)$ is connected. If $\mathcal{PS}_G(M)$ is disconnected, then there is a U clopen in $\mathcal{PS}_G(M)$ such that $U \neq \emptyset$ and $U \neq \mathcal{PS}_G(M)$. Since U is clopen in $\mathcal{PS}_G(M)$, then $U = \mathcal{PS}_G(M) - \nu_G(N_1) = \nu_G(N_2)$ for some $N_1, N_2 \leq_G M$. By Proposition 2.10, we have $\rho(U)$ is clopen in $\text{Spec}_G(\overline{R})$. But φ is continuous. So $\varphi^{-1}(\rho(U))$ is clopen in $\mathcal{PS}_G(M)$, and so $\varphi^{-1}(\rho(U)) = \emptyset$ or $\varphi^{-1}(\rho(U)) = \text{Spec}_G(M)$. If $\varphi^{-1}(\rho(U)) = \emptyset$, then $\varphi^{-1}(\rho(\nu_G(N_2))) = \emptyset$. Thus $\varphi^{-1}(V_G^{\overline{R}}(\overline{(N_2 :_R M)})) = \emptyset$. It follows that $V_G(N_2) = \emptyset$, which means that $(N_2 :_R M) \not\subseteq (P :_R M)$ for any $P \in \text{Spec}_G(M)$. As $U = \nu_G(N_2) \neq \emptyset$, then $\exists Q \in \mathcal{PS}_G(M)$ such that $(N_2 :_R M) \subseteq (Gr_M(Q) :_R M)$. Since $Gr_M(Q) \neq M$, then $\exists P' \in \text{Spec}_G(M)$ such that $Q \subseteq P'$. Therefore $(Gr_M(Q) :_R M) \subseteq (Gr_M(P') :_R M) = (P' :_R M)$. Hence $(N_2 :_R M) \subseteq (P' :_R M)$ which is a contradiction. Now, if $\varphi^{-1}(\rho(U)) = \text{Spec}_G(M)$, then $\text{Spec}_G(M) = \varphi^{-1}(\rho(\nu_G(N_2))) = V_G(N_2) = \text{Spec}_G(M) \cap \nu_G(N_2)$. It follows that $\text{Spec}_G(M) \subseteq \nu_G(N_2) = U = \mathcal{PS}_G(M) - \nu_G(N_1)$. As $U = \mathcal{PS}_G(M) - \nu_G(N_1) \neq \mathcal{PS}_G(M)$, then $\exists Q \in \mathcal{PS}_G(M) \cap \nu_G(N_1)$. Therefore $(N_1 :_R M) \subseteq (Gr_M(Q) :_R M)$ and $\exists P \in \text{Spec}_G(M)$ such that $Q \subseteq P$. It follows that $(N_1 :_R M) \subseteq (Gr_M(Q) :_R M) \subseteq (Gr_M(P) :_R M)$. Then $P \in \nu_G(N_1)$. But $P \in \text{Spec}_G(M) \subseteq \mathcal{PS}_G(M) - \nu_G(N_1)$. Therefore $P \notin \nu_G(N_1)$ which is a contradiction. Consequently, $\mathcal{PS}_G(M)$ is a connected space.

$(2) \Rightarrow (3)$: Since ρ is continuous surjective map and $\mathcal{PS}_G(M)$ is connected, then $\text{Spec}_G(\overline{R})$ is connected.

$(3) \Rightarrow (2)$: Assume by way of contradiction that $\mathcal{PS}_G(M)$ is disconnected. Then there is a U clopen in $\mathcal{PS}_G(M)$ such that $U \neq \emptyset$ and $U \neq \mathcal{PS}_G(M)$. Since ρ is surjective, then $\rho(U)$ is clopen in $\text{Spec}_G(\overline{R})$, and so $\rho(U) = \emptyset$ or $\rho(U) = \text{Spec}_G(\overline{R})$ as $\text{Spec}_G(\overline{R})$ is connected. Also, since U is open in $\mathcal{PS}_G(M)$, then $U = \mathcal{PS}_G(M) - \nu_G(N)$ for some $N \leq_G M$. Now, if $\rho(U) = \text{Spec}_G(\overline{R})$, then $\text{Spec}_G(\overline{R}) = \rho(\mathcal{PS}_G(M) - \nu_G(N)) = \text{Spec}_G(\overline{R}) - V_G^{\overline{R}}(\overline{(N :_R M)})$ by Proposition 2.10. Thus $V_G^{\overline{R}}(\overline{(N :_R M)}) = \emptyset$ which implies that $\emptyset = \rho^{-1}(\emptyset) = \rho^{-1}(V_G^{\overline{R}}(\overline{(N :_R M)})) = \nu_G(N)$. Therefore $\nu_G(N) = \emptyset$ and hence $U = \mathcal{PS}_G(M)$, a contradiction. Also if $\rho(U) = \emptyset$, then $U \subseteq \rho^{-1}(\rho(U)) = \emptyset$. It follows that $U = \emptyset$ which is also a contradiction. Therefore $\mathcal{PS}_G(M)$ is connected.

(ii) If φ is surjective, then it is clear that ρ is surjective and hence $(1) \Rightarrow (2) \Leftrightarrow (3)$ by (i). So it is enough to show that $(3) \Rightarrow (1)$. We prove it in a similar way to proof of $(3) \Rightarrow (2)$ using the properties of φ . So again, we assume by way of contradiction that $\text{Spec}_G(M)$ is disconnected, which means that there is a U clopen in $\text{Spec}_G(M)$ such that $U \neq \emptyset$ and $U \neq \text{Spec}_G(M)$. As $\text{Spec}_G(\overline{R})$ is connected and φ is surjective, then $\varphi(U) = \emptyset$ or $\varphi(U) = \text{Spec}_G(\overline{R})$. Also the open set U can be written as $U = \text{Spec}_G(M) - V_G(N)$ for some $N \leq_G M$. It is clear that

if $\varphi(U) = \emptyset$, then $U = \emptyset$ and we have a contradiction. If $\varphi(U) = \text{Spec}_G(\overline{R})$, then $\text{Spec}_G(\overline{R}) = \varphi(\text{Spec}_G(M) - V_G(N)) = \text{Spec}_G(\overline{R}) - V_G^R((N :_R M))$, and so $V_G^R((N :_R M)) = \emptyset$. Thus $V_G(N) = \varphi^{-1}(V_G^R((N :_R M))) = \varphi^{-1}(\emptyset) = \emptyset$. It follows that $U = \text{Spec}_G(M)$ which is a contradiction. Therefore $\text{Spec}_G(M)$ is a connected space and this completes the proof. \square

Let M and S be two G -graded R -modules. Recall that an R -module homomorphism $f : M \rightarrow S$ is called a G -graded R -module homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$, see [13]. Let $f : M \rightarrow M'$ be a G -graded module epimorphism between the two graded modules M and M' . If $N' \leq_G M'$, then $(N' :_R M') = (f^{-1}(N') :_R M)$. Also it is easy to check that, $(N :_R M) = (f(N) :_R M')$ and $f(\text{Gr}_M(N)) = \text{Gr}_{M'}(f(N))$ for any $N \leq_G M$ containing the kernel of f . We will denote the kernel of f by $\ker f$.

Lemma 2.13. *Let M and M' be G -graded R -modules. Let $f : M \rightarrow M'$ be a G -graded module epimorphism. Then the following hold:*

- (1) *If $Q' \in \mathcal{PS}_G(M')$, then $f^{-1}(Q') \in \mathcal{PS}_G(M)$.*
- (2) *If $Q \in \mathcal{PS}_G(M)$ and $\ker f \subseteq Q$, then $f(Q) \in \mathcal{PS}_G(M')$.*

Proof. (1) It is easy to verify that $f^{-1}(Q')$ is graded primary submodule of M and it remains to show that $(\text{Gr}_M(f^{-1}(Q')) :_R M) = \text{Gr}((f^{-1}(Q') :_R M))$. Since $\ker f \subseteq f^{-1}(Q') \subseteq \text{Gr}_M(f^{-1}(Q'))$ and $(\text{Gr}_{M'}(Q') :_R M') = \text{Gr}((Q' :_R M'))$, we obtain $(\text{Gr}_M(f^{-1}(Q')) :_R M) = (f(\text{Gr}_M(f^{-1}(Q')))) :_R M' = (\text{Gr}_{M'}(f(f^{-1}(Q')))) :_R M' = (\text{Gr}_{M'}(Q') :_R M') = \text{Gr}((Q' :_R M')) = \text{Gr}((f^{-1}(Q') :_R M))$ as required.

(2) First note that $f(Q)$ is a graded proper submodule of M' , since Q is a graded proper submodule of M containing $\ker f$. Let $rm' \in f(Q)$ for $r \in h(R)$ and $m' \in h(M')$. As f is a graded module epimorphism and $m' \in h(M')$, we get $\exists m \in h(M)$ such that $f(m) = m'$, which implies that $f(rm) \in f(Q)$. Thus $\exists t \in Q$ such that $rm - t \in \ker f \subseteq Q$. So $rm \in Q$, and so $m \in Q$ or $r \in \sqrt{(Q :_R M)} = \sqrt{(f(Q) :_R M')}$. Hence $f(Q)$ is a graded primary submodule of M' . Moreover, $(\text{Gr}_{M'}(f(Q)) :_R M') = (f(\text{Gr}_M(Q)) :_R M') = (\text{Gr}_M(Q) :_R M) = \text{Gr}((Q :_R M)) = \text{Gr}((f(Q) :_R M'))$. Therefore $f(Q) \in \mathcal{PS}_G(M')$. \square

Theorem 2.14. *Let M and M' be G -graded R -modules and $f : M \rightarrow M'$ be a graded module epimorphism. Then the mapping $\pi : \mathcal{PS}_G(M') \rightarrow \mathcal{PS}_G(M)$ by $\pi(Q') = f^{-1}(Q')$ is an injective continuous map. Moreover, if π is surjective map, then $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$.*

Proof. By Lemma 2.13, π is well-defined. Also, the injectivity of π is obvious. Now for any $O \in \mathcal{PS}_G(M')$ and any closed set $\nu_G(N)$ in $\mathcal{PS}_G(M)$, where $N \leq_G M$, we have $O \in \pi^{-1}(\nu_G(N)) = \pi^{-1}(\nu_G^*(\text{Gr}((N :_R M))M)) \Leftrightarrow \text{Gr}((N :_R M))M \subseteq \text{Gr}_M(f^{-1}(O)) \Leftrightarrow \text{Gr}((N :_R M)) \subseteq (\text{Gr}_M(f^{-1}(O)) :_R M) = \text{Gr}((f^{-1}(O) :_R M)) = \text{Gr}((O :_R M')) = (\text{Gr}_{M'}(O) :_R M') \Leftrightarrow \text{Gr}((N :_R M))M' \subseteq \text{Gr}_{M'}(O) \Leftrightarrow O \in \nu_G^*(\text{Gr}((N :_R M))M') = \nu_G(\text{Gr}((N :_R M))M')$. Therefore $\pi^{-1}(\nu_G(N)) = \nu_G(\text{Gr}((N :_R M))M')$ and hence π is continuous.

For the last statement, we assume that π is surjective and it is enough to show that π is closed. So let $\nu_G(N')$ be a closed set in $\mathcal{PS}_G(M')$, where $N' \leq_G M'$. As we have seen, $\pi^{-1}(\nu_G(N)) = \nu_G(\text{Gr}((N :_R M))M')$ for any $N \leq_G M$. It follows that $\pi^{-1}(\nu_G(f^{-1}(N'))) = \nu_G(\text{Gr}((f^{-1}(N') :_R M))M') = \nu_G(\text{Gr}((N' :_R M'))M') = \nu_G(N')$ and hence $\pi^{-1}(\nu_G(f^{-1}(N'))) = \nu_G(N')$. Thus $\pi(\nu_G(N')) = \nu_G(f^{-1}(N'))$ as π is surjective. Therefore π is closed, and so $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$. \square

Corollary 2.15. *Let M and M' be G -graded R -modules. Let $f : M \rightarrow M'$ be a G -graded module isomorphism. Then $\mathcal{PS}_G(M)$ is homeomorphic to $\mathcal{PS}_G(M')$.*

Proof. By Lemma 2.13 (2) and Theorem 2.14. \square

3. A BASE FOR THE ZARISKI TOPOLOGY ON $\mathcal{PS}_G(M)$

Let M be a G -graded R -module. In [14, Theorem 2.3], it has been proved that for each $r \in h(R)$, the set $D_r = \text{Spec}_G(R) - V_G^R(rR)$ is open in $\text{Spec}_G(R)$ and the family $\{D_r \mid r \in h(R)\}$ is a base for the Zariski topology on $\text{Spec}_G(R)$. In addition, each D_r is compact and thus $D_1 = \text{Spec}_G(R)$ is compact. In this section, we set $S_r = \mathcal{PS}_G(M) - \nu_G(rM)$ for each $r \in h(R)$ and prove that $S = \{S_r \mid r \in h(R)\}$ forms a base for the Zariski-topology on $\mathcal{PS}_G(M)$. Also, we show that each S_r is compact and hence $\mathcal{PS}_G(M)$ is compact.

Proposition 3.1. *For any G -graded R -module M , the set $S = \{S_r \mid r \in h(R)\}$ forms a base for the Zariski topology on $\mathcal{PS}_G(M)$.*

Proof. Let $U = \mathcal{PS}_G(M) - \nu_G(N)$ be an open set in $\mathcal{PS}_G(M)$, where $N \leq_G M$. Let $Q \in U$ and it is enough to find an element $r \in h(R)$ such that $Q \in S_r \subseteq U$. Since $Q \in U$, then $(N :_R M) \not\subseteq (\text{Gr}_M(Q) :_R M)$, and so there exists $x \in R$ and $g \in G$ such that $x_g \in (N :_R M) - (\text{Gr}_M(Q) :_R M)$. Take $r = x_g \in h(R)$. Therefore $(rM :_R M) \not\subseteq (\text{Gr}_M(Q) :_R M)$ and thus $Q \in S_r$. Now for any $Q' \in S_r$, we have $(rM :_R M) \not\subseteq (\text{Gr}_M(Q') :_R M)$, which implies that $(N :_R M) \not\subseteq (\text{Gr}_M(Q') :_R M)$. Thus $Q' \in U$ and hence $Q \in S_r \subseteq U$ which completes the proof. \square

Lemma 3.2 ([14, Theorem 2.3 (2)]). *Let R be a G -graded ring. Then $D_r \cap D_t = D_{rt}$ for any $r, t \in h(R)$.*

Let R be a G -graded ring. As usual, the nilradical of R and the set of all units of R will be denoted by $N(R)$ and $U(R)$, respectively.

Proposition 3.3. *Let M be a G -graded R -module and $r \in h(R)$. Then,*

- (1) $\rho^{-1}(D_{\bar{r}}) = S_r$
- (2) $\rho(S_r) \subseteq D_{\bar{r}}$. If ρ is surjective, then the equality holds.
- (3) $S_r \cap S_t = S_{rt}$, for any $r, t \in h(R)$.
- (4) If $r \in N(R)$, then $S_r = \emptyset$.
- (5) If $r \in U(R)$, then $S_r = \mathcal{PS}_G(M)$.

Proof. (1) $\rho^{-1}(D_{\bar{r}}) = \rho^{-1}(\text{Spec}_G(\bar{R}) - V_G^{\bar{R}}(\bar{r}\bar{R})) = \mathcal{PS}_G(M) - \rho^{-1}(V_G^{\bar{R}}(\bar{r}\bar{R})) = \mathcal{PS}_G(M) - \nu_G(rM) = S_r$ by Proposition 2.9.

(2) Trivial.

(3) For any $r, t \in h(R)$, we have $\bar{r}, \bar{t} \in h(\bar{R})$ and hence $D_{\bar{r}} \cap D_{\bar{t}} = D_{\bar{r}\bar{t}}$ by Lemma 3.2. It follows that $S_r \cap S_t = \rho^{-1}(D_{\bar{r}}) \cap \rho^{-1}(D_{\bar{t}}) = \rho^{-1}(D_{\bar{r}\bar{t}}) = S_{rt}$.

(4) Assume that $r \in N(R)$. It follows that $D_r = \emptyset$ by [18, Proposition 3.6 (2)], and thus $D_{\bar{r}} = \emptyset$. Therefore $S_r = \rho^{-1}(D_{\bar{r}}) = \emptyset$ by (1).

(5) Assume that $r \in U(R)$. By [18, Proposition 3.6 (3)], we have $D_r = \text{Spec}_G(R)$ and hence $D_{\bar{r}} = \text{Spec}_G(\bar{R})$. By (1), we obtain $S_r = \rho^{-1}(D_{\bar{r}}) = \rho^{-1}(\text{Spec}_G(\bar{R})) = \mathcal{PS}_G(M)$. \square

In part (a) of the next example, we see that if F is a G -graded field and M is a G -graded F -module, then the Zariski topology on $\mathcal{PS}_G(M)$ is the trivial topology. However, a G -graded ring R for which a G -graded R -module M , the Zariski topology on $\mathcal{PS}_G(M)$ is the indiscrete topology is not necessarily a G -graded field and this will be discussed in part (b).

Example 3.4. (a) Let F be a G -graded field and M be a G -graded F -module. Then any non-zero homogeneous element of F is unit. By Proposition 3.3 (5), we have $S_r = \mathcal{PS}_G(M)$ for any non-zero homogeneous element r of F . Also $S_0 = \mathcal{PS}_G(M) - \nu_G(0) = \emptyset$ and hence $S = \{S_r \mid r \in h(F)\} = \{\mathcal{PS}_G(M), \emptyset\}$. Therefore, the Zariski topology on $\mathcal{PS}_G(M)$ is the trivial topology on $\mathcal{PS}_G(M)$. (b) Let $R = \mathbb{Z}_8$ as a \mathbb{Z}_2 -graded \mathbb{Z}_8 module by $R_0 = \mathbb{Z}_8$ and $R_1 = \{0\}$. Note that $1, 3, 5, 7 \in h(\mathbb{Z}_8) \cap U(\mathbb{Z}_8)$. So $S_1 = S_3 = S_5 = S_7 = \mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)$ by Proposition 3.3 (5). Also $S_0 = S_2 = S_4 = S_6 = \emptyset$ by Proposition 3.3 (4), since $0, 2, 4, 6 \in N(\mathbb{Z}_8) \cap h(\mathbb{Z}_8)$. Now $S = \{S_r \mid r \in h(R)\} = \{\emptyset, \mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)\}$ and hence the Zariski topology on $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_8)$ is the trivial topology. But \mathbb{Z}_8 is not \mathbb{Z}_2 -graded field.

Theorem 3.5. *Let M be a G -graded R -module. If ρ is surjective, then the open set S_r in $\mathcal{PS}_G(M)$ for each $r \in h(R)$ is compact; in particular, the space $\mathcal{PS}_G(M)$ is compact.*

Proof. Let $r \in h(R)$ and $\zeta = \{S_t \mid t \in \Delta\}$ be a basic open cover for S_r , where Δ is a subset of $h(R)$. Then $S_r \subseteq \bigcup_{t \in \Delta} S_t$ and thus $D_{\bar{r}} = \rho(S_r) \subseteq \bigcup_{t \in \Delta} \rho(S_t) = \bigcup_{t \in \Delta} D_{\bar{t}}$ by Proposition 3.3 (2). Then $\bar{\zeta} = \{D_{\bar{t}} \mid t \in \Delta\}$ is a basic open cover for the compact set $D_{\bar{r}}$ and hence it has a finite subcover $\bar{\zeta} = \{D_{\bar{t}_i} \mid i = 1, \dots, n\}$, where $t_i \in \Delta$ for any $i = 1, \dots, n$. This means that $D_{\bar{r}} \subseteq \bigcup_{i=1}^n D_{\bar{t}_i}$ and it follows that $S_r = \rho^{-1}(D_{\bar{r}}) \subseteq \bigcup_{i=1}^n \rho^{-1}(D_{\bar{t}_i}) = \bigcup_{i=1}^n S_{t_i}$ by Proposition 3.3 (1). Therefore $\zeta = \{S_{t_i} \mid i = 1, \dots, n\} \subseteq \zeta$ is a finite subcover for S_r . For the other part of the theorem, since $\mathcal{PS}_G(M) = S_1$, then $\mathcal{PS}_G(M)$ is compact. \square

Theorem 3.6. *Let M be a G -graded R -module. If ρ is surjective, then the compact open sets of $\mathcal{PS}_G(M)$ are closed under finite intersection and form an open base.*

Proof. Let C_1, C_2 be quasi compact open sets of $\mathcal{PS}_G(M)$ and $\zeta = \{S_r \mid r \in \Delta\}$ be a basic open cover for $C_1 \cap C_2$, where Δ is a subset of $h(R)$. Since S is a base for the Zariski topology on $\mathcal{PS}_G(M)$, then the compact open sets C_1, C_2 can be written as a finite union of elements of S . So let $C_1 = \bigcup_{i=1}^n S_{t_i}$

and $C_2 = \bigcup_{j=1}^m S_{z_j}$. By Proposition 3.3 (3), we have $C_1 \cap C_2 = \bigcup_{i,j} (S_{t_i} \cap S_{z_j}) = \bigcup_{i,j} S_{t_i z_j} \subseteq \bigcup_{r \in \Delta} S_r$. Note that for any i, j we have $t_i z_j \in h(R)$ as $t_i, z_j \in h(R)$.

So without loss of generality we can assume that $C_1 \cap C_2 = \bigcup_{k=1}^L S_{h_k}$ where $h_k \in h(R)$, for $k = 1, \dots, L$. Then $S_{h_k} \subseteq \bigcup_{r \in \Delta} S_r$ for each k . Now each S_{h_k}

is compact by Theorem 3.5 and it follows that $S_{h_k} \subseteq \bigcup_{i=1}^{d_k} S_{r_{k,i}}$, where $d_k \geq 1$ depends on k and $r_{k,i} \in \Delta$, for any $k = 1, \dots, L$ and $i = 1, \dots, d_k$. Therefore $C_1 \cap C_2 = \bigcup_{k=1}^L S_{h_k} \subseteq \bigcup_{k=1}^L \bigcup_{i=1}^{d_k} S_{r_{k,i}}$ and thus $\bar{\zeta} = \{S_{r_{k,i}} \mid k = 1, \dots, L, i = 1, \dots, d_k\}$ is a finite subcover for $C_1 \cap C_2$. The other part of the theorem is trivially true. \square

4. IRREDUCIBILITY IN $\mathcal{PS}_G(M)$

Let M be a G -graded R -module and Y be a subset of $\mathcal{PS}_G(M)$. We will denote the closure of Y in $\mathcal{PS}_G(M)$ by $Cl(Y)$ and the intersection $\bigcap_{Q \in Y} Gr_M(Q)$ by $\eta(Y)$. If Z is a subset of $Spec_G(R)$ or $Spec_G(M)$, then the intersection of all members of Z will be expressed by $\gamma(Z)$.

Proposition 4.1. *Let M be a G -graded R -module and $Y \subseteq \mathcal{PS}_G(M)$. Then $Cl(Y) = \nu_G(\eta(Y))$. Thus, Y is closed in $\mathcal{PS}_G(M)$ if and only if $\nu_G(\eta(Y)) = Y$.*

Proof. Let $\nu_G(N)$ be any closed set containing Y , where $N \leq_G M$. Note that $Y \subseteq \nu_G(\eta(Y))$, and so it is enough to show that $\nu_G(\eta(Y)) \subseteq \nu_G(N)$. So let $Q \in \nu_G(\eta(Y))$. Then $(\eta(Y) :_R M) \subseteq (Gr_M(Q) :_R M)$. Note that for any $Q' \in Y$, we have $(N :_R M) \subseteq (Gr_M(Q') :_R M)$ and hence $(N :_R M) \subseteq \bigcap_{Q' \in Y} (Gr_M(Q') :_R M) = (\bigcap_{Q' \in Y} Gr_M(Q') :_R M) = (\eta(Y) :_R M) \subseteq (Gr_M(Q) :_R M)$. Thus $Q \in \nu_G(N)$. Therefore $\nu_G(\eta(Y))$ is the smallest closed set containing Y and hence $Cl(Y) = \nu_G(\eta(Y))$. \square

Recall that a topological space X is irreducible if any two non-empty open subsets of X intersect. Equivalently, X is irreducible if for any decomposition $X = F_1 \cup F_2$ with closed subsets F_i of X with $i = 1, 2$, we have $F_1 = X$ or $F_2 = X$. A subset X' of X is irreducible if it is an irreducible topological space with the induced topology. Let X be a topological space. Then a subset Y of X is irreducible if and only if its closure is irreducible. Also every singleton subset of X is irreducible, (see [6]).

Theorem 4.2. *Let M be a G -graded R -module. Then for each $Q \in \mathcal{PS}_G(M)$, the closed set $\nu_G(Q)$ is irreducible closed subset of $\mathcal{PS}_G(M)$. In particular, if $\{0\} \in \mathcal{PS}_G(M)$, then $\mathcal{PS}_G(M)$ is irreducible.*

Proof. For any $Q \in \mathcal{PS}_G(M)$, we have $Cl(\{Q\}) = \nu_G(\eta(\{Q\})) = \nu_G(Gr_M(Q)) = \nu_G(Q)$. Now $\{Q\}$ is irreducible in $\mathcal{PS}_G(M)$, then its closure $\nu_G(Q)$ is irreducible. The other part of the theorem follows from the equality $\nu_G(\{0\}) = \mathcal{PS}_G(M)$. \square

In Theorem 4.2, if we drop the condition that $Q \in \mathcal{PS}_G(M)$, then $\nu_G(Q)$ might not be irreducible. Actually, $\mathcal{PS}_G(M)$ itself is not always irreducible. For this, if we take $R = \mathbb{Z}_6$ as \mathbb{Z}_2 -graded \mathbb{Z}_6 module by $R_0 = R$ and $R_1 = \{0\}$. Then it can easily be checked that $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \{\{0, 3\}, \{0, 2, 4\}\}$, $\nu_{\mathbb{Z}_2}(3\mathbb{Z}_6) = \{\{0, 3\}\}$ and $\nu_{\mathbb{Z}_2}(2\mathbb{Z}_6) = \{\{0, 2, 4\}\}$. Now $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \nu_{\mathbb{Z}_2}(3\mathbb{Z}_6) \cup \nu_{\mathbb{Z}_2}(2\mathbb{Z}_6)$. But $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) \neq \nu_{\mathbb{Z}_2}(3\mathbb{Z}_6)$ and $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) \neq \nu_{\mathbb{Z}_2}(2\mathbb{Z}_6)$. Therefore $\mathcal{PS}_{\mathbb{Z}_2}(\mathbb{Z}_6) = \nu_{\mathbb{Z}_2}(\{0\})$ is not irreducible.

Now, we need the following lemma to prove the next theorem.

Lemma 4.3 ([10, Lemma 4.9]). *A subset Y of $Spec_G(R)$ for any graded ring R is irreducible if and only if $\gamma(Y)$ is a graded prime ideal of R .*

Proof. \Rightarrow : Let Y be irreducible subset of $Spec_G(R)$ and $r_1, r_2 \in h(R)$ with $r_1r_2 \in \gamma(Y)$. Then $r_1r_2 \in p$ for any $p \in Y$. Let $U_1 = Y \cap (Spec_G(R) - V_G^R(r_1R))$ and $U_2 = Y \cap (Spec_G(R) - V_G^R(r_2R))$. If U_1, U_2 are non-empty sets, then $U_1 \cap U_2 \neq \emptyset$ as Y is irreducible and U_1, U_2 are open sets in Y . So $\exists p \in Y$ such that $r_1R \not\subseteq p$ and $r_2R \not\subseteq p$. It follows that $r_1 \notin p$ and $r_2 \notin p$ and hence $r_1r_2 \notin p$ as $p \in Spec_G(R)$, a contradiction. Therefore $U_1 = \emptyset$ or $U_2 = \emptyset$. If $U_1 = \emptyset$, then $Y \subseteq V_G^R(r_1R)$. This implies that $r_1R \subseteq Q$ for any $Q \in Y$ and thus $r_1R \subseteq \bigcap_{Q \in Y} Q = \gamma(Y)$. Therefore $r_1 \in \gamma(Y)$. Similarly, if $U_2 \neq \emptyset$, then

$r_2 \in \gamma(Y)$. Hence $r_1 \in \gamma(Y)$ or $r_2 \in \gamma(Y)$.

\Leftarrow : Assume that $\gamma(Y)$ is a graded prime ideal of R , where $Y \subseteq Spec_G(R)$. Let $Y = F_1 \cup F_2$, where F_1, F_2 are closed sets in Y . Now $F_1 = V_G^R(I_1) \cap Y$ and $F_2 = V_G^R(I_2) \cap Y$ for some $I_1, I_2 \triangleleft_G R$. It follows that $Y = Y \cap V_G^R(I_1 \cap I_2)$ and hence $Y \subseteq V_G^R(I_1 \cap I_2)$, which implies that $I_1 \cap I_2 \subseteq p$ for any $p \in Y$. Thus $I_1 \cap I_2 \subseteq \gamma(Y)$. Since $\gamma(Y) \in Spec_G(R)$, then $I_1 \subseteq \gamma(Y)$ or $I_2 \subseteq \gamma(Y)$. If $I_1 \subseteq \gamma(Y)$, then $V_G^R(\gamma(Y)) \subseteq V_G^R(I_1)$ and thus $Y \subseteq V_G^R(I_1)$ as $Y \subseteq V_G^R(\gamma(Y))$. It follows that $F_1 = Y$. Similarly, if $I_2 \subseteq \gamma(Y)$, we obtain $F_2 = Y$. This proves that $F_1 = Y$ or $F_2 = Y$, hence, Y is irreducible. \square

Theorem 4.4. *Let M be a G -graded R -module and $Y \subseteq \mathcal{PS}_G(M)$. Then:*

- (1) *If $\eta(Y)$ is graded primary submodule of M , then Y is irreducible.*
- (2) *If Y is irreducible, then $\Upsilon = \{(Gr_M(Q) :_R M) \mid Q \in Y\}$ is an irreducible subset of $Spec_G(R)$, i.e., $\gamma(\Upsilon) = (\eta(Y) :_R M) \in Spec_G(R)$.*

Proof. (1) Assume that $\eta(Y)$ is a graded primary submodule of M , then it is easy to see that $\eta(Y) \in \mathcal{PS}_G(M)$. By Theorem 4.2 and Proposition 4.1, we have $\nu_G(\eta(Y)) = Cl(Y)$ is irreducible and hence Y is irreducible.

(2) Assume that Y is irreducible. Then $\rho(Y) = Y'$ is irreducible subset of $\text{Spec}_G(\bar{R})$ as ρ is continuous by Proposition 2.9. Note that $\gamma(Y') = \gamma(\rho(Y)) = \bigcap_{Q \in Y'} (\overline{Gr_M(Q) :_R M}) = (\bigcap_{Q \in Y} \overline{Gr_M(Q) :_R M}) = \overline{(\eta(Y) :_R M)}$ and hence $\gamma(Y') = \overline{(\eta(Y) :_R M)} \in \text{Spec}_G(\bar{R})$ by Lemma 4.3. It follows that $(\eta(Y) :_R M) \in \text{Spec}_G(R)$. Now $\gamma(\Upsilon) = \bigcap_{Q \in Y} (\overline{Gr_M(Q) :_R M}) = (\bigcap_{Q \in Y} \overline{Gr_M(Q) :_R M}) = \overline{(\eta(Y) :_R M)} \in \text{Spec}_G(R)$ and thus Υ is irreducible subset of $\text{Spec}_G(R)$ by Lemma 4.3 again. \square

Let X be a topological space and Y be a closed subset of X . An element $y \in Y$ is called a generic point if $Y = Cl(\{y\})$. An irreducible component of X is a maximal irreducible subset of X . The irreducible components of X are closed and they cover X , (see [9]).

Theorem 4.5. *Let M be a G -graded R -module. Let $Y \subseteq \mathcal{PS}_G(M)$ and ρ be surjective. Then Y is an irreducible closed subset of $\mathcal{PS}_G(M)$ if and only if $Y = \nu_G(Q)$ for some $Q \in \mathcal{PS}_G(M)$. Hence every irreducible closed subset of $\mathcal{PS}_G(M)$ has a generic point.*

Proof. Assume that Y is an irreducible closed subset of $\mathcal{PS}_G(M)$. Then $Y = \nu_G(N)$ for some $N \leq_G M$. Also $(\eta(Y) :_R M) = (\eta(\nu_G(N)) :_R M) \in \text{Spec}_G(R)$ by Theorem 4.4. It follows that $\overline{(\eta(Y) :_R M)} \in \text{Spec}_G(\bar{R})$ and hence $\exists Q \in \mathcal{PS}_G(M)$ such that $(\overline{Gr_M(Q) :_R M}) = \overline{(\eta(\nu_G(N)) :_R M)}$ as ρ is surjective. So $Gr((\eta(Y) :_R M)) = Gr(\overline{(\eta(\nu_G(N)) :_R M)})$ and so $\nu_G(Q) = \nu_G(\eta(\nu_G(N))) = Cl(\nu_G(N)) = \nu_G(N) = Y$ by Proposition 4.1 and Lemma 2.4 (3). Conversely, if $Y = \nu_G(Q)$ for some $Q \in \mathcal{PS}_G(M)$, then Y is irreducible by Theorem 4.2. \square

Theorem 4.6. *Let M be a G -graded R -module and $Q \in \mathcal{PS}_G(M)$. If $(\overline{Gr_M(Q) :_R M})$ is a minimal graded prime ideal of \bar{R} , then $\nu_G(Q)$ is irreducible component of $\mathcal{PS}_G(M)$. The converse is true if ρ is surjective.*

Proof. Note that $\nu_G(Q)$ is irreducible by Theorem 4.2 and it remains to show that it is a maximal irreducible. Let Y be irreducible subset of $\mathcal{PS}_G(M)$ with $\nu_G(Q) \subseteq Y$ and if we show that $Y = \nu_G(Q)$, then we are done. Since $Q \in \nu_G(Q) \subseteq Y$, then $Q \in Y$ and thus $\overline{(\eta(Y) :_R M)} \subseteq \overline{(\overline{Gr_M(Q) :_R M})}$. It follows that $\overline{(\eta(Y) :_R M)} = \overline{(\overline{Gr_M(Q) :_R M})}$ as $\overline{(\overline{Gr_M(Q) :_R M})}$ is a minimal graded prime ideal of \bar{R} and $\overline{(\eta(Y) :_R M)} \in \text{Spec}_G(\bar{R})$ by Theorem 4.4 (2). Hence $V_G^{\bar{R}}(\overline{(\eta(Y) :_R M)}) = V_G^{\bar{R}}(\overline{(\overline{Gr_M(Q) :_R M})})$, which implies that $\nu_G(\eta(Y)) = \rho^{-1}(V_G^{\bar{R}}(\overline{(\eta(Y) :_R M)})) = \rho^{-1}(V_G^{\bar{R}}(\overline{(\overline{Gr_M(Q) :_R M})})) = \nu_G(Gr_M(Q)) = \nu_G(Q)$ by Proposition 2.9 and Lemma 2.4 (4). Since $Y \subseteq \nu_G(\eta(Y))$, then $Y \subseteq \nu_G(Q)$ and thus $Y = \nu_G(Q)$. For the converse, we assume that ρ is surjective. Since $Q \in \mathcal{PS}_G(M)$, then $\overline{(\overline{Gr_M(Q) :_R M})} \in \text{Spec}_G(\bar{R})$. Let $\bar{J} \in \text{Spec}_G(\bar{R})$ with $\bar{J} \subseteq \overline{(\overline{Gr_M(Q) :_R M})}$ and it is enough to show that $\bar{J} = \overline{(\overline{Gr_M(Q) :_R M})}$. Note that $\exists Q' \in \mathcal{PS}_G(M)$ such that $\rho(Q') = \bar{J}$ as ρ is surjective. So we have $J = (Gr_M(Q') :_R M)$. Now $(Gr_M(Q') :_R M) \subseteq (\overline{Gr_M(Q) :_R M})$ and thus

$\nu_G(Q) \subseteq \nu_G(Q')$. Since $\nu_G(Q)$ is irreducible component and $\nu_G(Q')$ is irreducible by Theorem 4.2, then $\nu_G(Q) = \nu_G(Q')$. By Lemma 2.4 (3), we get $(Gr_M(Q) :_R M) = (Gr_M(Q') :_R M) = J$ and thus $\overline{J} = \overline{(Gr_M(Q) :_R M)}$. \square

Corollary 4.7. *Let M be a G -graded R -module and $\mathcal{K} = \{Q \in \mathcal{PS}_G(M) \mid \overline{(Gr_M(Q) :_R M)} \text{ is a minimal graded prime ideal of } \overline{R}\}$. If ρ is surjective, then the following hold:*

- (1) $T = \{\nu_G(Q) \mid Q \in \mathcal{K}\}$ is the set of all irreducible components of $\mathcal{PS}_G(M)$.
- (2) $\mathcal{PS}_G(M) = \bigcup_{Q \in \mathcal{K}} \nu_G(Q)$.
- (3) $Spec_G(\overline{R}) = \bigcup_{Q \in \mathcal{K}} V_G^{\overline{R}}(\overline{(Q :_R M)})$.
- (4) $Spec_G(M) = \bigcup_{Q \in \mathcal{K}} V_G(Q)$.
- (5) If $\{0\} \in Spec_G(M)$, then the only irreducible component subset of $\mathcal{PS}_G(M)$ is $\mathcal{PS}_G(M)$ itself.

Proof. (1) follows from Theorem 4.5 and Theorem 4.6.

(2) Since any topological space is the union of its irreducible components, then $\mathcal{PS}_G(M) = \bigcup_{Y \in T} Y = \bigcup_{Q \in \mathcal{K}} \nu_G(Q)$.

(3) Since ρ is surjective, then $Spec_G(\overline{R}) = \rho(\mathcal{PS}_G(M)) = \rho(\bigcup_{Q \in \mathcal{K}} \nu_G(Q)) =$

$\bigcup_{Q \in \mathcal{K}} \rho(\nu_G(Q)) = \bigcup_{Q \in \mathcal{K}} V_G^{\overline{R}}(\overline{(Q :_R M)})$ by Proposition 2.10.

(4) $Spec_G(M) = \mathcal{PS}_G(M) \cap Spec_G(M) = (\bigcup_{Q \in \mathcal{K}} \nu_G(Q)) \cap Spec_G(M) = \bigcup_{Q \in \mathcal{K}} V_G(Q)$

by Lemma 2.4 (1).

(5) Assume that $\{0\} \in Spec_G(M)$. Then $\overline{(\{0\} :_R M)} \in Spec_G(\overline{R})$. For any $Q \in \mathcal{K}$, we have $\overline{(\{0\} :_R M)} \subseteq \overline{(Gr_M(Q) :_R M)}$ and hence $\overline{(\{0\} :_R M)} = \overline{(Gr_M(Q) :_R M)}$ as $\overline{(Gr_M(Q) :_R M)}$ is a minimal graded prime ideal of \overline{R} . Therefore $Gr((Q :_R M)) = Gr((\{0\} :_R M))$ and thus $\nu_G(Q) = \nu_G(\{0\}) = \mathcal{PS}_G(M)$ by Lemma 2.4 (3). By (1), the set of all irreducible components of $\mathcal{PS}_G(M)$ is $T = \{\nu_G(Q) \mid Q \in \mathcal{K}\} = \{\mathcal{PS}_G(M)\}$ which completes the proof. \square

Proposition 4.8. *Let R be a G -graded principal ideal domain and M be a multiplication graded R -module. Let $Y \subseteq \mathcal{PS}_G(M)$. If $\eta(Y)$ is a non-zero graded primary submodule of M , then $Y \subseteq \mathcal{PS}_G^p(M)$ for some graded maximal ideal p of R .*

Proof. Clearly, $\eta(Y) = Gr_M(\eta(Y))$. Since $\eta(Y)$ is a graded primary submodule of the graded multiplication module M , then $\eta(Y) \in Spec_G(M)$ by [15, Theorem 13] and hence $(\eta(Y) :_R M) \in Spec_G(R)$. If $(\eta(Y) :_R M) = \{0\}$, then $\eta(Y) = (\eta(Y) :_R M)M = \{0\}$, a contradiction. So $(\eta(Y) :_R M)$ is a non-zero graded prime ideal in the graded principle ideal domain R and thus $(\eta(Y) :_R M)$ is a graded maximal ideal of R . It follows that $\eta(Y)$ is a graded

maximal submodule of M as M is a graded multiplication module. Now for any $Q \in Y \subseteq \mathcal{PS}_G(M)$, we have $\eta(Y) \subseteq Gr_M(Q) \neq M$ and thus $\eta(Y) = Gr_M(Q)$. This implies that $(Gr_M(Q) :_R M) = (\eta(Y) :_R M)$ for any $Q \in Y$. Take $p = (\eta(Y) :_R M)$. Therefore $Y \subseteq \mathcal{PS}_G^p(M)$. \square

A topological space X is called a T_1 -space if every singleton subset of X is closed. A G -graded R -module M is called graded finitely generated R -module if there are $m_1, m_2, \dots, m_k \in h(M)$ such that $M = \sum_{i=1}^k Rm_i$.

Proposition 4.9. *Let M be a G -graded finitely generated R -module. If $\mathcal{PS}_G(M)$ is a T_1 -space, then $\mathcal{PS}_G(M) = Max_G(M) = Spec_G(M)$, where $Max_G(M)$ is the set of all graded maximal submodule of M .*

Proof. It is clear that $Max_G(M) \subseteq \mathcal{PS}_G(M)$. Now, let $Q \in \mathcal{PS}_G(M)$. Since $\mathcal{PS}_G(M)$ is a T_1 -space, then $Cl(\{Q\}) = \{Q\}$ and so $\nu_G(Q) = \{Q\}$ by Proposition 4.1. As $M \neq Q$ is a graded finitely generated module, we obtain M/Q is a non-zero graded finitely generated module and hence $\exists N \leq_G M$ with $Q \subseteq N$ such that $N/Q \in Max_G(M/Q)$ by [4, Lemma 2.7 (ii)]. Now, it is easy to see that $N \in Max_G(M) \subseteq \mathcal{PS}_G(M)$. Since $(Q :_R M) \subseteq (N :_R M) = (Gr_M(N) :_R M)$ and $N \in \mathcal{PS}_G(M)$, then $N \in \nu_G(Q) = \{Q\}$ and thus $N = Q \in Max_G(M)$. Therefore $Max_G(M) = \mathcal{PS}_G(M)$. Now, $Spec_G(M) \subseteq \mathcal{PS}_G(M) = Max_G(M) \subseteq Spec_G(M)$. Hence $Spec_G(M) = \mathcal{PS}_G(M) = Max_G(M)$. \square

A topological space is called a T_0 -space if the closure of any two distinct points are distinct. A topological space is called spectral space if it is homeomorphic to the prime spectrum of a ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster[9, Proposition 4] as the topological spaces X which satisfy the following conditions:

- (a) X is a T_0 -space.
- (b) X is compact.
- (c) The compact open subsets of X are closed under finite intersection and form an open base.
- (d) each irreducible closed subset of X has a generic point.

Theorem 4.10. *Let M be a G -graded R -module and ρ be surjective. Then the following statements are equivalent:*

- (1) $\mathcal{PS}_G(M)$ is a T_0 -space.
- (2) If whenever $\nu_G(Q) = \nu_G(Q')$ with $Q, Q' \in \mathcal{PS}_G(M)$, then $Q = Q'$.
- (3) ρ is injective.
- (4) $|\mathcal{PS}_G^p(M)| \leq 1$ for every $p \in Spec_G(R)$.
- (5) $\mathcal{PS}_G(M)$ is a spectral space.

Proof. The equivalence of (2), (3) and (4) is proved in Proposition 2.6. Also (1), (5) are equivalent by Theorem 3.5, Theorem 3.6 and Theorem 4.5. For (1) \Rightarrow (2), assume that $\nu_G(Q) = \nu_G(Q')$ for $Q, Q' \in \mathcal{PS}_G(M)$, then $Cl(\{Q\}) = \nu_G(Q) = \nu_G(Q') = Cl(\{Q'\})$ and hence $Q = Q'$ as $\mathcal{PS}_G(M)$ is T_0 space. For

(2) \Rightarrow (1), let $Q, Q' \in \mathcal{PS}_G(M)$ with $Q \neq Q'$, then by the assumption (2) we have $\nu_G(Q) \neq \nu_G(Q')$. Hence $Cl(\{Q\}) \neq Cl(\{Q'\})$. Therefore $\mathcal{PS}_G(M)$ is a T_0 -space. \square

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REFERENCES

- [1] K. Al-Zoubi, The graded primary radical of a graded submodules, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 1 (2016), 395–402.
- [2] K. Al-Zoubi, I. Jaradat and M. Al-Dolat, On graded P-compactly packed modules, Open Mathematics 13, no. 1 (2015), 487–492.
- [3] S. E. Atani, On graded prime submodules, Chiang Mai J. Sci. 33, no. 1 (2006), 3–7.
- [4] S. E. Atani and F. Farzalipour, Notes on the graded prime submodules, Int. Math. Forum. 1, no. 38 (2006), 1871–1880.
- [5] S. E. Atani and F. Farzalipour, On graded secondary modules, Turk. J. Math. 31 (2007), 371–378.
- [6] N. Bourbaki, Commutative Algebra. Chapter 1–7. Springer-Verlag, Berlin, 1989.
- [7] A. Y. Darani, Topology on $Spec_g(M)$, Buletinul Academiei De Stiinte 3 (67) (2011), 45–53.
- [8] J. Escoriza and B. Torrecillas, Multiplication objects in commutative Grothendieck categories, Comm. in Algebra 26, no. 6 (1998), 1867–1883.
- [9] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43–60.
- [10] M. Jaradat and K. Al-Zoubi, The Quasi-Zariski topology on the graded quasi-primary spectrum of a graded module over a graded commutative ring, preprint ([arXiv:2106.09519](https://arxiv.org/abs/2106.09519)).
- [11] C. Nastasescu and V. F. Oystaeyen, Graded and filtered rings and modules, Lecture Notes in Mathematics Vol. 758, Springer-Verlag, Berlin, 2004.
- [12] C. Nastasescu and F. Van Oystaeyen, Graded ring theory, Mathematical Library 28, North Holland, Amsterdam, 1982.
- [13] C. Nastasescu and V. F. Oystaeyen, Methods of Graded Rings, Lecture Notes in Math., Vol. 1836, Berlin-Heidelberg: Springer-Verlag, 2004.
- [14] N. A. Ozkirisci, K. H. Oral and U. Tekir, Graded prime spectrum of a graded module, Iran. J. Sci. Technol. 37A3 (2013), 411–420.
- [15] K. H. Oral, U. Tekir and A. G. Agargun, On graded prime and primary submodules, Turk. J. Math. 35 (2011), 159–167.
- [16] M. Refai, On properties of $G\text{-spec}(R)$, Sci. Math. Jpn. 53, no. 3 (2001), 411–415.
- [17] M. Refai and K. Al-Zoubi, On graded primary ideals, Turk. J. Math. 28 (2004), 217–229.
- [18] M. Refai, M. Hailat and S. Obiedat, Graded radicals on graded prime spectra, Far East J. of Math. Sci., part I (2000), 59–73.