

On w -Isbell-convexity

OLIVIER OLELA OTAFUDU^a AND KATLEGO SEBOGODI^b

Dedicated to the memory of Prof. Hans-Peter Künzi

^a School of Mathematical and Statistical Sciences North-West University, Potchefstroom Campus, Potchefstroom 2520, South Africa. (olivier.olelaotafudu@nwu.ac.za)

^b Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa. (7katlego3@gmail.com)

Communicated by M. A. Sánchez-Granero

ABSTRACT

Chistyakov introduced and developed the concept of modular metric for an arbitrary set in order to generalise the classical notion of modular on a linear space. In this article, we introduce the theory of hyperconvexity in the setting of modular pseudometric that is herein called w -Isbell-convexity. We show that on a modular set, w -Isbell-convexity is equivalent to hyperconvexity whenever the modular pseudometric is continuous from the right on the set of positive numbers.

2020 MSC: 45E35; 46A80.

KEYWORDS: modular pseudometric; Isbell-convexity; w -Isbell-convexity.

1. INTRODUCTION

Modular metric spaces were introduced by Chistyakov [8] in 2010. He developed the theory of modular metric on an arbitrary set and investigated the theory of metric spaces induced by a modular metric. He defined a modular metric in the following way. Let X be a nonempty set, then the function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is called a *modular metric* if it satisfies (a) $w(\lambda, x, y) = 0$ if and only if $x = y$ whenever $\lambda > 0$, (b) $w(\lambda, x, y) = w(\lambda, y, x)$ whenever $x, y \in X$ and $\lambda > 0$ and (c) $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y)$ whenever $x, z, y \in X$ and $\lambda, \mu > 0$.

Furthermore, the function w is said to be *modular pseudometric* on X if instead of (a), the function satisfies (d) $w(\lambda, x, x) = 0$ for all $\lambda > 0$ and $x \in X$. For $a \in X$, the modular set $X_w(a)$ is defined by $X_w = X_w(a) = \{x \in X : \lim_{\lambda \rightarrow \infty} w(\lambda, a, x) = 0\}$. Chistyakov equipped X_w with the metric q_w , where $q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$ whenever $x, y \in X_w$.

In [6], Chistyakov introduced a topology $\tau(w)$ on X_w in the following sense. A subset V of X_w is $\tau(w)$ -open if for any $\lambda > 0$ and $x \in V$, there exists $\mu > 0$ such that the entourage set $B_{\lambda, \mu}(x) = \{y \in X_w : w(\lambda, x, y) \leq \mu\} \subset V$. He studied the Hausdorff modular pseudometric on a power set of a nonempty set equipped with a modular pseudometric. In addition, Chistyakov provided an application of modular metrics which consists of an extended kind of Helly's theorem on the pointwise selection principle. This was obtained by building a special modular space, the set of all bounded and regulated mappings on an interval. Furthermore, Chistyakov considered the description of superposition operators acting in modular spaces, the existence of regular selections of set-valued maps, the new interpretation of Lipschitzian and absolutely continuous maps, and the existence of solutions to the Carathéodory-type ordinary differential equations in Banach spaces with the right-hand side from the Orlicz space.

Since Chistyakov developed the theory of metric spaces generated by a modular metric defined on an arbitrary set, the interest of this concept has grown up in the community of mathematicians, especially in operator theory where many authors are applying modular metrics to study the existence and uniqueness of fixed points of self-maps on a modular set that satisfy some particular properties (see for instance [4, 5, 20, 28]). Let mention some important and popular recent works on different type of modular metric spaces [1, 3, 10, 11, 12, 14, 15].

In addition, the well-known and important concept of hyperconvexity in a metric space has been successfully investigated and applied in many areas of mathematics and other fields. For instance the theory of hyperconvexity has been introduced in the framework of quasi-pseudometric spaces which is called Isbell-convexity (or q -hyperconvexity) (See for instance [16, 19, 21, 23, 25]). Naturally this has led us to believe that Isbell-convexity on a set equipped with a modular pseudometric should be investigated. The aim of this article is to introduce and study the theory of Isbell-convexity in the setting of modular pseudometrics and we call it w -Isbell-convexity. For instance, we study connections between hyperconvexity in a metric space and w -Isbell-convexity on a modular set. In addition, we discuss the boundedness (w -boundedness) of a set endowed with a modular pseudometric. We eventually show that a nonexpansive self-map (w -nonexpansive map) on a w -Isbell-convex modular set has a fixed point. In addition, its fixed points set is w -Isbell convex whenever its modular pseudometric is continuous from the right on the set of positive numbers.

2. PRELIMINARIES

For the comfort of the reader and in preparation of the terminology that we are going through this article, we recall the following concepts that can be found in [5, 6, 7, 8].

Definition 2.1. Consider a nonempty set X . A function $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is said to be a *modular pseudometric* on X if it satisfies the following conditions:

- (a) $w(\lambda, x, x) = 0$ for all $x \in X$ and $\lambda \in (0, \infty)$,
- (b) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $x, y \in X$ and $\lambda \in (0, \infty)$,
- (c) $w(\lambda + \mu, x, y) \leq w(\lambda, x, z) + w(\mu, z, y)$ for all $x, y, z \in X$ and $\lambda, \mu \in (0, \infty)$.

We shall say that w is a *modular metric* provided that w satisfies also the following condition: for all $x, y \in X$ and $\lambda \in (0, \infty)$,

- (d) $w(\lambda, x, y) = 0$ for all $\lambda > 0$ imply $x = y$.

Let w be a modular pseudometric on a nonempty set X . For any $x \in X$, we denote by

$$X_w = X_w(x) := \{y \in X : \lim_{\lambda \rightarrow \infty} w(\lambda, x, y) = 0\}.$$

Let us fix an element $x_0 \in X$. Then set $X_w^*(x_0)$ defined by

$$X_w^* = X_w^*(x_0) = \{x \in X : w(\lambda, x, x_0) < \infty \text{ for some } \lambda > 0\}$$

is called a *modular set* (around x_0) and x_0 is called the *center* of X_w^* .

It has been observed in [8] that it is easy to see that $X_w(x_0) \subseteq X_w^*(x_0)$. For further details on modular sets and examples of these sets, we refer the reader to [8].

The function q_w defined by

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\}$$

whenever $x, y \in X_w$ is a (pseudo) metric on X_w whenever w is modular (pseudo) metric on X . If $x, y \in X$, then q_w is an extended (pseudo) metric on X .

For any $x \in X_w$ and λ and $\mu > 0$, we define the sets $B_{\lambda, \mu}^w(x)$ and $C_{\lambda, \mu}^w(x)$ by

$$B_{\lambda, \mu}^w(x) := \{z \in X_w : w(\lambda, x, z) < \mu\}$$

and

$$C_{\lambda, \mu}^w(x) := \{z \in X_w : w(\lambda, x, z) \leq \mu\}.$$

Then the set $B_{\lambda, \mu}^w(x)$ is called a $w_{<}$ -*entourage* about x relative to λ and μ and the set $C_{\lambda, \mu}^w(x)$ is called a w_{\leq} -*entourage* about x relative to λ and μ .

We need the following two examples in the sequel.

Example 2.2 ([8, Example 2.4 (a)]). Let \mathbb{R} be equipped with the modular metric w defined by

$$w(\lambda, x, y) = \begin{cases} \infty & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

whenever $x, y \in \mathbb{R}$ and $\lambda > 0$.

It is readily checked that $X_w = X_w(x_0) = \{x_0\}$, where $x_0 \in \mathbb{R}$ and $q_w(x, y) = 0$ for all $x, y \in X_w$. Moreover, for any $r > 0$, we have

$$B_{q_w}(x_0, r) = \{y \in X_w : 0 = q_w(x_0, y) < r\} = \{x_0\}$$

and

$$B_{r,r}^w(x_0) = \{y \in X_w : w(r, x_0, y) < r\} = \{x_0\} = C_{r,r}^w(x_0).$$

Example 2.3 ([8, Example 2.7 (b)]). Consider a pseudometric space (X, q) .

If we equip X with the modular pseudometric $w(\lambda, x, y) = \frac{q(x, y)}{\lambda^p}$ whenever $x, y \in X$ and $\lambda > 0$, where p is a strictly positive constant. Then it follows that $X_w = X$ and $q_w(x, y) = [q(x, y)]^{\frac{1}{p+1}}$.

Furthermore, for any $\lambda > 0$, we have

$$\begin{aligned} B_{q_w}(x, \lambda) &= \left\{ y \in X_w : [q(x, y)]^{\frac{1}{p+1}} < \lambda \right\} \\ &= \left\{ y \in X_w : \frac{q(x, y)}{\lambda^p} < \lambda \right\} \\ &= \{y \in X_w : w(\lambda, x, y) < \lambda\} = B_{\lambda, \lambda}^w(x). \end{aligned}$$

If w is a modular pseudometric on a nonempty set X , then the topology induced by w (denoted by $\tau(w)$) is defined by the following:

A subset A of X_w is said to be $\tau(w)$ -open (or w -open) if for any $x \in A$ and $\lambda > 0$, there exists $\mu := \mu(x, \lambda) > 0$ such that $B_{\lambda, \mu}^w(x) \subset A$. Note that $B_{\lambda, \mu}^w(x)$ is not $\tau(w)$ -open, in general.

Lemma 2.4 ([5]). *Let w be a modular pseudometric on X and $x \in X_w$. Then, whenever $\lambda > 0$ we have*

- (a) $B_{q_w}(x, \lambda) \subseteq B_{\lambda, \lambda}^w(x)$,
- (b) $C_{q_w}(x, \lambda) \subseteq C_{\lambda, \lambda}^w(x)$, where $B_{q_w}(x, \lambda)$ and $C_{q_w}(x, \lambda)$ are respectively open and closed balls with centre x and radius λ with respect to the pseudometric q_w .

The following important concept of continuity of a modular pseudometric space was introduced in [6].

Definition 2.5. Let w be a modular pseudometric on a set X . Given $x, y \in X$,

- (a) the limit from the right $w_{+0}(\lambda, x, y)$ of w at a point $\lambda > 0$ is defined by

$$w_{+0}(\lambda, x, y) := \lim_{\mu \rightarrow \lambda^+} w(\mu, x, y) = \sup\{w(\mu, x, y) : \mu > \lambda\}.$$

- (b) The *limit from the left* $w_{-0}(\lambda, x, y)$ of w at a point $\lambda > 0$ is defined by
- $$w_{-0}(\lambda, x, y) := \lim_{\mu \rightarrow \lambda^-} w(\mu, x, y) = \inf\{w(\mu, x, y) : 0 < \mu < \lambda\}.$$

Moreover,

- (c) we say that w is *continuous from the right* on $(0, \infty)$ if for any $\lambda > 0$ we have

$$w(\lambda, x, y) = w_{+0}(\lambda, x, y).$$

- (d) We say that w is *continuous from the left* on $(0, \infty)$ if for any $\lambda > 0$ we have

$$w(\lambda, x, y) = w_{-0}(\lambda, x, y).$$

- (e) We say that w is *continuous* on $(0, \infty)$ if w is continuous from the right and continuous from the left on $(0, \infty)$.

Lemma 2.6 ([6]). *Let w be a modular pseudometric on a set X . If $x, y \in X_w$ and $0 < \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\} < \infty$, then*

$$q_w(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq \lambda\} > \lambda \text{ if and only if } w_{+0}(\lambda, x, y) > \lambda.$$

The following remark is a consequence of Definition 2.5.

Remark 2.7. If w is continuous from the right on $(0, \infty)$, then for any $x, y \in X_w$ and $\lambda > 0$, we have that $q_w(x, y) \leq \lambda$ if and only if $w(\lambda, x, y) \leq \lambda$.

Let us recall the well-known concept of hyperconvexity.

Definition 2.8. A pseudometric space (X, q) is called *hyperconvex* provided that for any $(x_i)_{i \in I}$ and family $(r_i)_{i \in I}$ of nonnegative real numbers satisfying $q(x_i, x_j) \leq r_i + r_j$ whenever $i, j \in I$, the following condition holds:

$$\bigcap_{i \in I} \left[C_q(x_i, r_i) \right] \neq \emptyset.$$

Let (X, q) be a pseudometric space. Then X is said to be *metrically convex* if for any points $x, y \in X$ and positive numbers r and s such that $q(x, y) \leq r + s$, there exists $z \in X$ such that $q(x, z) \leq r$ and $q(z, y) \leq s$.

Furthermore, a family of balls $(C_q(x_i, r_i))_{i \in I}$ is said to have the *mixed binary intersection property* if for all indices $i \in I$,

$$C_q(x_i, r_i) \neq \emptyset.$$

Definition 2.9. A pseudometric space (X, q) is called *hypercomplete* if every family $(C_q(x_i, r_i))_{i \in I}$ of balls having the mixed binary intersection property satisfies

$$\bigcap_{i \in I} \left[C_q(x_i, r_i) \right] \neq \emptyset.$$

The following is a well-known characterisation of hyperconvexity in terms of metric convexity and hypercompleteness.

Proposition 2.10 ([26]). *A pseudometric space (X, q) is hyperconvex if and only if it is metrically convex and hypercomplete.*

3. ISBELL-CONVEXITY

In this section, we introduce the concept of hyperconvexity in modular metric spaces.

Definition 3.1. Let w be a modular pseudometric on a nonempty set X . We say that X_w is w -Isbell-convex if for any family of points $(x_i)_{i \in I}$ in X_w and family of points $(\lambda_i)_{i \in I}$ in $(0, \infty)$ such that

$$w(\lambda_i + \lambda_j, x_i, x_j) \leq \lambda_i + \lambda_j, \quad \text{whenever } i, j \in I,$$

then

$$\bigcap_{i \in I} \left[C_{\lambda_i, \lambda_i}^w(x_i) \right] \neq \emptyset.$$

Example 3.2. Let (X, d) be a metric space. For any $x, y \in X$ and $\lambda > 0$, it is well-known from [8, Example 2.4] that the function w defined by $w(\lambda, x, y) = \frac{d(x, y)}{\varphi(\lambda)}$, where $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a bounded nondecreasing function is a modular metric on X . Furthermore, whenever $x_0 \in X$ the set $X_w(x_0) = \{x_0\}$ is w -Isbell-convex. Indeed, for any $\lambda, \mu > 0$ such that

$$0 = w(\lambda + \mu, x_0, x_0) < \lambda + \mu,$$

then we have

$$x_0 \in B_{\lambda, \lambda}(x_0) \cap B_{\mu, \mu}(x_0) \subseteq C_{\lambda, \lambda}(x_0) \cap C_{\mu, \mu}(x_0).$$

Definition 3.3. Let w be a modular pseudometric on a nonempty set X . We say that X_w is w -metrically convex if for any points $x, y \in X_w$ so that

$$w(\lambda + \mu, x, y) \leq \lambda + \mu \quad \text{whenever } \lambda, \mu > 0,$$

there exists $z \in X_w$ such that $w(\lambda, x, z) \leq \lambda$ and $w(\mu, z, y) \leq \mu$.

Example 3.4. Let $X = \mathbb{R}$ be equipped with the modular metric $w(\lambda, x, y) = \frac{d(x, y)}{\lambda}$ for any $x, y \in X$, where d is the discrete metric on \mathbb{R} . Then it is obvious that $X_w = \mathbb{R}$. Moreover X_w is not w -metrically convex since

$$w\left(1, \frac{1}{2}, 1\right) = \frac{d\left(\frac{1}{2}, 1\right)}{1} = 1 \leq \frac{1}{2} + \frac{1}{2}$$

but there is no $z \in \mathbb{R}$ such that

$$w\left(\frac{1}{2}, \frac{1}{2}, z\right) = \frac{d\left(\frac{1}{2}, z\right)}{\frac{1}{2}} \leq \frac{1}{2}$$

and

$$w\left(\frac{1}{2}, z, 1\right) = \frac{d(z, 1)}{\frac{1}{2}} \leq \frac{1}{2}.$$

If such z exists it would satisfy $z = 1$ and $z = \frac{1}{2}$.

Remark 3.5. Let w be a modular metric on a set X . It is easy to see that if (X_w, q_w) is metrically convex, then X_w is w -metrically convex too.

Proof. Let $x, y \in X_w$ and $\lambda, \mu > 0$ such that $w(\lambda + \mu, x, y) \leq \lambda + \mu$. It follows that $q_w(x, y) \leq \lambda + \mu$. By metric convexity of (X_w, q_w) , there exists $z \in X_w$ such that $z \in C_{q_w}(x, \lambda) \cap C_{q_w}(y, \mu) \subseteq C_{\lambda, \lambda}^w(x) \cap C_{\mu, \mu}^w(y)$. Hence there exists $z \in X_w$ such that $w(\lambda, x, z) \leq \lambda$ and $w(\mu, z, y) \leq \mu$. Therefore, X_w is w -metrically convex. \square

Definition 3.6. Let w be a modular pseudometric on a nonempty set X . A family $(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}$ of w_{\leq} -entourages with $\lambda_i > 0$ and $x_i \in X_w$ for all $i \in I$ is said to have the *mixed binary intersection property* provided that $C_{\lambda_i, \lambda_i}^w(x_i) \neq \emptyset$ whenever $i \in I$.

Definition 3.7. Let w be a modular pseudometric on a nonempty set X . The modular set X_w is called *w -Isbell-complete* if every family $(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}$ of w_{\leq} -entourages, where $\lambda_i > 0$ and $x_i \in X_w$ for all $i \in I$, having the mixed binary intersection property satisfies

$$\bigcap_{i \in I} \left[C_{\lambda_i, \lambda_i}^w(x_i) \right] \neq \emptyset.$$

Proposition 3.8. *If w is a modular pseudometric on X , then X_w is w -Isbell-convex if and only if X_w is w -metrically-convex and w -Isbell-complete.*

Proof. (\implies) Suppose that X_w be w -Isbell-convex. Let $x_1, x_2 \in X_w$ and $\lambda_1, \mu_2 > 0$ such that $w(\lambda_1 + \mu_2, x_1, x_2) \leq \lambda_1 + \mu_2$. Then, we set $\lambda_2 = \mu_1 = w(\lambda_2 + \mu_1, x_2, x_1)$. By w -Isbell-convexity of X_w , there exists

$$a \in C_{\lambda_1, \lambda_1}^w(x_1) \cap C_{\mu_2, \mu_2}^w(x_2).$$

It follows that $w(\lambda_1, x_1, a) \leq \lambda_1$ and $w(\mu_2, a, x_2) \leq \mu_2$. Hence X_w is w -metrically convex.

Consider the family $(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}$ of w_{\leq} -entourages having mixed binary intersection property. Then there exists

$$z \in C_{\lambda_i, \lambda_i}^w(x_i) \cap C_{\lambda_j, \lambda_j}^w(x_j) \text{ whenever } i, j \in I.$$

Furthermore, whenever $i, j \in I$ we have

$$w(\lambda_i + \lambda_j, x_i, x_j) \leq w(\lambda_i, x_i, z) + w(\lambda_j, z, x_j) \leq \lambda_i + \lambda_j.$$

Thus $\bigcap_{i \in I} \left[C_{\lambda_i, \lambda_i}^w(x_i) \right] \neq \emptyset$ by w -Isbell-convexity of X_w .

Therefore, X_w is w -Isbell-complete.

(\impliedby) Let X_w be w -metrically convex and w -Isbell-complete. Let $(x_i)_{i \in I}$ be a family of points in X_w and $(\lambda_i)_{i \in I}$ be a family of points in $(0, \infty)$ such that

$$w(\lambda_i + \lambda_j, x_i, x_j) \leq \lambda_i + \lambda_j \text{ whenever } i, j \in I.$$

By w -metrical convexity of X_w there exists $z \in X_w$ such that $w(\lambda_i, x_i, z) \leq \lambda_i$ and $w(\lambda_j, z, x_j) \leq \lambda_j$ whenever $i, j \in I$.

Then the family $(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}$ has the mixed binary intersection property. Since X_w is w -Isbell-complete, we have

$$\bigcap_{i \in I} [C_{\lambda_i, \lambda_i}^w(x_i)] \neq \emptyset.$$

□

Proposition 3.9. *Let w be a modular pseudometric on a nonempty set X . If (X_w, q_w) be a hyperconvex pseudometric space, then X_w is w -Isbell-convex.*

Proof. Suppose that (X_w, q_w) be a hyperconvex pseudometric space. Let $(x_i)_{i \in I}$ be a family of points in X_w and $(\lambda_i)_{i \in I}$ be a family of points in $(0, \infty)$ such that

$$w(\lambda_i + \lambda_j, x_i, x_j) \leq \lambda_i + \lambda_j \quad \text{whenever } i, j \in I.$$

Then

$$q_w(x_i, x_j) = \inf\{\lambda > 0 : w(\lambda, x_i, x_j) \leq \lambda\} \leq \lambda_i + \lambda_j \quad \text{whenever } i, j \in I.$$

By hyperconvexity of (X_w, q_w) , we have

$$\bigcap_{i \in I} [C_{q_w}(x_i, \lambda_i)] \neq \emptyset.$$

Since by Lemma 2.4, $C_{q_w}(x_i, \lambda_i) \subseteq C_{\lambda_i, \lambda_i}^w(x_i)$, it follows that

$$\emptyset \neq \bigcap_{i \in I} [C_{q_w}(x_i, \lambda_i)] \subseteq \bigcap_{i \in I} [C_{\lambda_i, \lambda_i}^w(x_i)].$$

Therefore, X_w is w -Isbell-convex. □

Remark 3.10. Let w be a modular pseudometric on a set X . If w be continuous from the right on $(0, \infty)$, then $q_w(x, y) \leq \lambda$ if and only if $w(\lambda, x, y) \leq \lambda$ for any $x, y \in X_w$ and positive number λ .

It is natural to wonder about the converse of Proposition 3.9.

Lemma 3.11. *Let w be a modular pseudometric on X . If w be continuous from the right on $(0, \infty)$, then X_w is w -metrically convex if and only if (X_w, q_w) is a metrically convex pseudometric space.*

Proof. (\implies) Suppose that X_w be w -metrically convex. Let $x, y \in X_w$ and $\lambda, \mu > 0$ such that $q_w(x, y) \leq \lambda + \mu$.

Since w is continuous from the right on $(0, \infty)$ we have $w(\lambda + \mu, x, y) \leq \lambda + \mu$. It follows that there exists $a \in X_w$ such that $w(\lambda, x, a) \leq \lambda$ and $w(\mu, a, y) \leq \mu$. Then $q_w(x, a) \leq \lambda$ and $q_w(a, y) \leq \mu$ by the right continuity of w . So (X_w, q_w) is a metrically convex.

(\impliedby) Follows from Remark 3.5. □

We leave the proof of the following lemma to the reader.

Lemma 3.12. *Let w be a modular pseudometric on X . If w be continuous from the right on $(0, \infty)$, then X_w is w -Isbell-complete if and only if (X_w, q_w) is an Isbell-complete pseudometric space.*

Theorem 3.13. *Let w be a modular pseudometric on X . If w be continuous from the right on $(0, \infty)$, then X_w is w -Isbell-convex if and only if (X_w, q_w) is a hyperconvex pseudometric space.*

Proof. Suppose that X_w be w -Isbell-convex. Then, by Proposition 3.8 X_w is w -Isbell-complete and w -metrically convex. Thus (X_w, q_w) is an hypercomplete and metrically convex pseudometric space by Lemma 3.11 and Lemma 3.12. Therefore, (X_w, q_w) is a hyperconvex pseudometric space.

The converse follows by a similar argument. □

4. NONEXPANSIVE MAPS

We are aware of [2], where the concept of boundedness of a subset of modular set was introduced in order to study the existence of fixed point of modular contractive maps in modular metric spaces. This is a motivation for our next definitions.

Let w be a modular pseudometric on X . We say that a nonempty subset A of X_w is w -bounded if there exists $x \in X_w$ such that

$$A \subseteq C_{\lambda, \lambda}^w(x) \text{ for some } \lambda > 0.$$

Remark 4.1. If w be a modular pseudometric on a set X , then boundeness on (X_w, q_w) implies w -boundeness. This observation follows from the fact that $C_{q_w}(x, \lambda) \subseteq C_{\lambda, \lambda}^w(x)$ whenever $\lambda > 0$ and $x \in X_w$.

Definition 4.2. Let A be a w -bounded subset of X_w . Then we denote by $\text{diam}_w(A)$ the w -diameter of A and it is defined by

$$\text{diam}_w(A) := \sup\{w(\lambda, x, y) : x, y \in A\}$$

for some $\lambda > 0$.

Lemma 4.3. *Let w be a modular pseudometric on X . If A is a w -bounded subset of X_w , then $\text{diam}_w(A) < \infty$.*

Proof. Suppose that A is w -bounded. Then for some $x \in X_w$, we have $A \subseteq C_{\lambda, \lambda}^w(x)$ for some $\lambda > 0$.

If $z, y \in A$ then

$$w(\lambda, x, z) \leq \lambda \quad \text{and} \quad w(\lambda, x, y) \leq \lambda.$$

It follows that

$$w(\lambda + \lambda, z, y) \leq w(\lambda, z, x) + w(\lambda, x, y) \leq \lambda + \lambda.$$

Hence for $\lambda' = \lambda + \lambda > 0$, we have

$$\sup\{w(\lambda', z, y) : z, y \in A\} \leq \lambda'.$$

Therefore, $\text{diam}_w(A) < \infty$. □

Lemma 4.4. *Let w be a modular pseudometric on X . If w be continuous from the right on $(0, \infty)$, then boundedness on (X_w, q_w) is equivalent to w -boundedness.*

Proof. We only prove the sufficient condition since the necessary condition follows from Remark 4.1. Suppose that A is a w -bounded subset of X_w . Then there exists $x \in X_w$ such that $A \subseteq C_{\lambda, \lambda}^w(x)$ for some $\lambda > 0$.

Let $y \in A$. Then $w(\lambda, x, y) \leq \lambda$. By the right continuity of w on $(0, \infty)$, we have

$$q_w(x, y) \leq \lambda \quad \text{for some } x \in X_w \text{ and } \lambda > 0.$$

Thus $A \subseteq C_{q_w}(x, \lambda)$. Therefore, A is bounded in (X_w, q_w) . □

Proposition 4.5. *Let w be a modular pseudometric on X . Let X_w be w -Isbell-convex and $(x_i)_{i \in I}$ is a family of points in X_w and $(\lambda_i)_{i \in I}$ is a family of positive real numbers such that $w(\lambda_i + \lambda_j, x_i, x_j) \leq \lambda_i + \lambda_j$ whenever $i, j \in I$. Then the set $A := \bigcap_{i \in I} [C_{\lambda_i, \lambda_i}^w(x_i)]$ is nonempty and w -Isbell-convex.*

Proof. Observe that the set A is nonempty by w -Isbell-convexity of X_w . Now we show that the set A is w -Isbell-convex. Let $(x_\alpha)_{\alpha \in \Gamma}$ be a family of points in A and $(\lambda_\alpha)_{\alpha \in \Gamma}$ be a family of positive real numbers such that

$$w(\lambda_\alpha + \lambda_\beta, x_\alpha, x_\beta) \leq \lambda_\alpha + \lambda_\beta \quad \text{whenever } \alpha, \beta \in \Gamma.$$

We have to show that the family of w_{\leq} -entourages

$$((C_{\lambda_\alpha, \lambda_\alpha}^w(x_\alpha))_{\alpha \in \Gamma}; (C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I})$$

satisfies the hypothesis of w -Isbell-convexity.

Then, in particular, for all $\alpha \in \Gamma$ and $i \in I$, we have

$$w(\lambda_i, x_i, x_\alpha) \leq \lambda_i < \lambda_i + \lambda_\alpha$$

since $x_\alpha \in A$.

By w -Isbell-convexity of X_w , it follows that

$$\begin{aligned} \emptyset &\neq \bigcap_{\alpha \in \Gamma} [C_{\lambda_\alpha, \lambda_\alpha}^w(x_\alpha)] \cap \bigcap_{i \in I} [C_{\lambda_i, \lambda_i}^w(x_i)] \\ &= A \cap \bigcap_{\alpha \in \Gamma} [C_{\lambda_\alpha, \lambda_\alpha}^w(x_\alpha)]. \end{aligned}$$

Hence A is w -Isbell-convex. □

For a w -bounded subset A of X_w , we set

$$(4.1) \quad \text{cov}_w(A) := \bigcap \left\{ C_{\lambda, \lambda}^w(x) : A \subseteq C_{\lambda, \lambda}^w(x), x \in X_w, \lambda > 0 \right\}.$$

Furthermore, we set:

$$r_{x, \lambda}^w(A) := \sup_{y \in A} \{w(\lambda, x, y) : \text{for some } \lambda > 0\}, \quad \text{where, } x \in X_w.$$

Proposition 4.6 (compare [13, Lemma 3.3]). *Let w be a modular pseudometric on X and A be a w -bounded subset of X_w . Then we have:*

- (a) $\text{cov}_w(A) = \bigcap_{x \in X} \left[C_{r_{x,\lambda}^w(A), r_{x,\lambda}^w(A)}^w(x) \right]$.
- (b) $r_x(\text{cov}_w(A)) = r_x(A)$.

Definition 4.7 (compare [13, Definition 3.4]). Let w be a modular pseudometric on X . A nonempty and w -bounded subset A of X_w is called w -admissible if $A = \text{cov}_w(A)$.

In the sequel, we will denote by $\mathcal{A}_w(X_w)$, the set of all w -admissible subsets of X_w .

Remark 4.8. Observe that a w -admissible subset of X_w can be written as the intersection of a family of the form $C_{\lambda,\lambda}^w(x)$, where $x \in X_w$ and $\lambda > 0$.

Lemma 4.9. Let w be a modular pseudometric on X which is continuous from the right on $(0, \infty)$. Then

$$C_{q_w}(x, \lambda) = C_{\lambda,\lambda}^w(x)$$

whenever $\lambda > 0$ and $x \in X_w$.

Proof. Since we know that $C_{q_w}(x, \lambda) \subseteq C_{\lambda,\lambda}^w(x)$, then we only prove the reverse inclusion.

If $a \in C_{\lambda,\lambda}^w(x)$, then $w(\lambda, x, a) \leq \lambda$ which is equivalent to $q_w(x, a) \leq \lambda$ by the right continuity of w . Thus $a \in C_{q_w}(x, \lambda)$. \square

Corollary 4.10. Let w be a modular pseudometric on X which is continuous from the right on $(0, \infty)$ and $A \subseteq X_w$. Then A is w -admissible if and only if A is q_w -admissible.

Definition 4.11. Let w be a modular pseudometric on X . Given a subset A of X_w , we define for $\lambda > 0$ the λ -parallel set of A as

$$P_\lambda(A) = \bigcup_{a \in A} \left[C_{\lambda,\lambda}^w(a) \right].$$

Proposition 4.12 (compare [17, Lemma 4.2]). Let w be a modular pseudometric on X . If X_w is w -Isbell-convex and A is a w -admissible subset of X_w , that is $A = \bigcap_{i \in I} C_{\lambda_i,\lambda_i}^w(x_i)$ with $x_i \in X_w$ and $\lambda_i > 0$ for each $i \in I \neq \emptyset$, then

$$P_\lambda(A) = \bigcap_{i \in I} \left[C_{\lambda_i+\lambda,\lambda_i+\lambda}^w(x_i) \right]$$

whenever $\lambda > 0$.

Proof. Let $y \in P_\lambda(A)$. Then, for some $a \in A$ we have $w(\lambda, a, y) \leq \lambda$. Furthermore, for each $i \in I$,

$$w(\lambda_i + \lambda, x_i, y) \leq w(\lambda_i, x_i, a) + w(\lambda, a, y) \leq \lambda_i + \lambda.$$

It follows that $y \in C_{\lambda_i+\lambda,\lambda_i+\lambda}^w(x_i)$ whenever $i \in I$. Hence

$$P_\lambda(A) \subseteq \bigcap_{i \in I} \left[C_{\lambda_i+\lambda,\lambda_i+\lambda}^w(x_i) \right].$$

We now suppose that $y \in \bigcap_{i \in I} \left[C_{\lambda_i + \lambda, \lambda_i + \lambda}^w(x_i) \right]$. Hence, for any $i \in I$,

$$w(\lambda_i + \lambda, x_i, y) \leq \lambda_i + \lambda.$$

Thus the family of w_{\leq} -entourages $\left[(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}, C_{\lambda, \lambda}^w(y) \right]$ satisfies the hypothesis of w -Isbell-convexity of X_w . Then

$$\begin{aligned} \emptyset &\neq \left[\bigcap_{i \in I} C_{\lambda_i, \lambda_i}^w(x_i) \right] \bigcap C_{\lambda, \lambda}^w(y) \\ &= A \bigcap C_{\lambda, \lambda}^w(y). \end{aligned}$$

It follows that $w(\lambda, y, a) \leq \lambda$ for some $a \in A$. Therefore, $y \in P_{\lambda}(A)$. □

Definition 4.13. Let w be a modular pseudometric on a set X . Then we say that a map $T : X_w \rightarrow X_w$ is w -Lipschitz if there exists a $k > 0$ such that

$$w(k\lambda, T(x), T(y)) \leq w(\lambda, x, y) \text{ for all } \lambda > 0 \text{ and } x, y \in X_w.$$

If $k = 1$, then the map T is called a w -nonexpansive map.

Remark 4.14. Let w be a modular pseudometric on X . In light of [5, Theorem 5.1] and [5, Theorem 5.2], one can easily prove that for any w -Lipschitz map $T : X_w \rightarrow X_w$, we have that $w(k\lambda, T(x), T(y)) \leq w(\lambda, x, y)$ implies $q_w(T(x), T(y)) \leq kq_w(x, y)$ for some $k > 0$ and whenever $\lambda > 0$ and $x, y \in X_w$.

Theorem 4.15 (compare [17, Lemma 4.3]). *Let w be a modular pseudometric on X which is continuous from the right on $(0, \infty)$ and X_w be w -Isbell-convex. If A is a w -admissible subset of X_w , then there is a w -nonexpansive retraction R of $P_{\lambda}(A)$ onto A such that $w(\lambda, x, R(x)) \leq \lambda$ whenever $x \in P_{\lambda}(A)$ and $\lambda > 0$.*

Proof. Suppose that A is w -admissible. Then

$$A = \bigcap_{i \in I} C_{\lambda_i, \lambda_i}^w(x_i) \text{ with } I \neq \emptyset.$$

Since $P_{\lambda}(A)$ is an intersection of w_{\leq} -entourages from Proposition 4.12, then $P_{\lambda}(A)$ is w -admissible in X_w . Moreover, from Proposition 4.5 we have $P_{\lambda}(A)$ is w -Isbell-convex.

Let us consider the family Ω defined by

$\Omega = \{(D, R_D) : A \subseteq D \subseteq P_{\lambda}(A) \text{ and } R_D : D \rightarrow A \text{ is a } w\text{-nonexpansive retraction such that } w(\lambda, R_D(x), x) \leq \lambda \text{ whenever } x \in D\}$.

If the map I_A be the identity on A , then $(A, I_A) \in \Omega$. Hence $\Omega \neq \emptyset$. Furthermore, one can ordered the family Ω by the partial order $(D, R_D) \leq (E, R_E)$ if and only if $D \subseteq E$ and the w -nonexpansive map R_D is an extension of the w -nonexpansive map R_E .

It follows that every chain in (Ω, \leq) is bounded above. Thus by Zorn's lemma, Ω has a maximal element. Suppose that (D, R_D) is the maximal element of (Ω, \leq) .

Let us prove that $D = P_\lambda(A)$. Suppose that there exists an element $x \in P_\lambda(A) \setminus D$ such that $w(\lambda_d, d, x) = \lambda_d$ whenever $d \in D$ and $\lambda_d > 0$. We consider the set

$$\mathcal{C} := \left[\bigcap_{d \in D} C_{w(\lambda_d, d, x), w(\lambda_d, d, x)}^w(R_D(d)) \right] \cap \left[\bigcap_{i \in I} C_{\lambda_i, \lambda_i}^w(x_i) \right] \cap [C_{\lambda, \lambda}^w(x)].$$

It is easy to see that $\mathcal{C} \neq \emptyset$ since the family

$$[(C_{\lambda_i, \lambda_i}^w(x_i))_{i \in I}, (C_{\lambda, \lambda}^w(x)), (C_{w(\lambda_d, d, x), w(\lambda_d, d, x)}^w(R_D(d)))_{d \in D}]$$

of \leq -entourages has the mixed binary intersection property in light of Proposition 3.8.

Since $\emptyset \neq \mathcal{C} \subseteq A$. We now suppose that $z \in \mathcal{C}$ and we define the map $R' : D \cup \{x\} \rightarrow A$ by $R'(d) = R_D(d)$ if $d \in D$ and $R'(x) = z$. Then, we have for any $d \in D$

$$w(\lambda_d, R'(d), R'(x)) = w(\lambda_d, R_D(d), z) \leq w(\lambda_d, d, x).$$

So R' is w -nonexpansive.

Moreover,

$$w(\lambda, R'(x), x) = w(\lambda, z, x) \leq \lambda$$

since $z \in \mathcal{C}$.

Hence $(D \cup \{x\}, R') \in \Omega$ which contradicts the maximality of D . Therefore, $D = P_\lambda(A)$. \square

Theorem 4.16. *Let w be a modular pseudometric on X . If X_w be w -bounded w -Isbell-convex and w be continuous from the right on $(0, \infty)$ and $T : X_w \rightarrow X_w$ be a w -nonexpansive map, then the fixed point set $\text{Fix}(T)$ is nonempty and w -Isbell-convex.*

Proof. Since $T : X_w \rightarrow X_w$ is a w -nonexpansive map, then $T : (X_w, q_w) \rightarrow (X_w, q_w)$ is a nonexpansive map. Moreover, X_w is q_w -bounded by Lemma 4.4.

Hence for any $\lambda > 0$, we have $w(T(x), T(y)) \leq w(\lambda, x, y)$ whenever $x, y \in X_w$. Then from Remark 4.14 when $k = 1$, we have $q_w(T(x), T(y)) \leq q_w(x, y)$ whenever $x, y \in X_w$.

Observe that (X_w, q_w) is a hyperconvex pseudometric space by Theorem 3.13. Furthermore, we have $\text{Fix}(T)$ is nonempty and Isbell-convex by [13, Theorem 6.1]. Therefore, $\text{Fix}(T)$ is w -Isbell-convex by Theorem 3.13. \square

Definition 4.17 (compare [19, Theorem 5.5]). Let w be a modular pseudometric on X and $T : X_w \rightarrow X_w$ be a map. For $\lambda_1, \lambda_2 > 0$, we define the set $F_{\lambda_1, \lambda_2}(T)$ by

$$F_{\lambda_1, \lambda_2}(T) = \{x \in X_w : w(\lambda_2, x, T(x)) \leq \lambda_2 \text{ and } w(\lambda_1, T(x), x) \leq \lambda_1\}.$$

Corollary 4.18 (compare [17, Theorem 4.11]). *Let w be a modular pseudometric on X and X_w be w -Isbell-convex. If w be continuous from the right on $(0, \infty)$ and $T : X_w \rightarrow X_w$ be a w -nonexpansive map, then the set $F_{\lambda_1, \lambda_2}(T)$ is w -Isbell-convex whenever $F_{\lambda_1, \lambda_2}(T)$ is nonempty.*

The conclusion leads us to list some of the open problems that can be studied in future for the continuation of this work.

Problem 4.19. *Let w be a modular metric on a set X . Does there exist a w -Isbell-convex hull in the category of modular metric spaces and w -nonexpansive maps?*

Problem 4.20. *Is it possible to introduce the concept of geodesic bicomplete in the framework of the modular spaces?*

REFERENCES

- [1] A. A. N. Abdou, Fixed points of Kannan maps in modular metric spaces, *AIMS Maths* 5 (2020), 6395–6403.
- [2] A. A. N. Abdou and M. A. Khamsi, Fixed point results of pointwise contractions in modular metric spaces, *Fixed Point Theory Appl.* 2013 (2013):163.
- [3] C. Alaca, M. E. Ege and C. Park, Fixed point results for modular ultrametric spaces, *J. Comput. Anal. Appl.* 20 (2016), 1259–1267.
- [4] A. H. Ansari, M. Demma, L. Guran, J. R. Lee and C. Park, Fixed point results for C -class functions in modular metric spaces. *J. Fixed Point Theory Appl.* 20, no. 3 (2018), Paper No. 103.
- [5] V. V. Chistyakov, A fixed point theorem for contractions in modular metric spaces, [arXiv:1112.5561](https://arxiv.org/abs/1112.5561).
- [6] V. V. Chistyakov, *Metric modular spaces: Theory and applications*, SpringerBriefs in Mathematics, Springer, Switzerland, 2015.
- [7] V. V. Chistyakov, Modular metric spaces generated by F -modulars, *Folia Math.* 15 (2008), 3–24.
- [8] V. V. Chistyakov, Modular metric spaces, I: Basic concepts, *Nonlinear Anal.* 72 (2010), 1–14.
- [9] S. Cobzas, *Functional Analysis in Asymmetric Normed Spaces*, *Frontiers in Mathematics*, Springer, Basel, 2012.
- [10] M. E. Ege and C. Alaca, Fixed point results and an application to homotopy in modular metric spaces, *J. Nonlinear Sci. Appl.* 8 (2015), 900–908.
- [11] M. E. Ege and C. Alaca, Some properties of modular S -metric spaces and its fixed point results, *J. Comput. Anal. Appl.* 20 (2016), 24–33.
- [12] M. E. Ege and C. Alaca, Some results for modular b -metric spaces and an application to system of linear equations, *Azerb. J. Math.* 8 (2018), 3–14.
- [13] R. Espínola and M. A. Khamsi, Introduction to hyperconvex spaces, in: *Handbook of Metric Fixed Point Theory*, Kluwer Academic, Dordrecht, The Netherlands (2001), pp. 39135
- [14] A. Gholidahneh, S. Sedghi, O. Ege, Z. D. Mitrovic and M. de la Sen, The Meir-Keeler type contractions in extended modular b -metric spaces with an application, *AIMS Math.* 6 (2021), 1781–1799.
- [15] H. Hosseinzadeh and V. Parvaneh, Meir-Keeler type contractive mappings in modular and partial modular metric spaces, *Asian-Eur. J. Math.* 13 (2020): 2050087.
- [16] E. Kemajou, H.-P. Künzi and O. Olela Otafudu, The Isbell-hull of di-space, *Topology Appl.* 159 (2012), 2463–2475.
- [17] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.
- [18] H.-P. Künzi, An introduction to quasi-uniform spaces, *Contemp. Math.* 486 (2009), 239–304.

- [19] H.-P. Künzi and O. Olela Otafudu, q -hyperconvexity in quasipseudometric spaces and fixed point theorems, *J. Funct. Spaces Appl.* 2012 (2012): Art. ID 765903.
- [20] N. Kumar and R. Chugh, Convergence and stability results for new three step iteration process in modular spaces, *Aust. J. Math. Anal. Appl.* 14 (2017): 14.
- [21] Y. Mutemwa, O. Olela Otafudu and H. Sabao, On gluing of quasi-pseudometric spaces, *Khayyam J. Math.* 6 (2020), 129–140.
- [22] O. Olela Otafudu, On one-local retract in quasi-metric spaces, *Topology Proc.* 45 (2015), 271–281.
- [23] O. Olela Otafudu and H. Sabao, Set-valued contractions and q -hyperconvex spaces, *J. Nonlinear Convex Anal.* 18 (2017), 1609–1617.
- [24] R. C. Sine, On nonlinear contraction semigroups in sup norm spaces, *Nonlinear Anal.* 3 (1979), 885–890.
- [25] H. Sabao and O. Olela Otafudu, On soft quasi-pseudometric spaces, *Appl. Gen. Topol.* 22 (2021), 17–30.
- [26] S. Salbany, Injective objects and morphisms, in: *Categorical Topology and Its Relation to Analysis, Algebra and Combinatorics*, Prague, 1988, World Sci. Publ., Teaneck, NJ, 1989, pp. 394–409.
- [27] S. Yamamuro, On conjugate space of Nakano space, *Trans. Amer. Math. Soc.* 90 (1959), 291–311.
- [28] C. I. Zhu, J. Chen, X. J. Huang and J. H. Chen, Fixed point theorems in modular spaces with simulation functions and altering distance functions with applications, *J. Nonlinear Convex Anal.* 21 (2020), 1403–1424.