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On the Menger and almost Menger properties in locales

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Abstract

The Menger and the almost Menger properties are extended to locales. Regarding the former, the extension is conservative (meaning that a space is Menger if and only if it is Menger as a locale), and the latter is conservative for sober T_D-spaces. Non-spatial Menger (and hence almost Menger) locales do exist, so that the extensions genuinely transcend the topological notions. We also consider projectively Menger locales, and show that, as in spaces, a locale is Menger precisely when it is Lindelöf and projectively Menger. Transference of these properties along localic maps (via direct image or pullback) is considered.

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Introduction

Recall that a topological space X is Menger if for every sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X we can select, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $\bigcup_{n\in\mathbb{N}}\mathscr{V}_n$ is a cover of X. This definition is purely in terms of the lattice of open subsets, and can thus be extended to frames almost verbatim. That is exactly what we do. It then turns that the extension of the Menger property to frames is conservative.

On the other hand, a topological space is called almost Menger if for every sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X we can select, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$

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such that $\bigcup\{\overline{V} \mid V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n\} = X$. Although this definition is not solely in terms of the lattice of open sets (because of the appearance of closures), it can be adapted to frames by working within the lattice of sublocales, with the union replaced by the join. That is precisely what we do to define almost Menger frames.

Our aim in this paper is to initiate the study of the Menger-type properties in pointfree topology. Some of the results we obtain not only extend the known topological ones to frames, but also sharpen the topological ones. There are various weaker forms of the Menger property in spaces, but we restrict ourselves to extensions of the Menger property and the almost Menger property.

Here is a brief overview of the paper. Since the theory of frames and locales has by now come of age, the preliminaries in Section 1 are written tersely; the main purpose being just to fix notation and recall the concepts that are used most throughout the paper.

In Section 2 we study some properties of Menger frames. We start by observing that (as already been mentioned) a topological space X is Menger if and only if the frame $\Omega(X)$ is Menger, and that non-spatial Menger frames do exist, so that our extension to frames of this property is a genuine extension covering more objects than topologies of Menger spaces.

Since the contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ preserves and reflects the Menger property, one may ask about its right adjoint $\Sigma\colon \mathbf{Frm} \to \mathbf{Top}$. A frame whose spatial reflection is a codense sublocale is Menger if and only if its spectrum is Menger (Proposition 2.6). It is perhaps worth underscoring that a frame whose spatial reflection is a codense sublocale is not necessarily spatial.

Defining a frame to be projectively Menger if every subframe with a countable base is Menger, we have that a frame is Menger precisely when it is Lindelöf and projectively Menger (Corollary 2.14). A completely regular normal countably paracompact frame is projectively Menger if and only if its Lindelöf coreflection is Menger (Corollary 2.16).

In Section 3 we consider almost Menger frames. Our definition, adapted from spaces as indicated above, turns out to be conservative for sober T_D-spaces (Theorem 3.3). Although our definition invokes the lattice of sublocales, we have a characterisation (Proposition 3.6) solely in terms of elements.

1. Preliminaries

We assume familiarity with frames and locales. Our references are [12] and [15]. In this section we recall just a few of the concepts that we shall need. Our notation is standard, and is, by and large, that of our references.

1.1. **Frames and spatiality.** Throughout this section, L denotes a frame. We denote by $\Omega(X)$ the frame of open subsets of a topological space X. An element $p \in L$ is called a *point* (or a *prime*) if it satisfies the property that

$$p < 1 \quad \text{and} \quad (\forall x, y \in L)(x \wedge y \le p \implies x \le p \text{ or } y \le p).$$

We write $\mathrm{Pt}(L)$ for the set of points of L. A frame is *spatial* if it is isomorphic to $\Omega(X)$ for some space X. This is the case precisely when every element is a meet of primes.

We view the *spectrum* of L as the topological space whose underlying set is $\mathrm{Pt}(L)$ with the topology

$$\Omega(\Sigma L) = \{\Sigma_a \mid a \in L\} \quad \text{where, for each } a \in L, \quad \Sigma_a = \{p \in \mathrm{Pt}(L) \mid a \not\leq p\}.$$

The map $\eta_L\colon L \to \Omega(\Sigma L)$ given by $\eta_L(a) = \Sigma_a$ is an onto frame homomorphism, and is the reflection map from L to spatial frames.

As usual, we shall write $\prec$ and $\prec\!\!\prec$, respectively, for the *rather below* and the *completely below* relations, and recall that L is called *regular* (resp. *completely regular*) if every element of L is the join of the elements that are rather below (resp. completely below) it.

1.2. **Sublocales and localic maps.** The lattice of sublocales of L, ordered by inclusion, is a coframe denoted by $\mathcal{S}(L)$. For later use, we recall that joins in $\mathcal{S}(L)$ are given by

$$\bigvee_{i\in I} S_i = \Big\{\bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i\Big\}.$$

Since $\mathcal{S}(L)$ is a coframe, when turned upside down it is a frame, denoted $\mathcal{S}(L)^{\mathrm{op}}$, whose top element is the *void sublocale* $\mathsf{O} = \{1\}$. A sublocale of L is called a *one-point* sublocale if it is of the form $\{p, 1\}$ for some $p \in \mathrm{Pt}(L)$. Spatial frames are precisely those that are joins of their one-point sublocales.

The open sublocale associated with $a \in L$ is denoted by $\mathfrak{o}_L(a)$, and the closed one by $\mathfrak{c}_L(a)$. We shall drop the subscript if no confusion may result from that. The *closure* of a sublocale S of L, denoted $\overline{S}$ or $\mathrm{cl}_L\, S$, is the sublocale

$$\overline{S} = \mathfrak{c}_L\Big(\bigwedge S\Big).$$

In particular, $\overline{\mathfrak{o}_L(a)} = \mathfrak{c}_L(a^*)$. A sublocale S of L is *dense* if $\overline{S} = L$. If S and T are sublocales of L and $S \subseteq T$, then S is a sublocale of T. The closure of S in T will be denoted by $\mathrm{cl}_T\, S$, and $\overline{S}$ (unadorned) will be understood to be the closure in L.

A localic map $f\colon L \to M$ gives rise to two maps

$$f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M) \quad \text{and} \quad f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$$

given by

$$f[S] = \{f(x) \mid x \in S\} \quad \text{and} \quad f_{-1}[T] = \bigvee\{A \in \mathcal{S}(L) \mid A \subseteq f^{-1}[T]\}.$$

The map $f[-]$ preserves all joins and $f_{-1}[-]$ preserves all meets (recall that they are intersections) and all binary joins. For any $S \in \mathcal{S}(L)$ and $T \in \mathcal{S}(M)$,

$$f[S] \subseteq T \quad \Longleftrightarrow \quad S \subseteq f_{-1}[T].$$

Writing h for the left adjoint of f, we have that, for any $b \in M$,

$$f_{-1}[\mathfrak{o}_M(b)] = \mathfrak{o}_L(h(b)) \quad \text{and} \quad f_{-1}[\mathfrak{c}_M(b)] = \mathfrak{c}_L(h(b)).$$

This then shows that the map $f_{-1}[-]$ also preserves arbitrary joins of open sublocales. For, if $\{b_i \mid i \in I\} \subseteq M$, then

$$f_{-1}\Big[\bigvee_{i\in I}\mathfrak{o}_M(b_i)\Big] = f_{-1}\Big[\mathfrak{o}_M\Big(\bigvee_{i\in I} b_i\Big)\Big] = \mathfrak{o}_L\Big(h\Big(\bigvee_{i\in I} b_i\Big)\Big)$$
$$= \mathfrak{o}_L\Big(\bigvee_{i\in I} h(b_i)\Big) = \bigvee_{i\in I}\mathfrak{o}_L\big(h(b_i)\big) = \bigvee_{i\in I} f_{-1}[\mathfrak{o}_M(b_i)].$$

1.3. **Covers and coverings.** By a *cover* of L we mean a set $C \subseteq L$ such that $\bigvee C = 1$. On the other hand, to avoid possible confusion, we say a collection $\mathscr{C}$ of sublocales of L is a *covering* of L if $\bigvee\{C \mid C \in \mathscr{C}\} = L$, where the join is calculated in $\mathcal{S}(L)$. This terminology is not standard. A cover consists of elements of L, whereas a covering consists of sublocales of L. If every sublocale in a covering $\mathscr{C}$ of L is open, then $\mathscr{C}$ is an *open covering* of L. There is a bijection between covers and open coverings given by

$$C \mapsto \mathscr{C}^C = \{\mathfrak{o}_L(c) \mid c \in C\} \qquad \text{and} \qquad \mathscr{C} \mapsto C^{\mathscr{C}} = \{x \in L \mid \mathfrak{o}_L(x) \in \mathscr{C}\}.$$

A cover C of L is said to *refine* a cover D if for every $c \in C$ there is a $d \in D$ such that $c \leq d$. In this case, C is called a *refinement* of D.

2. Menger locales

We aim to define Menger locales in such a way that a space X is Menger precisely when the frame $\Omega(X)$ is Menger. Our definition will be localic, and we will then cast it in frame terms, which will enable us to show easier that the definition is conservative. Throughout, every sequence is indexed by $\mathbb{N}$.

Definition 2.1. A frame L is *Menger* if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}}\mathscr{D}_n$ is a covering of L. In this case, we say the sequence $(\mathscr{D}_n)$ is a *Menger witness* for $(\mathscr{C}_n)$.

From the bijection between covers and coverings, this definition could equivalently have been stated in terms of covers. The reason is that if C is a cover of L and D is a finite subset of C, then, in the notation of Subsection 1.3, $\mathscr{C}^D$ is a finite subset of $\mathscr{C}^C$. Conversely, if $\mathscr{C}$ is an open covering of L and $\mathscr{D}$ is a finite subset of $\mathscr{C}$, then $C^{\mathscr{D}}$ is a finite subset of $C^{\mathscr{C}}$ because the mapping $u \mapsto \mathfrak{o}_L(u)$ is one-one.

Proposition 2.2. *A frame L is Menger iff for every sequence (C_n) of covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L.*

As with coverings, we shall say such a sequence (D_n) is a Menger witness for the sequence (C_n). This proposition makes it most apparent that every Menger frame is Lindelöf, and every compact frame (in fact, every σ-compact one – meaning one that is a join of countably many compact sublocales) is Menger. Since there are non-spatial compact frames (see [12, p. 89]), it follows that:

A Menger frame need not be spatial.

Since every cover of a subframe is a cover of the ambient frame, we deduce that:

Every subframe of a Menger frame is Menger. Thus, a localic image of a Menger frame is Menger.

Since a collection of open subsets of a space X is a cover of the frame $\Omega(X)$ if and only if it is an open cover of the space X, we deduce the following from Proposition 2.2.

Corollary 2.3. *A topological space X is Menger iff $\Omega(X)$ is Menger.*

Recall that the *sobrification* of a topological space is the spectrum of its frame of open sets. Since a space and its sobrification have isomorphic frames of open sets, we have the following result.

Corollary 2.4. *A topological space is Menger iff its sobrification is Menger.*

In light of the dual adjunction

$$\mathbf{Top} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathbf{Frm}$$

and the result in Corollary 2.3, one may ask if it is the case that a frame is Menger if and only if its spectrum is Menger. We address this for some types of frames. As is well known, a frame L is spatial if and only if the frame homomorphism $\eta_L\colon L \to \Omega(\Sigma L)$ is one-one. We will show that for frames L for which η_L is *codense* (meaning that $\eta_L(a) = 1_{\Omega(\Sigma L)}$ implies $a = 1_L$) the spectrum analogue of Corollary 2.3 holds. We reiterate that such frames need not be spatial, as the following example shows.

Example 2.5. Let L be a frame with no points, such as the smallest dense sublocale of $\Omega(\mathbb{R})$. Let $\tilde{L}$ be the frame obtained from L by adjoining a new top element $1_{\tilde{L}} > 1_L$. Then $\tilde{L}$ is not spatial and $\mathrm{Pt}(\tilde{L}) = \{1_L\}$. From the latter, it is not hard to see that $\eta_{\tilde{L}}$ is codense.

Proposition 2.6. *A frame whose spatial reflection is a codense sublocale is Menger iff its spectrum is Menger.*

Proof. Let L be such a frame. For any $A \subseteq L$ we set $\Sigma_A = \{\Sigma_a \mid a \in A\}$. Since $\Sigma_{\bigvee_{i\in I} a_i} = \bigcup_{i\in I}\Sigma_{a_i}$ for any collection $\{a_i \mid i \in I\}$ of elements of L, it follows that Σ_C is an open cover of ΣL whenever C is a cover of L. On the other hand, the part of the hypothesis that says η_L is codense ensures that every open cover of ΣL is of the form Σ_C for some cover C of L.

Now assume that ΣL is Menger. We apply Proposition 2.2 to show that L is Menger. Let (C_n) be a sequence of covers of L. Then (Σ_{C_n}) is a sequence of open covers of ΣL. So for each n there exists a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}}\Sigma_{D_n} = \Sigma L$. Since $\bigcup_{n\in\mathbb{N}}\Sigma_{D_n} = \Sigma_{\bigcup_{n\in\mathbb{N}} D_n}$, we deduce from the codensity part of the hypothesis that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L. Therefore L is Menger.

The converse is shown similarly. $\square$

When working with covers or coverings, it is at times convenient to deal with directed ones. When we say a subset of a poset is *directed*, we mean that it is up-directed. If C is a cover of L and $\mathscr{C}$ is an open covering of L, we set

$$C^{\times} = \left\{ \bigvee F \mid F \text{ is a finite subset of } C \right\}$$

and

$$\mathscr{C}^{\times} = \left\{ \bigvee \mathscr{F} \mid \mathscr{F} \text{ is a finite subset of } \mathscr{C} \right\},$$

and observe that $C^{\times}$ is a directed cover of L and $\mathscr{C}^{\times}$ is directed (open) covering of L.

Proposition 2.7. *The following are equivalent for a frame L.*

(1) *L is Menger.*
(2) *For every sequence (C_n) of directed covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.*
(3) *For every sequence $(\mathscr{U}_n)$ of directed open coverings of L, there exists, for each n, a sublocale $U_n \in \mathscr{U}_n$ such that $\{U_n \mid n \in \mathbb{N}\}$ is a covering of L.*

Proof. (2) $\Leftrightarrow$ (3): This equivalence comes from the bijection between covers and open coverings, together with the observation that this bijection induces a bijection between directed covers and directed open coverings.

(1) $\Leftrightarrow$ (2): Assume that L is Menger, and let (C_n) be a sequence of directed covers of L. Let (D_n) be a Menger witness for (C_n). Since each D_n is a finite subset of C_n and C_n is directed, there exists an element $c_n \in C_n$ such that $\bigvee D_n \leq c_n$. Therefore the cover $\bigcup_{n\in\mathbb{N}} D_n$ refines $\{c_n \mid n \in \mathbb{N}\}$, showing that the latter is a cover of L.

Conversely, let (C_n) be a sequence of covers of L, and consider the sequence $(C_n^{\times})$ of directed covers of L. The current hypothesis furnishes, for each n, a finite $B_n \subseteq C_n$ such that the set $\{\bigvee B_n \mid n \in \mathbb{N}\}$ is a cover of L. Clearly, this makes the sequence (B_n) a Menger witness for (C_n), and hence L is a Menger frame. □

Now, let us observe that the Menger property is preserved under finite joins. In the proof we shall use the fact that families of open sublocales are distributive in the coframe of sublocales, that is, if $(U_\alpha)_{\alpha\in A}$ is a family of *open* sublocales of L and T is any sublocale of L, then

$$T \cap \bigvee_{\alpha\in A} U_\alpha = \bigvee_{\alpha\in A} (T \cap U_\alpha).$$

Recall that open sublocales of a sublocale A of a frame L are precisely the intersections with A of the open sublocales of L.

Proposition 2.8. *The join of finitely many Menger sublocales of a given frame is Menger.*

Proof. Let us first show that a sublocale A of L is Menger if and only if whenever $(\mathscr{U}_n)$ is a sequence of families of open sublocales of L such that $A \subseteq \bigvee \mathscr{U}_n$ for every n, then there exists, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $A \subseteq \bigvee \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$. To see this, let us introduce the ad hoc notation that if $\mathscr{F}$ is a family of sublocales of L, we write

$$A \cap \mathscr{F} = \{A \cap F \mid F \in \mathscr{F}\}.$$

Now, for the "if" part, the equality $A = \bigvee\{A \cap U \mid U \in \mathscr{U}_n\}$ implies that $A \cap \mathscr{U}_n$ is an open covering of A for each n. Since A is Menger, we can find, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $A \cap \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ is a covering of A. Thus,

$$A = \bigvee \Big\{ A \cap S \mid S \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n \Big\} = A \cap \bigvee \Big\{ S \mid S \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n \Big\},$$

which implies $A \subseteq \bigvee \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$, as desired. The "only if" part is proved similarly, taking into account the fact every open covering of A is of the form $A \cap \mathscr{U}$, where $\mathscr{U}$ is a family of open sublocales of L with $A \subseteq \bigvee \mathscr{U}$.

Now let A and B be Menger sublocales of L, and consider a sequence $(\mathscr{C}_n)$ of families of open sublocales of L with $A \vee B \subseteq \bigvee \mathscr{C}_n$ for each n. Then $A \subseteq \bigvee \mathscr{C}_n$ and $B \subseteq \bigvee \mathscr{C}_n$ for each n. So we can find a finite $\mathscr{D}_n^A \subseteq \mathscr{C}_n$ and a finite $\mathscr{D}_n^B \subseteq \mathscr{C}_n$ such that

$$A \subseteq \bigvee \bigcup_{n\in\mathbb{N}} \mathscr{D}_n^A \quad \text{and} \quad B \subseteq \bigvee \bigcup_{n\in\mathbb{N}} \mathscr{D}_n^B.$$

Consequently, for each n, $\mathscr{D}_n^A \cup \mathscr{D}_n^B$ is a finite subset of $\mathscr{C}_n$ such that

$$A \vee B \subseteq \bigvee \big\{ S \mid S \in \mathscr{D}_n^A \cup \mathscr{D}_n^B \big\},$$

which proves that $A \vee B$ is Menger. The general case follows by induction because the binary join is an associative operation on $\mathcal{S}(L)$. □

The sublocales that inherit the Menger property include the closed ones because for any frame L and $a \in L$, a cover of $\mathfrak{c}(a)$ is a cover of L. In fact, we have a stronger result. Recall that a frame homomorphism is called *perfect* if its right adjoint preserves directed joins. In [8], a frame homomorphism is called *weakly perfect* if its right adjoint preserves directed covers.

Perfect homomorphisms are weakly perfect. Weak perfectness is strictly weaker than perfectness. Indeed, as observed in [7, Example 3.11], if L is a compact frame which is not Boolean, then the right adjoint of the join map $\mathfrak{J}L \to L$ (where $\mathfrak{J}L$ denotes the frame of ideals of L) takes covers to covers (and hence is weakly perfect), but it is not perfect. On the other hand, weak perfectness does not imply that the right adjoint takes covers to covers. A counterexample (also sourced from [7]) is the embedding of the two-element chain in the four-element Boolean algebra.

We can summarise these "pictorially" using the acronyms that a frame homomorphism satisfies:

- (DJ) if its right adjoint preserves directed joins;
- (DC) if its right adjoint sends directed covers to covers; and
- (CC) if its right adjoint sends covers to covers.

Then

$$(\mathrm{DJ}) \not\Longrightarrow (\mathrm{CC}) \not\Longrightarrow (\mathrm{DJ}); \quad (\mathrm{CC}) \Longrightarrow (\mathrm{DC}) \not\Longrightarrow (\mathrm{CC})$$

and

$$(\mathrm{DJ}) \Longrightarrow (\mathrm{DC}) \not\Longrightarrow (\mathrm{DJ}).$$

Thus, weak perfectness is the least restricted of these properties of homomorphisms. So a property of frames that is preserved or reflected by weakly perfect homomorphisms is also done so by the other types of homomorphisms.

In [6], a cover B of a frame L is called a *strong refinement* of a cover C if for every $b \in B$ there is a $c \in C$ such that $b \prec c$. Then L is called *cover regular* if every cover of L has a strong refinement. Regular frames are cover regular. Any finite chain with more than two elements is a cover regular frame which is not regular. Note that if a directed cover has a strong refinement, then it has a strong refinement which is directed because whenever $b_i \prec c_i$ for each $i \in \{1, \dots, n\}$, with $n \in \mathbb{N}$, then $(b_1 \vee \cdots \vee b_n) \prec (c_1 \vee \cdots \vee c_n)$.

Recall that a frame homomorphism $h\colon L \to M$ is called *dense* if the zero of its domain is the only element mapped to the zero of its codomain. This is precisely when $h_*(0) = 0$, where h_* denotes the right adjoint of h. It is well known that if h is a dense frame homomorphism, then $h_*(h(x)) \leq a$ whenever $x \prec a$ in the domain of h.

Corollary 2.9. *Let $h\colon L \to M$ be a frame homomorphism.*

(a) *If h is weakly perfect and L is Menger, then M is Menger.*
(b) *If h is dense and weakly perfect, L is cover regular, and M is Menger, then L is Menger.*

Proof. (a) Let (C_n) be a sequence of directed covers of M. Since h is weakly perfect, $h_*[C_n]$ is a directed cover of L for each n. Since L is Menger, by Proposition 2.7 there exists, for each n, an element $c_n \in C_n$ such that $\{h_*(c_n) \mid n \in \mathbb{N}\}$ is a cover of L. Therefore $\{hh_*(c_n) \mid n \in \mathbb{N}\}$ is a cover of M, and hence $\{c_n \mid n \in \mathbb{N}\}$ is a cover of M because $hh_*(x) \leq x$ for every $x \in M$. Proposition 2.7 again shows that M is Menger.

(b) Let (C_n) be a sequence of directed covers of L, and, for each n, let B_n be a directed strong refinement of C_n. Then $h[B_n]$ is a directed cover of M for each n. Since M is Menger, Proposition 2.7 enables us to find, for each $n \in \mathbb{N}$, an element $b_n \in B_n$ such that $\{h(b_n) \mid n \in \mathbb{N}\}$ is a cover of M. Since h_* takes directed covers to covers, the set $C = \{h_*h(b_n) \mid n \in \mathbb{N}\}$ is a cover of L. Since B_n is a strong refinement of C_n, for each n, there exists some $c_n \in C_n$ such that $b_n \prec c_n$, and since h is dense, we have $h_*h(b_n) \leq c_n$. So the cover C refines $\{c_n \mid n \in \mathbb{N}\}$, whence we deduce that L is Menger. □

Part (a) of this corollary enables us to say a word about coproducts. We do not need to recall the construction of coproducts.

Corollary 2.10. *If L is compact and M is Menger, then $L \oplus M$ is Menger.*

Proof. It is shown in [11, Lemma 2] that if L is compact, then the coproduct injection $M \to L \oplus M$ is a perfect map (actually, it is a proper map – we will recall the definition later). Since any perfect homomorphism is weakly perfect, the result follows from Corollary 2.9(a) because compact frames are Menger. ☐

We recall that a topological space X is called "projectively Menger" if every continuous second countable image of X is Menger. In [4], it is shown that a Tychonoff space is Menger if and only if it is Lindelöf and projectively Menger. We extend this result to frames without restricting to completely regular ones. Let us first recall some frame-theoretic notions. A subset S of a frame L is said to be a *generating set* if L is the smallest (under inclusion) subframe containing S. This is the case precisely when each $a \in L$ is of the form

$$a = \bigvee\{x \in L \mid x \text{ is the meet of some finite } F \subseteq S\}.$$

A *base* of a frame L is a subset B with the property that every element of L is a join of some elements of B.

Definition 2.11. A frame L is *projectively Menger* if every subframe of L with a countable base is Menger.

This definition can clearly be rephrased to say L is projectively Menger in case whenever $h\colon M \to L$ is a one-one frame homomorphism and M has a countable base, then M is Menger. Let us record the following easy observation, which should certainly be known, but for which we provide a proof as we do not have a reference.

Lemma 2.12. *Every frame with a countable base is Lindelöf.*

Proof. Let B be a countable base of a frame L. Let C be a cover of L. For each $c \in C$, let $B^{(c)}$ be a subset of B such that $c = \bigvee B^{(c)}$. Then $\bigcup_{c\in C} B^{(c)}$ is a countable cover of L refining C. Therefore C has a countable subcover. ☐

In what follows, we say a countable cover $C = \{c_1, c_2, \dots\}$ is *increasing* if $c_n \leq c_{n+1}$ for each n. This is standard terminology. Of course, an increasing cover is directed. For any given countable cover $C = \{c_1, c_2, \dots\}$, we denote by

$$C^+ = \{c_1, c_1 \vee c_2, c_1 \vee c_2 \vee c_3, \dots\}$$

the increasing cover constructed from C as indicated.

Proposition 2.13. *The following are equivalent for a frame L.*

(1) *L is projectively Menger.*
(2) *Every Lindelöf subframe of L is Menger.*
(3) *For every sequence (C_n) of countable covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L.*
(4) *For every sequence (C_n) of increasing countable covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.*
(5) *For every sequence $(\mathscr{C}_n)$ of countable open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathscr{D}_n$ is a covering of L.*

Proof. (1) ⇒ (4): Assume that L is projectively Menger, and let (C_n) be a sequence of increasing countable covers of L. Put $C = \bigcup_{n\in\mathbb{N}} C_n$. Then C is a countable subset of L. Let M be the subframe of L generated by C. Then the set C^F whose elements are the meets of finite subsets of C is a base for M. Since C is countable, C^F is countable, and so M has a countable base. Therefore M is Menger, by hypothesis. Since each $C_n \subseteq M$, (C_n) is a sequence of increasing covers of M, so for each n there exists some $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of M, and hence of L. Therefore (1) implies (4).

(4) ⇒ (3): Assume that condition (4) holds, and let (C_n) be a sequence of countable covers of L. Consider the sequence (C_n^+) of increasing covers of L. By (4), there exists, for each n, some $d_n \in C_n^+$ such that $\{d_n \mid n \in \mathbb{N}\}$ is a cover of L. Since each d_n is a join of some finite $D_n \subseteq C_n$, $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L.

(3) ⇒ (2): Assume condition (3), and let M be a Lindelöf subframe of L. Let (U_n) be a sequence of covers of M. Since M is Lindelöf, we can find, for each n, a countable $C_n \subseteq U_n$ such that C_n is a cover of M. Then of course C_n is a cover of L, and so, by (3), there exists a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L, and hence of M. Thus, M is Menger, and therefore L is projectively Menger.

(2) ⇒ (1): This follows from the fact that every frame with a countable base is Lindelöf.

(4) ⇔ (5): This follows from the bijection between covers and open coverings. □

Since every Menger frame is Lindelöf and every subframe of a Menger frame is Menger, we deduce the following characterisation of Menger frames from this proposition.

Corollary 2.14. *A frame is Menger iff it is Lindelöf and projectively Menger.*

We now turn to some subclass of completely regular frames. Recall from [6] that a frame L is *countably paracompact* if for every countable cover C of L, there is a cover W of L such that each element of W misses all but a finite number of elements of C. Let us also recall that a countable cover $D = \{d_n \mid n \in \mathbb{N}\}$ is called a *shrinking* of a countable cover $C = \{c_n \mid n \in \mathbb{N}\}$ if $d_n \prec c_n$ for every $n \in \mathbb{N}$. By a *cozero cover* of a frame we mean a cover consisting entirely of cozero elements.

Theorem 2.15. *The following are equivalent for a normal countably paracompact completely regular frame L.*

(1) *L is projectively Menger.*
(2) *For every sequence (C_n) of countable cozero covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} D_n$ is a cover of L.*
(3) *For every sequence (C_n) of increasing countable cozero covers of L, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 2.13, and the implication (2) $\Rightarrow$ (3) is almost immediate.

(3) $\Rightarrow$ (1): Let (U_n) be a sequence of increasing countable covers of L. Now fix $n \in \mathbb{N}$, and write

$$U_n = \{u_{n1}, u_{n2}, \dots\} \qquad \text{with} \qquad u_{n1} \leq u_{n2} \leq \cdots .$$

Since L is countably paracompact, U_n has a shrinking [5, Corollary to Proposition 1]. Since $x \vee y \prec a$ whenever $x \prec a$ and $y \prec a$, and since $x \leq a \prec b$ implies $x \prec b$, the cover U_n actually has an *increasing* shrinking

$$V_n = \{v_{n1}, v_{n2}, \dots\} \qquad \text{with} \qquad v_{nk} \prec u_{nk} \quad \text{for each } k \in \mathbb{N}.$$

By normality, $v_{nk} \prec\!\!\prec u_{nk}$ for each k, and hence, by [2, Corollary 1], there is a cozero element z_{nk} such that $v_{nk} \prec\!\!\prec z_{nk} \prec\!\!\prec u_{nk}$. Since the join of finitely many (actually, countably many) cozero elements is a cozero element, the sequence $(z_{nk})_{k\in\mathbb{N}}$ can be chosen so that it is increasing. Thus, if for each $n \in \mathbb{N}$ we let Z_n be the set

$$Z_n = \{z_{nk} \mid k \in \mathbb{N}\},$$

then (Z_n) is a sequence of countable increasing cozero covers of L, and so by (3) there exists, for each n, an element $z_n \in Z_n$ such that $\{z_n \mid n \in \mathbb{N}\}$ is a cover of L. Since each Z_n refines U_n, there exists, for each n, an element $u_n \in U_n$ such that $\{u_n \mid n \in \mathbb{N}\}$ is cover of L. Therefore L is projectively Menger by Proposition 2.13. □

Let L be a completely regular frame and λL be its Lindelöf coreflection (see [14] for details). As shown in [1, Corollary 8.2.13], if we let $h\colon \lambda L \to L$ be the coreflection map to L from completely regular Lindelöf frames, then for any countable cozero cover C of L, $h_*[C]$ is a (countable) cozero cover of λL. Since, as is well known, $\mathrm{Coz}(\lambda L) = \{h_*(c) \mid c \in \mathrm{Coz}\, L\}$, the countable cozero covers of λL are precisely the covers $h_*[C]$, for C a countable cozero cover of L. Furthermore, $h_*[C]$ is increasing if and only if C is increasing. In all, this yields the following corollary.

Corollary 2.16. *A normal countably paracompact completely regular frame L is projectively Menger iff λL is Menger.*

Proof. Suppose that L is projectively Menger, and denote by $h\colon \lambda L \to L$ the coreflection map to L from completely regular Lindelöf frames. Let $(h_*[C_n])$ be a sequence of increasing countable cozero covers of λL. Then (C_n) is a sequence of increasing countable cozero covers of L. By Theorem 2.15, there exists, for each n, an element $c_n \in C_n$ such that $\{c_n \mid n \in \mathbb{N}\}$ is a cover of L. Thus, for each n, there exists $d_n \in h_*[C_n]$ such that $\{d_n \mid n \in \mathbb{N}\}$ is a cover of λL. Therefore λL is projectively Menger by Theorem 2.15, and hence it is Menger by Corollary 2.14 because it is Lindelöf.

The converse is proved similarly. □

3. Almost Menger frames

Apart from the projective Menger property that we discussed in the previous section (solely for purposes of characterising Menger frames), we shall also consider a weaker form of the Menger property in frames. As with the case of the Menger property, we will define it by a condition lifted straight from spaces, mutatis mutandis. We will show that the localic extension is conservative for sober T_D-spaces. Recall that a space is *sober* if it is a T_0-space and the complements of the closures of its singletons are exactly its meet-irreducible open sets. That is, a space X is sober if and only if it is a T_0-space and

$$\mathrm{Pt}(\Omega(X)) = \{X \smallsetminus \overline{\{x\}} \mid x \in X\}.$$

On the other hand, X is a T_D*-space* if each $x \in X$ has an open neighbourhood U such that $U \smallsetminus \{x\}$ is open. Hausdorff spaces are sober T_D-spaces.

Let us recall that a space X is called almost Menger if for every sequence $(\mathscr{C}_n)$ of open covers of X, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigcup\big\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\big\} = X$.

Definition 3.1. A frame L is *almost Menger* if for every sequence $(\mathscr{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that $\bigvee\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n\big\} = L$. In this case, we say the sequence $(\mathscr{D}_n)$ is an *almost Menger witness* for the sequence $(\mathscr{C}_n)$.

It is clear that every Menger frame is almost Menger. Before investigating some of the properties of almost Menger frames, we show that, among sober T_D-spaces, the definition of almost Menger frames is conservative. We need some background. Recall (from [16], for instance) that a prime element p of a frame L is said to be a *covered prime* if whenever $p = \bigwedge S$ for some $S \subseteq L$, then $p \in S$. Since for any sober space X the prime elements of $\Omega(X)$ are precisely the open sets $X \smallsetminus \overline{\{x\}}$, for $x \in X$, we deduce from [16, Proposition 1.6.2] that if X is a sober T_D-space, then all its prime elements are covered.

We recalled in the Preliminaries that

$$L \textit{ is spatial iff } L = \bigvee\big\{\{p, 1\} \mid p \in \mathrm{Pt}(L)\big\}.$$

If X is sober, then the one-point sublocales of $\Omega(X)$ are exactly the sublocales $\{X \smallsetminus \overline{\{x\}}, 1_{\Omega(X)}\}$, with $x \in X$. Hence, if X is sober, then

$$\Omega(X) = \bigvee\big\{\{X \smallsetminus \overline{\{x\}}, 1_{\Omega(X)}\} \mid x \in X\big\}.$$

In fact, this equality holds without sobriety, as observed in [17]. The argument goes as follows. If X is any topological space and $U \subseteq X$ is open, then $U = \bigcap\big\{X \smallsetminus \overline{\{w\}} \mid w \notin U\big\}$, so that, calculating in $\Omega(X)$ and $\mathcal{S}(\Omega(X))$,

$$U = \mathrm{int}\left(\bigcap_{w\notin U}\big(X \smallsetminus \overline{\{w\}}\big)\right) = \bigwedge_{w\notin U}\big(X \smallsetminus \overline{\{w\}}\big) \in \bigvee\big\{\{X \smallsetminus \overline{\{x\}}, 1_{\Omega(X)}\} \mid x \in X\big\}.$$

Lemma 3.2. *Let $\{U_\alpha \mid \alpha \in A\}$ be a family of open subsets of a topological space X.*

(a) *If $\bigcup_{\alpha\in A}\overline{U_\alpha} = X$, then $\bigvee\{\overline{\mathfrak{o}_{\Omega(X)}(U_\alpha)} \mid \alpha \in A\} = \Omega(X)$.*
(b) *If X is a sober T_D-space and $\bigvee\{\overline{\mathfrak{o}_{\Omega(X)}(U_\alpha)} \mid \alpha \in A\} = \Omega(X)$, then $\bigcup_{\alpha\in A}\overline{U_\alpha} = X$.*

Proof. (a) Note that

$$\overline{\mathfrak{o}_{\Omega(X)}(U_\alpha)} = \mathfrak{c}_{\Omega(X)}(U_\alpha^*) = \mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{U_\alpha}).$$

Now, given $p \in X$, since $\bigcup_{\alpha\in A}\overline{U_\alpha} = X$, there is an index $\gamma \in A$ such that $p \in \overline{U_\gamma}$, which implies $\overline{\{p\}} \subseteq \overline{U_\gamma}$, and hence $X \smallsetminus \overline{U_\gamma} \subseteq X \smallsetminus \overline{\{p\}}$. Consequently, the element $X \smallsetminus \overline{\{p\}}$ of the frame $\Omega(X)$ belongs to the sublocale $\mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{U_\gamma})$, whence we deduce that

$$\{X \smallsetminus \overline{\{p\}}, 1_{\Omega(X)}\} \subseteq \mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{U_\gamma})$$

and therefore,

$$\begin{aligned}\Omega(X) &= \bigvee\{\{X \smallsetminus \overline{\{x\}}, 1_{\Omega(X)}\} \mid x \in X\}\\ &\subseteq \bigvee\{\mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{U_\alpha}) \mid \alpha \in A\}\\ &= \bigvee\{\overline{\mathfrak{o}_{\Omega(X)}(U_\alpha)} \mid \alpha \in A\}\\ &\subseteq \Omega(X),\end{aligned}$$

which proves the claimed equality.

(b) Assume that X is a sober T_D-space and $\bigvee\{\overline{\mathfrak{o}_{\Omega(X)}(U_\alpha)} \mid \alpha \in A\} = \Omega(X)$. Then, for any $x \in X$, the element $X \smallsetminus \overline{\{x\}}$ of $\Omega(X)$ belongs to this join, so that

$$X \smallsetminus \overline{\{x\}} \in \bigvee\{\mathfrak{c}_{\Omega(X)}(X \smallsetminus \overline{U_\alpha}) \mid \alpha \in A\}.$$

By the way joins of sublocales are computed, for each α there is an open subset V_α of X such that $X \smallsetminus \overline{U_\alpha} \subseteq V_\alpha$ and $X \smallsetminus \overline{\{x\}} = \bigwedge_\alpha V_\alpha$. Since primes are covered here, there is an index $\gamma \in A$ such that $X \smallsetminus \overline{\{x\}} = V_\gamma$. Therefore $x \in X \smallsetminus V_\gamma \subseteq \overline{U_\gamma}$, which then shows that $\bigcup_{\alpha\in A}\overline{U}_\alpha = X$. ☐

Theorem 3.3. *Let X be a topological space.*

(a) *If X is almost Menger, then $\Omega(X)$ is almost Menger.*
(b) *If X is a sober T_D-space, then $\Omega(X)$ is almost Menger iff X is almost Menger.*

Proof. (a) Let $(\mathscr{C}_n)$ be a sequence of open coverings of $\Omega(X)$. For each n, there is an open cover $\mathscr{U}_n$ of X such that $\mathscr{C}_n = \{\mathfrak{o}_{\Omega(X)}(U) \mid U \in \mathscr{U}_n\}$. Now, since X is almost Menger, there exists, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $\bigcup\{\overline{V} \mid V \in \bigcup_{n\in\mathbb{N}}\mathscr{V}_n\} = X$. For each n, put

$$\mathscr{D}_n = \{\mathfrak{o}_{\Omega(X)}(V) \mid V \in \mathscr{V}_n\}.$$

We will show that $(\mathscr{D}_n)$ is an almost Menger witness for $(\mathscr{C}_n)$. Clearly, each $\mathscr{D}_n$ is finite and $\mathscr{D}_n \subseteq \mathscr{C}_n$. Let us check the final condition. Since $\bigcup\{\overline{V} \mid V \in \bigcup_{n\in\mathbb{N}}\mathscr{V}_n\} = X$, we deduce from Lemma 3.2(a) that

$$\Omega(X) = \bigvee\Big\{\overline{\mathfrak{o}_{\Omega(X)}(V)} \mid V \in \bigcup_{n\in\mathbb{N}}\mathscr{V}_n\Big\} = \bigvee\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}}\mathscr{D}_n\},$$

which shows that the final condition to make $(\mathscr{D}_n)$ an almost Menger witness for $(\mathscr{C}_n)$ is satisfied. Therefore $\Omega(X)$ is almost Menger.

(b) Assume that X is a sober T_D-space and the frame $\Omega(X)$ is almost Menger. Let $(\mathscr{U}_n)$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, put

$$\mathscr{U}'_n = \{\mathfrak{o}_{\Omega(X)}(U) \mid U \in \mathscr{U}_n\},$$

and observe that $\mathscr{U}'_n$ is an open covering of $\Omega(X)$; and so we have a sequence $(\mathscr{U}'_n)$ of open coverings of the almost Menger frame $\Omega(X)$. In accordance with the definition, for each n, there is a finite $\mathscr{V}'_n \subseteq \mathscr{U}'_n$ such that the sequence $(\mathscr{V}'_n)$ is an almost Menger witness for $(\mathscr{U}'_n)$, and hence

$$(\dagger) \qquad \bigvee\Big\{\overline{S} \mid S \in \bigcup_{n\in\mathbb{N}}\mathscr{V}'_n\Big\} = \Omega(X).$$

For each n, put

$$\mathscr{V}_n = \{V \in \Omega(X) \mid \mathfrak{o}_{\Omega(X)}(V) \in \mathscr{V}'_n\}.$$

Since the mapping $\mathfrak{o}_{\Omega(X)} \colon \Omega(X) \to \mathcal{S}(\Omega(X))$ is injective and $\mathscr{V}'_n$ is a finite set, it follows that $\mathscr{V}_n$ is a finite set. Furthermore, $\mathscr{V}_n \subseteq \mathscr{U}_n$ because if $V \in \mathscr{V}_n$, then $\mathfrak{o}_{\Omega(X)}(V) \in \mathscr{V}'_n \subseteq \mathscr{U}'_n$, so that $\mathfrak{o}_{\Omega(X)}(V) = \mathfrak{o}_{\Omega(X)}(U)$ for some $U \in \mathscr{U}_n$, whence $V = U$. Observe that, for any sublocale T of L,

$$T \in \bigcup_{n\in\mathbb{N}}\mathscr{V}'_n \quad\iff\quad T = \mathfrak{o}_{\Omega(X)}(V) \text{ for some } V \in \bigcup_{n\in\mathbb{N}}\mathscr{V}_n,$$

and so, by Lemma 3.2(b), the equality in (†) implies

$$\bigcup\Big\{\overline{V} \mid V \in \bigcup_{n\in\mathbb{N}}\mathscr{V}_n\Big\} = X,$$

which then shows that X is almost Menger. ☐

Next, we show that in the definition of almost Menger frames, "open coverings" can be replaced with "directed open coverings" without violating the concept. In the proof we are going to use the fact that

$$\overline{S_1 \vee \cdots \vee S_n} = \overline{S_1} \vee \cdots \vee \overline{S_n}$$

for any collection of finitely many sublocales [15, Proposition III.8.1].

Proposition 3.4. *A frame L is almost Menger iff for every sequence $(\mathscr{C}_n)$ of directed open coverings of L, there exists, for each n, a sublocale $C_n \in \mathscr{C}_n$ such that $\bigvee\{\overline{C_n} \mid n \in \mathbb{N}\} = L$.*

Proof. Suppose, first, that L is almost Menger. Let $(\mathscr{C}_n)$ be a sequence of directed open coverings of L, and let $(\mathscr{D}_n)$ be an almost Menger witness for $(\mathscr{C}_n)$. Since each $\mathscr{D}_n$ is a finite subset of $\mathscr{C}_n$ and $\mathscr{C}_n$ is directed, there exists some $C_n \in \mathscr{C}_n$ such that $D \subseteq C_n$ for every $D \in \mathscr{D}_n$. It follows therefore that

$$L = \bigvee \Big\{ \overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n \Big\} \subseteq \bigvee \big\{ \overline{C_n} \mid n \in \mathbb{N} \big\} \subseteq L,$$

which proves the left-to-right implication.

Conversely, suppose that the stated condition holds, and let $(\mathscr{C}_n)$ be a sequence of open coverings of L. For each n, consider the collection

$$\mathscr{C}_n^{\times} = \Big\{ \bigvee \mathscr{F} \mid \mathscr{F} \text{ is a finite subset of } \mathscr{C}_n \Big\}.$$

Then $(\mathscr{C}_n^{\times})$ is a sequence of directed open coverings of L. By hypothesis, there exists, for each n, some $W_n \in \mathscr{C}_n^{\times}$ such that

$$\bigvee \{ \overline{W_n} \mid n \in \mathbb{N} \} = L.$$

Now, for each $n \in \mathbb{N}$, there exists some $k(n) \in \mathbb{N}$ and elements $F_n^{(1)}, \ldots, F_n^{k(n)}$ of $\mathscr{C}_n$ such that

$$W_n = F_n^{(1)} \vee \cdots \vee F_n^{k(n)} \qquad \text{and hence} \qquad \overline{W_n} = \overline{F_n^{(1)}} \vee \cdots \vee \overline{F_n^{k(n)}}.$$

Consequently, if for each n we let $\mathscr{C}'_n = \big\{ F_n^{(1)}, \ldots, F_n^{k(n)} \big\}$, then each $\mathscr{C}'_n$ is a finite subset of $\mathscr{C}_n$ such that

$$L = \bigvee \big\{ \overline{W_n} \mid n \in \mathbb{N} \big\} \subseteq \bigvee \Big\{ \overline{T} \mid T \in \bigcup_{n\in\mathbb{N}} \mathscr{C}'_n \Big\} \subseteq L,$$

which then shows that $(\mathscr{C}'_n)$ is an almost Menger witness for $(\mathscr{C}_n)$, and hence L is almost Menger. □

Although we could have proved directly from the definition that a localic image of an almost Menger frame is almost Menger, we shall use this result. If $f\colon L \to M$ is a localic map and $\mathscr{U}$ is a collection of sublocales of M, we write $f_{-1}[\mathscr{U}]$ for the set $\{f_{-1}[U] \mid U \in \mathscr{U}\}$.

Proposition 3.5. *A localic image of any almost Menger frame is itself almost Menger.*

Proof. Let $f\colon L \to M$ be an onto localic map with L almost Menger. Let $(\mathscr{U}_n)$ be a sequence of directed open coverings of M. Since $f_{-1}[-]$ preserves joins of open sublocales, $(f_{-1}[\mathscr{U}_n])_{n\in\mathbb{N}}$ is a sequence of open coverings of L, and since $f_{-1}[-]$ preserves order, this sequence is directed. Since L is almost Menger, the foregoing proposition furnishes, for each n, an element $U_n \in \mathscr{U}_n$ such that $\bigvee\{\overline{f_{-1}[U_n]} \mid n \in \mathbb{N}\} = L$. Since $f[-]$ preserves joins and $f[-]\circ f_{-1}[-] \leq \mathrm{id}_{\mathcal{S}(M)}$, we have (in light of f being onto)

$$M = f[L] = f\Big[\bigvee_{n\in\mathbb{N}} \overline{f_{-1}[U_n]}\Big] = \bigvee_{n\in\mathbb{N}} f\big[\overline{f_{-1}[U_n]}\big]$$
$$\subseteq \bigvee_{n\in\mathbb{N}} f\big[f_{-1}\big[\overline{U_n}\big]\big] \subseteq \bigvee_{n\in\mathbb{N}} \overline{U_n} \subseteq M,$$

which shows that L is almost Menger by Proposition 3.4. □

As with the case of Menger frames, the almost Menger ones can also be characterised frame-theoretically without invoking sublocales. To do this, given a collection $\{\mathfrak{o}_L(a_\alpha) \mid \alpha \in A\}$ of open sublocales of L, we note that

$$\bigvee_{\alpha\in A} \overline{\mathfrak{o}_L(a_\alpha)} = L \quad\Longleftrightarrow\quad \bigvee_{\alpha\in A} \mathfrak{c}_L(a^*_\alpha) = L$$
$$\Longleftrightarrow\quad (\forall a \in L)\Big(a = \bigwedge_{\alpha\in A} t_\alpha,\ \text{for some } t_\alpha \geq a^*_\alpha\Big);$$

where we have surreptitiously used the fact that

$$L = \Big\{\bigwedge M \mid M \subseteq \bigcup_{a\in A} \mathfrak{c}_L(a^*_\alpha)\Big\}$$

and that if $M \subseteq L$ is expressible as a union $M = \bigcup_{i\in I} M_i$ of some subsets, then, setting $m_i = \bigwedge M_i$ for each i, we have

$$\bigwedge M = \bigwedge_{i\in I} m_i.$$

Given a sequence (C_n) of covers of L, suppose that, for each n, there is a finite $D_n \subseteq C_n$ such that every element a of L is expressible as $a = \bigwedge_\alpha t_\alpha$ where each $t_\alpha \geq d^*_\alpha$ for some $d_\alpha \in \bigcup_{n\in\mathbb{N}} D_n$. We then say the sequence (D_n) is an *almost Menger witness* for the sequence (C_n). Recall the bijection between covers and coverings. The calculation above shows that a sequence of covers has an almost Menger witness if and only if the corresponding sequence of open coverings has an almost Menger witness. Consequently we have the following result.

Proposition 3.6. *A frame L is almost Menger iff for every sequence (C_n) of covers of L, there exists, for each n, a finite $D_n \subseteq C_n$ such that every element a of L is expressible as $a = \bigwedge_\alpha t_\alpha$ where each $t_\alpha \geq d^*_\alpha$ for some $d_\alpha \in \bigcup_{n\in\mathbb{N}} D_n$.*

We have seen that working with directed covers (or coverings) is often neater. After all, informally speaking, selecting an element is easier and quicker than selecting a finite subset. The frame-theoretic characterisation just stated can be couched in terms of directed covers.

Corollary 3.7. *A frame L is almost Menger iff for every sequence (C_n) of directed covers of L, we can select, for each n, an element $c_n \in C_n$ such that any $a \in L$ is expressible as $a = \bigwedge_{n\in\mathbb{N}} t_n$ for some elements $t_n \in L$ with each $t_n \geq c^*_n$.*

Proof. This follows from Proposition 3.6, the equivalences displayed in the paragraph following the proof of Proposition 3.5, and the fact that if (C_n) is a sequence of directed covers of L, then $(\mathscr{C}^{C_n})_{n\in\mathbb{N}}$ is a sequence of directed open coverings of L, and, conversely, if $(\mathscr{C}_n)$ is a sequence of directed open coverings of L, then $(C^{\mathscr{C}_n})_{n\in\mathbb{N}}$ is a sequence of directed covers of L. □

It is known that regular-closed subspaces of almost Menger spaces need not be almost Menger [20, Example 3.1], but clopen subspaces inherit the almost Menger property [13, Proposition 3.3]. In frames we present a formally stronger result. We are going to impose conditions on an onto frame homomorphism which we first show by an example not to be so stringent as to make the homomorphism an isomorphism. Recall that an element a of a frame L is *co-linear* in case $a \vee \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \vee x_i)$ for all families $\{x_i\}_{i\in I}$ of elements of L.

Example 3.8. Let a be a co-linear element of L and let $\kappa_a \colon L \to {\uparrow}a$ be the map given by $\kappa_a(x) = a \vee x$. Then κ_a is an onto, weakly perfect (actually, perfect) frame homomorphism preserving meets (since a is co-linear). Note though that κ_a is not an isomorphism if $a \neq 0$.

Now let us recall from [10, Remark 7.1] that:

> *if $h\colon L \to M$ is a perfect frame homomorphism, then $h_*(a^*) \leq h_*(a)^*$ for every $a \in M$.*

Proposition 3.9. *Let $h\colon L \to M$ be a meet-preserving perfect onto frame homomorphism. If L is an almost Menger frame, then so is M.*

Proof. Let (C_n) be a sequence of directed covers of M. Then $(h_*[C_n])$ is a sequence of directed covers of L. Since L is almost Menger, there exists, for each n, an element $u_n \in h_*[C_n]$ such that the set $\{u_n \mid n \in \mathbb{N}\}$ has the property stated in Corollary 3.7. Each u_n is of the form $h_*(c_n)$ for some $c_n \in C_n$. We show that the set $\{c_n \mid n \in \mathbb{N}\}$ has the desired property as per Corollary 3.7. Let $a \in M$. For each n, we can select $t_n \in L$ such that $t_n \geq h_*(c_n)^*$ and $h_*(a) = \bigwedge_{n\in\mathbb{N}} t_n$. By the result cited above from [10], for each n we have

$$h(t_n) \geq h\big(h_*(c_n)^*\big) \geq h\big(h_*(c_n^*)\big) = c_n^*$$

because h is onto. Now, using the fact that h preserves meets, we see that the elements $h(t_n)$ of M, for $n \in \mathbb{N}$ have the property that

$$a = h(h_*(a)) = \bigwedge_{n\in\mathbb{N}} h(t_n) \qquad \text{and} \qquad h(t_n) \geq c_n^* \text{ for each n },$$

so it follows that M is almost Menger. □

Recall that a *coframe* is a complete lattice in which binary joins distributive over meets. A frame which is simultaneously a coframe need not be Boolean.

Corollary 3.10. *If L is almost Menger, then $\mathfrak{c}_L(a)$ is almost Menger for every co-linear $a \in L$. In particular, every closed sublocale of an almost Menger frame which is also a coframe is almost Menger.*

We saw in the proof of Proposition 2.8 that if A is a sublocale of L, then A is Menger if and only if whenever $(\mathscr{U}_n)$ is a sequence of families of open sublocales of L such that $A \subseteq \bigvee \mathscr{U}_n$ for every n, then there exists, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $A \subseteq \bigvee \bigcup_{n \in \mathbb{N}} \mathscr{V}_n$. We present an almost similar result for the almost Menger property, but only for dense complemented sublocales. In the proof we shall use the fact that if S is a dense sublocale of L and U is an open sublocale of L, then $\overline{S \cap U} = \overline{U}$ [15, XIII.1.2.3]. Let us also recall that if $T \subseteq S$ are sublocales of L, then the closure of T in S is given by $\mathrm{cl}_S T = S \cap \overline{T}$ [15, III.8.5]. We also recall that if S is a sublocale of L and $(T_i)_{i \in I}$ is a family of sublocales of S, then

$$\bigvee^{\mathcal{S}(S)} \{T_i \mid i \in I\} = \bigvee^{\mathcal{S}(L)} \{T_i \mid i \in I\}.$$

In the upcoming proof, the unadorned joins will be in $\mathcal{S}(L)$.

Theorem 3.11. *The following are equivalent for a complemented dense sublocale A of L.*

1. *A is almost Menger.*
2. *Whenever $(\mathscr{U}_n)$ is a sequence of families of open sublocales of L with $A \subseteq \bigvee \mathscr{U}_n$ for every n, then there exists, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $A \subseteq \bigvee \{\overline{V} \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n\}$.*

Proof. Assume that A is almost Menger, and let $(\mathscr{U}_n)$ be a sequence of families of open sublocales of L with $A \subseteq \bigvee \mathscr{U}_n$ for every n. Using the notation in the proof of Proposition 2.8, we have that $(A \cap \mathscr{U}_n)$ is a sequence of open coverings of A. Since A is almost Menger, for each n, there is a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that

$$\begin{aligned} A &= \bigvee^{\mathcal{S}(A)} \Big\{ \mathrm{cl}_A(A \cap V) \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n \Big\} \\ &= \bigvee^{\mathcal{S}(A)} \Big\{ A \cap \overline{A \cap V} \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n \Big\} \\ &\subseteq \bigvee \Big\{ \overline{V} \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n \Big\}, \end{aligned}$$

which shows that (1) implies (2).

Conversely, assume that (2) holds. Let $(\mathscr{C}_n)$ be a sequence of open coverings of A. Then, for each n, there is a family $\mathscr{U}_n$ of open sublocales of L such that $\mathscr{C}_n = A \cap \mathscr{U}_n$. Then $A \subseteq \bigvee \mathscr{U}_n$. By hypothesis, there exists, for each n, a finite $\mathscr{V}_n \subseteq \mathscr{U}_n$ such that $A \subseteq \bigvee \{\overline{V} \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n\}$, so that

$$A = \bigvee \{A \cap \overline{V} \mid V \in \bigcup_{n \in \mathbb{N}} \mathscr{V}_n\} \qquad (\ddagger)$$

because A is complemented. For each n, put

$$\mathscr{D}_n = \{A \cap V \mid V \in \mathscr{V}_n\},$$

and observe that $\mathscr{D}_n$ is a finite subset of $\mathscr{C}_n$. Now, for each $V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$, $A \cap \overline{V} = A \cap \overline{A \cap V}$ because $\overline{A \cap V} = \overline{V}$ as A is dense and V is open. Since

$$\Big\{ \overline{A \cap V} \mid V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n \Big\} = \Big\{ \overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n \Big\},$$

we deduce from (‡) that

$$A = \bigvee^{\mathcal{S}(A)} \Big\{ A \cap \overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n \Big\} = \bigvee^{\mathcal{S}(A)} \Big\{ \mathrm{cl}_A\, D \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n \Big\},$$

which shows that A is almost Menger. ☐

As in spaces, regular almost Menger frames are Menger. In fact, there is a stronger result. Note that if D is a strong refinement of a cover C of a frame L, then the open covering $\mathscr{D} = \{\mathfrak{o}(d) \mid d \in D\}$ of L has the property that for each $U \in \mathscr{D}$ there exists a $V \in \mathscr{C}$, where $\mathscr{C} = \{\mathfrak{o}(c) \mid c \in C\}$, such that $\overline{U} \subseteq V$ because $d \prec c$ implies $\overline{\mathfrak{o}(d)} \subseteq \mathfrak{o}(c)$. As in the case of covers, let us say an open covering $\mathscr{U}$ is a *strong refinement* of an open covering $\mathscr{V}$ if for each $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that $\overline{U} \subseteq V$.

Proposition 3.12. *A cover regular almost Menger frame is Menger.*

Proof. Let L be a cover regular almost Menger frame, and let $(\mathscr{C}_n)$ be a sequence of open coverings of L. For each n, let $\widehat{\mathscr{C}_n}$ be a strong refinement of $\mathscr{C}_n$, so that we have the sequence $(\widehat{\mathscr{C}_n})$ of open coverings of L. Since L is almost Menger, we can choose, for each n, a finite $\mathscr{V}_n \subseteq \widehat{\mathscr{C}_n}$ such that

$$\bigvee \Big\{ \overline{V} \mid V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n \Big\} = L.$$

Since each $\widehat{\mathscr{C}_n}$ is a strong refinement of $\mathscr{C}_n$, and since $\mathscr{V}_n$ is finite, there is a finite $\mathscr{D}_n \subseteq \mathscr{C}_n$ such that the closure of each sublocale in $\mathscr{V}_n$ is contained in some sublocale in $\mathscr{D}_n$. Consequently,

$$L = \bigvee \Big\{ \overline{V} \mid V \in \bigcup_{n\in\mathbb{N}} \mathscr{V}_n \Big\} \subseteq \bigvee \Big\{ D \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{D}_n \Big\} \subseteq L,$$

showing that $(\mathscr{D}_n)$ is a Menger witness for $(\mathscr{C}_n)$. Therefore L is Menger. ☐

We shall now present a result which is, in a way, an analogue of [20, Proposition 3.7]. We recall some pertinent terminology and facts. A frame is called *scattered* [18] just in case every sublocale of it is complemented. As observed in [18, p. 315]:

> *If* $f\colon L \to M$ *is a localic map with* M *scattered, then* $f_{-1}\Big[\bigvee_{i\in I} S_i\Big] = \bigvee_{i\in I} f_{-1}[S_i]$ *for every family* $\{S_i\}_{i\in I}$ *of sublocales of* M.

A localic map $f\colon L \to M$ is called *nearly open* if $f_{-1}[\overline{V}] = \overline{f_{-1}[V]}$ for every open sublocale V of M. This is a conservative extension of Pták's [19] notion of nearly open continuous maps, and it is equivalent to saying $h(a^*) = h(a)^*$

for every $a \in L$, where h is the left adjoint of f. Incidentally, this latter condition was used by Banaschewski and Pultr [3] to define nearly open frame homomorphisms. Recall that a localic map is called *proper* if it is closed and preserves directed joins.

Theorem 3.13. *Let $f\colon L \to M$ be a nearly open proper map of locales. Suppose that M is scattered and covered by its compact sublocales. If M is almost Menger, then L is almost Menger.*

Proof. Let $(\mathscr{C}_n)$ be a sequence of directed open coverings of L. Write the set of the compact sublocales of M as an indexed family $\{K_\alpha \mid \alpha \in A\}$. Since f is a proper map, each $f_{-1}[K_\alpha]$ is a compact sublocale of L [21, Corollary 4.3]. Fix $n \in \mathbb{N}$. Since $\mathscr{C}_n$ is an open covering of L, for each $\alpha \in A$ we have the containment $f_{-1}[K_\alpha] \subseteq \bigvee \mathscr{C}_n$, which, by compactness and the fact that $\mathscr{C}_n$ is an increasing open covering of L, implies $f_{-1}[K_\alpha] \subseteq C_{n\alpha}$, for some $C_{n\alpha} \in \mathscr{C}_n$. Since $C_{n\alpha}$ is an open sublocale of L, there exists an element $c_{n\alpha} \in L$ such that $C_{n\alpha} = \mathfrak{o}_L(c_{n\alpha})$. Since M is scattered and covered by its compact sublocales,

$$L = f_{-1}[M] = f_{-1}\Big[\bigvee_{\alpha \in A} K_\alpha\Big] = \bigvee_{\alpha \in A} f_{-1}[K_\alpha] \subseteq \bigvee_{\alpha \in A} \mathfrak{o}_L(c_{n\alpha}),$$

which implies that the collection $\mathscr{U}_n = \{\mathfrak{o}_L(c_{n\alpha}) \mid \alpha \in A\}$ is an open covering of L. We show from this that the collection

$$\mathscr{W}_n = \{\mathfrak{o}_M(f(c_{n\alpha})) \mid \alpha \in A\}$$

is an open covering of M; and the idea for that is to show that, for each $\alpha \in A$, $K_\alpha \subseteq \mathfrak{o}_M(f(c_{n\alpha}))$. Since $f_{-1}[K_\alpha] \subseteq \mathfrak{o}_L(c_{n\alpha})$, upon taking supplements, we have

$$\mathfrak{c}_L(c_{n\alpha}) = L \smallsetminus \mathfrak{o}_L(c_{n\alpha}) \subseteq L \smallsetminus f_{-1}[K_\alpha].$$

Taking direct images, and using the fact that f is a closed map, we obtain

$$\mathfrak{c}_M(f(c_{n\alpha})) = f[\mathfrak{c}_L(c_{n\alpha})] \subseteq f\big[L \smallsetminus f_{-1}[K_\alpha]\big] \subseteq M \smallsetminus K_\alpha;$$

where the last containment is obtained from [9, Equation (5.2)]. Since every sublocale of M is complemented, taking supplements in the containment above yields

$$K_\alpha \;=\; M \smallsetminus (M \smallsetminus K_\alpha) \;\subseteq\; M \smallsetminus \mathfrak{c}_M(f(c_{n\alpha})) \;=\; \mathfrak{o}_M(f(c_{n\alpha})),$$

whence we deduce that $\mathscr{W}_n$ is an open covering of M. Since M is almost Menger, the sequence $(\mathscr{W}_n)$ has an almost Menger witness, $(\mathscr{W}_n')$, say. Thus, for each $n \in \mathbb{N}$, there exists some $k(n) \in \mathbb{N}$ and indices $\alpha_{(n,1)}, \dots, \alpha_{(n,k(n))}$ in A such that

$$\mathscr{W}_n' = \big\{\mathfrak{o}_M\big(f(c_{n,\alpha(n,1)})\big), \dots, \mathfrak{o}_M\big(f(c_{n,\alpha(n,k(n))})\big)\big\}$$

and

$$(\#) \qquad \bigvee\Big\{\overline{W} \mid W \in \bigcup_{n \in \mathbb{N}} \mathscr{W}_n'\Big\} = M.$$

We claim that the sequence $(\mathscr{C}'_n)$, where, for each n,

$$\mathscr{C}'_n = \big\{\mathfrak{o}_L(c_{n,\alpha(n,1)}), \ldots, \mathfrak{o}_L(c_{n,\alpha(n,k(n))})\big\}$$

is an almost Menger witness for $(\mathscr{C}_n)$. It is clear that each $\mathscr{C}'_n$ is a finite subset of $\mathscr{C}_n$. Since M is scattered and f is nearly open, from (#) we obtain

$$L = f_{-1}\Big[\bigvee\Big\{\overline{W} \mid W \in \bigcup_{n\in\mathbb{N}} \mathscr{W}'_n\Big\}\Big] = \bigvee\Big\{f_{-1}\big[\overline{W}\big] \mid W \in \bigcup_{n\in\mathbb{N}} \mathscr{W}'_n\Big\}.$$

Now, if $W \in \mathscr{W}'_m$ for some $m \in \mathbb{N}$, then $W = \mathfrak{o}_M\big(f(c_{m,\alpha(m,i)})\big)$, for some $i \in \{1, \ldots, k(m)\}$, so that $\overline{W} = \mathfrak{c}_M\big(f(c_{m,\alpha(m,i)})^*\big)$. Since $h(f(b)) \leq b$, so that $b^* \leq h(f(b))^*$, for any $b \in M$, and since f is nearly open, we therefore have

$$\begin{aligned} f_{-1}\big[\overline{W}\big] = \mathfrak{c}_L\big(h(f(c_{m,\alpha(m,i)})^*)\big) &= \mathfrak{c}_L\big(h(f(c_{m,\alpha(n,i)}))^*\big) \\ &\subseteq \mathfrak{c}_L(c^*_{m,\alpha(m,i)}) \\ &\subseteq \bigvee\Big\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{C}'_n\Big\}, \end{aligned}$$

because $\mathfrak{o}_L(c_{m,\alpha(m,i)}) \in \mathscr{C}'_m$. It follows therefore that

$$\bigvee\Big\{\overline{D} \mid D \in \bigcup_{n\in\mathbb{N}} \mathscr{C}'_n\Big\} = L,$$

which proves that L is almost Menger. □

The condition that M is scattered was used, among other things, to ensure that joins (actually, only those that cover the codomain) are preserved under pullback. It is well known that coverings are generally not preserved under pullback. However, since in the first part of the proof it is a special type of a covering (by compact sublocales) that is pulled back along a special type of a localic map (a nearly open one), it is perhaps worth pointing out that we have not over hypothesised by requiring the codomain to be scattered. Here is an example demonstrating the point.

Example 3.14. Consider the frame $\Omega(\mathbb{R})$, and let $j\colon \mathfrak{B}(\Omega(\mathbb{R})) \to \Omega(\mathbb{R})$ be the inclusion of its smallest dense sublocale. For any frame L, the frame homomorphism $(-)^{**}\colon L \to \mathfrak{B}L$ is nearly open because the pseudocomplement of an element in any dense sublocale calculated in the sublocale is exactly its pseudocomplement calculated in the frame. Thus j is a nearly open localic map. For any $x \in \mathbb{R}$, denote by $\tilde{x}$ the prime element $\mathbb{R} \smallsetminus \{x\}$ of $\Omega(\mathbb{R})$. By spatiality,

$$\Omega(\mathbb{R}) = \bigvee\big\{\{\tilde{r}, 1_{\Omega(\mathbb{R})}\} \mid r \in \mathbb{R}\big\},$$

so that $\Omega(\mathbb{R})$ is covered by its (compact) one-point sublocales. For any $r \in \mathbb{R}$, the set-theoretic inverse image $j^{-1}[\{\tilde{r}, 1_{\Omega(\mathbb{R})}\}] = \{1_{\Omega(\mathbb{R})}\}$ because $\tilde{r}^{**} \neq \tilde{r}$. Hence $j_{-1}[\{\tilde{r}, 1_{\Omega(\mathbb{R})}\}] = \mathsf{O}$, which then says

$$\mathsf{O} = \bigvee_{r\in\mathbb{R}} j_{-1}[\{\tilde{r}, 1_{\Omega(\mathbb{R})}\}] \qquad \text{whereas} \qquad j_{-1}\Big[\bigvee_{r\in\mathbb{R}}\{\tilde{r}, 1_{\Omega(\mathbb{R})}\}\Big] = \mathfrak{B}(\Omega(\mathbb{R})),$$

showing that, generally, j_{-1} fails to preserve joins of (covering) compact sublocales.

We close with a result that shows that we can replace open coverings with regular-open coverings in the definition of almost Menger frames. To recall, a sublocale of L is called *regular-open* if it is of the form $\mathfrak{o}_L(a)$ with $a = a^{**}$. An element of L of the form x^{**} is called *regular*. Clearly, the bijection between covers and open coverings restricts to a bijection between covers consisting entirely of regular elements and open coverings consisting entirely of regular-open sublocales. For any cover C we will write $C^{**} = \{c^{**} \mid c \in C\}$, and observe that C^{**} is also a cover, consisting of regular elements.

Proposition 3.15. *A frame is almost Menger iff every sequence of open coverings consisting entirely of regular-open sublocales has an almost Menger witness.*

Proof. Only one implication needs proving. We do it via covers. So, suppose that every sequence of covers of L consisting entirely of regular elements has an almost Menger witness. Let (C_n) be a sequence of covers of L, and then consider the sequence (C_n^{**}). By our supposition, (C_n^{**}) has an almost Menger witness; so, for each n, there exists a finite $U_n \subseteq C_n^{**}$ such that any $a \in L$ is expressible as

$$a = \bigwedge_\alpha t_\alpha, \text{ where each } t_\alpha \geq u_\alpha^* \text{ for some } u_\alpha \in \bigcup_{n\in\mathbb{N}} U_n.$$

For each n, there exists a positive integer k_n and finitely many elements $c_{n1}, \ldots, c_{nk_n}$ in C_n such that $U_n = \{c_{n1}^{**}, \ldots, c_{nk_n}^{**}\}$. Let $D_n = \{c_{n1}, \ldots, c_{nk_n}\}$. Since $(x^{**})^* = x^*$ always, it follows that (D_n) is an almost Menger witness for (C_n). Therefore L is almost Menger. □

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