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Sum connectedness in proximity spaces

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ABSTRACT

The notion of sum δ-connected proximity spaces which contain the category of δ-connected and locally δ-connected spaces is defined. Several characterizations of it are substantiated. Weaker forms of sum δ-connectedness are also studied.

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KEYWORDS: *sum δ-connected; δ-connected; δ-component; locally δ-connected.*

1. INTRODUCTION

The notion of proximity was introduced by Efremovic [4, 5] as a natural generalization of metric spaces and topological groups. Smirnov [10, 11] and Naimpally [8, 9] did the most significant and extensive work in this area. In 2009, Bezhanishvili [1] defined zero-dimensional proximities and zero-dimensional compactifications.

Mrówka *et al.* [7] introduced the theory of δ-connectedness (or equiconnectedness) in proximity spaces. Consequently, Dimitrijević *et al.* [2, 3] defined local δ-connectedness, δ-component and the treelike proximity spaces. In 1978, Kohli [6] introduced the notion of sum connectedness in topological spaces.

We discuss sum δ-connectedness in proximity spaces in this paper. Some necessary definitions and the results which are used in further sections, are recalled in Section 2. In Section 3, sum δ-connectedness is defined and its relations with other kinds of connectedness are determined. Several characterizations of

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it are established. It is shown that sum δ-connectedness is equivalent to local δ-connectedness in a zero-dimensional proximity space. Further, the Stone-Čech compactification of a separated proximity space X is sum δ-connected if and only if X is sum δ-connected and it has finitely many δ-components. For a sum δ-connected proximity space to be sum connected, a sufficient condition is deduced. In the last section, weaker forms of sum δ-connectedness are defined. Finally, if a sum δ-connected space is δ-padded, then it is also locally δ-connected.

2. Preliminaries

Definition 2.1 ([9])**.** A binary relation δ on the power set $\mathcal{P}(X)$ of X is said to be a proximity on X, if the following axioms are satisfied for all P, Q, R in $\mathcal{P}(X)$:

(*i*) $(\phi, P) \notin \delta$;
(*ii*) If $P \cap Q \neq \phi$, then $(P, Q) \in \delta$;
(*iii*) If $(P, Q) \in \delta$, then $(Q, P) \in \delta$;
(*iv*) $(P, Q \cup R) \in \delta$ if and only if $(P, Q) \in \delta$ or $(P, R) \in \delta$;
(*v*) If $(P, Q) \notin \delta$, then there exists a subset R of X such that $(P, R) \notin \delta$ and $(X \backslash R, Q) \notin \delta$.

The pair (X, δ) is called a proximity space.

Throughout this paper, we simply write proximity space (X, δ) as X whenever there is no confusion of the proximity δ.

Definition 2.2 ([8, 9])**.** A proximity space X is said to be separated if $x = y$ whenever $(\{x\}, \{y\}) \in \delta$ for $x, y \in X$.

Proposition 2.3 ([9])**.** *Let X be a proximity space and P be a subset of X. If P is δ-closed if and only if $x \in P$ whenever $(\{x\}, P) \in \delta$, then the collection of the complements of all δ-closed sets forms a topology $\mathcal{T}_\delta$ on X.*

Proposition 2.4 ([9])**.** *Let X be a proximity space. Then the closure $C(P)$ of P with respect to $\mathcal{T}_\delta$ is given by $C(P) = \{x \in X : (\{x\}, P) \in \delta\}$.*

Corollary 2.5 ([9])**.** *Let X be a proximity space. Then $M \in \mathcal{T}_\delta$ if and only if $(\{x\}, X \backslash M) \notin \delta$ for every $x \in M$.*

Using Proposition 2.4, a set F is δ-closed if $C(F) = F$. From Corollary 2.5, a set U is δ-open, if $(\{x\}, X \backslash U) \notin \delta$ for every $x \in U$.

Definition 2.6 ([9])**.** Let X be a proximity space and $\mathcal{T}$ be a topology on X. Then δ is said to be compatible with $\mathcal{T}$ if the generated topology $\mathcal{T}_\delta$ and $\mathcal{T}$ are equal, that is, $\mathcal{T}_\delta = \mathcal{T}$.

Definition 2.7 ([9])**.** Let X be a proximity space. Then a subset N of X is said to be a δ-neighbourhood of $M \subset X$ if $(M, X \backslash N) \notin \delta$. It is denoted by $M \ll_\delta N$.

Definition 2.8 ([9]). Let (X, δ) and (Y, δ') be two proximity spaces. Then a map $f : (X, \delta) \longrightarrow (Y, \delta')$ is said to be δ-continuous (or p-continuous) if $(f(P), f(Q)) \in \delta'$ whenever $(P, Q) \in \delta$, for all $P, Q \subset X$.

Definition 2.9 ([7]). Let X be a proximity space. Then X is said to be δ-connected if every δ-continuous map from X to a discrete proximity space is constant.

Theorem 2.10 ([7]). *Let X be a proximity space. Then the following statements are equivalent:*

(i) X is δ-connected.
(ii) $(P, X\backslash P) \in \delta$ for each nonempty subset P with $P \neq X$.
(iii) For every δ-continuous real-valued function f, the image $f(X)$ is dense in some interval of R.
(iv) If $X = P \cup Q$ and $(P, Q) \notin \delta$, then either $P = \phi$ or $Q = \phi$.

Definition 2.11 ([2]). Let X be a proximity space and $x \in X$. Then the δ-component of a point x is defined as the union of all δ-connected subsets of X containing x. It is denoted by $C_\delta(x)$.

Definition 2.12 ([2]). Let X be a proximity space and $x \in X$. Then the δ-quasi component of x is the equivalence class of x with respect to the equivalence relation $\sim$ defined on X as " $x \sim y$ if and only if there do not exist the sets M, N such that $x \in M$ and $y \in N$ with $X = M \cup N$ and $(M, N) \notin \delta$".

Definition 2.13 ([2]). A proximity space X is called locally δ-connected if for every point x of X and for every δ-neighbourhood N of x, there exists some δ-connected δ-neighbourhood M of x such that $x \in M \subset N$.

Definition 2.14 ([12]). Let (X, δ) be a proximity space and $f : X \longrightarrow Y$ be a surjective map, where Y is any set. Then the quotient proximity on Y is the finest proximity such that the map f is δ-continuous. When Y has the quotient proximity, f is called δ-quotient map.

Proposition 2.15 ([12]). *Let (X, δ) be a proximity space and $f : X \longrightarrow Y$ be a surjective map, where Y be any set. Then the quotient proximity δ' on Y is given by $P \ll_{\delta'} Q$ if and only if for each binary rational $s \in [0, 1]$, there is some $P_s \subseteq Y$ such that $P_0 = P$, $P_1 = Q$ and $s < t$ implies $f^{-1}(P_s) \ll_\delta f^{-1}(P_t)$.*

Proposition 2.16 ([12]). *Let (X, δ) be a proximity space and $f : X \longrightarrow Y$ be a surjective map such that $f^{-1}(f(M)) = M$ for each δ-open set M of X, where Y be any set. Then the quotient proximity δ' on Y is given by $(P, Q) \in \delta'$ if and only if $(f^{-1}(P), f^{-1}(Q)) \in \delta$.*

Definition 2.17 ([1]). A proximity space X is said to be zero-dimensional if the proximity δ satisfies the following axiom:

If $(P, Q) \notin \delta$, then there is a subset R of X such that $(R, X\backslash R) \notin \delta$, $(P, R) \notin \delta$ and $(X\backslash R, Q) \notin \delta$.

Definition 2.18 ([6]). A topological space X is said to be sum connected at $x \in X$, if there exists an open connected neighbourhood of x. If X is sum connected at each of its points, then X is called sum connected.

Proposition 2.19 ([6]). *Let X^* be the Stone-Čech compactification of a Tychonoff space X. Then X is sum connected and has finitely many components, if X^* is sum connected.*

3. Sum δ-connectedness

Definition 3.1. A proximity space X is said to be sum δ-connected at $x \in X$ if there exists a δ-connected δ-open δ-neighbourhood of x. If X is sum δ-connected at each of its points, then it is said to be sum δ-connected.

Definition 3.2. Let $(X_i, \delta_i)_{i \in \mathcal{I}}$ be a family of proximity spaces, where $\mathcal{I}$ is an index set. A proximity space (X, δ) is said to be a far proximity sum of $(X_i)_{i \in \mathcal{I}}$ if $X = \bigcup_{i \in \mathcal{I}} X_i$ and $(X_i, X_j) \notin \delta$ for all $i \neq j$ in $\mathcal{I}$ with $\delta|_{X_i} = \delta_i$ for all $i \in \mathcal{I}$.

Note that a proximity space X is sum δ-connected if and only if each of its δ-component is δ-open. Therefore, every δ-connected proximity space is sum δ-connected.

Example 3.3.

(*i*) Let X be any discrete proximity space with $|X| \geq 2$. Then X is sum δ-connected but not δ-connected.

(*ii*) Let $X = (0,1) \cup (2,3)$ with usual subspace proximity of $\mathbb{R}$. Then X is sum δ-connected but not δ-connected.

Every sum connected proximity space is sum δ-connected. But, converse may not be true. However, in compact separated proximity spaces, the notion of sum connectedness and sum δ-connectedness coincides.

Example 3.4. The space $\mathbb{Q}$ of rationals with the usual proximity is sum δ-connected. But, it is not sum connected.

Every locally δ-connected proximity space is sum δ-connected. Converse may not be true.

Example 3.5. Consider $T = \{(x, \sin\frac{1}{x}) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$ the closed Topologist's Sine curve with subspace proximity induced from $\mathbb{R}^2$. Let X be the far proximity sum of two copies of T. Then X is sum δ-connected but it is neither δ-connected nor locally δ-connected.

Example 3.6. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ be a proximity space. Since each $\{\frac{1}{n}\}$ is δ-clopen in X, there does not exists any δ-connected δ-neighbourhood of 0 in X because every δ-neighbourhood of 0 contains infinitely many members of $X \backslash \{0\}$. Thus, X is not sum δ-connected at 0.

Thus, we have following relationship among several connectednesses in proximity space.

$$\begin{array}{ccccc} \text{connected} & \Longrightarrow & \text{sum connected} & \Longleftarrow & \text{locally connected} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \delta-\text{connected} & \Longrightarrow & \text{sum } \delta-\text{connected} & \Longleftarrow & \text{locally } \delta-\text{connected} \end{array}$$

The following theorem gives some necessary and sufficient conditions for sum δ-connectedness.

Theorem 3.7. *For a proximity space X, the following statements are equivalent:*

- (i) *X is sum δ-connected.*
- (ii) *For each $x \in X$ and each δ-clopen set U which contains x, there exists a δ-open δ-connected set W containing x such that $W \subset U$.*
- (iii) *δ-components of δ-clopen sets in X are δ-open in X.*

Proof. $(i) \Longrightarrow (ii)$. Let $x \in X$ and U be a δ-clopen set such that $x \in U$. Let $C_\delta(x)$ be the δ-component of X containing x. By hypothesis, $C_\delta(x)$ is δ-open. So, $C_\delta(x) \cap U$ is δ-clopen. Therefore, $((C_\delta(x) \cap U), C_\delta(x) \backslash (C_\delta(x) \cap U)) \notin \delta$ as $C_\delta(x) \cap U \subset C_\delta(x) \subset X$. Also, since $C_\delta(x)$ is δ-connected, $C_\delta(x) \cap U = C_\delta(x)$. Hence, $C_\delta(x)$ is a δ-open δ-connected such that $C_\delta(x) \subset U$.

$(ii) \Longrightarrow (iii)$. Let U be any δ-clopen set in X and C_δ be a δ-component of U. Then, by hypothesis, for each $x \in C_\delta$ there exists a δ-open δ-connected set W such that $x \in W \subset U$. Therefore, $W \subset C_\delta$ as C_δ is δ-component. Hence, C_δ is δ-open.

$(iii) \Longrightarrow (i)$. Since X is δ-clopen, the result follows. □

Proposition 3.8. *Let Y be a dense proximity subspace of X and $x \in Y$. Then X is sum δ-connected at x if Y is sum δ-connected at x.*

Proof. Let W be a δ-open δ-connected δ-neighbourhood of x in Y. Therefore, $W = U \cap Y$, where U is δ-open δ-neighbourhood of x in X. Thus, $W \subset U$ and $U \subset Cl_X(U) = Cl_X(W)$ as Y is dense in X. Note that $Cl_X(W)$ is δ-connected. Hence, U is δ-open δ-connected δ-neighbourhood of x in X. □

Next example shows that the closure of sum δ-connected proximity space may not be sum δ-connected.

Example 3.9. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ be a proximity subspace of $\mathbb{R}$. Then each δ-component $\{\frac{1}{n}\}$ is δ-clopen in X. So, X is sum δ-connected. But, note that $Cl(X) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is not sum δ-connected at 0 by Example 3.6.

Proposition 3.10. *Let X be a sum δ-connected proximity space and $f : (X, \delta) \longrightarrow (Y, \delta^*)$ be a δ-quotient map such that $f^{-1}(f(U)) = U$ for each δ-open subset U of X. Then Y is sum δ-connected.*

Proof. Let C_δ be any δ-component of Y and $y \in C_\delta$. We have to show that $(y, Y \backslash C_\delta) \notin \delta^*$. By definition of δ-quotient proximity δ^*, it suffices to show that $(f^{-1}(y), X \backslash f^{-1}(C_\delta)) \notin \delta$. Let $x \in f^{-1}(y)$, then the δ-component C_x of x in X, be δ-open in X. Therefore, $(z, X \backslash C_x) \notin \delta$ for every $z \in C_x$. Since

f is δ-continuous, $f(C_x)$ is δ-connected. Thus, $y = f(x) \in f(C_x) \cap C_\delta$. So $f(C_x) \subseteq C_\delta$, which implies $C_x \subseteq f^{-1}(C_\delta)$. Then, $(z, X \backslash f^{-1}(C_\delta)) \notin \delta$ for every $z \in C_x$. In particular, $(f^{-1}(y), X \backslash f^{-1}(C_\delta) \notin \delta$. $\square$

Corollary 3.11. *Let $f : (X, \delta) \longrightarrow (Y, \delta^*)$ be a δ-continuous, δ-closed, surjection such that $f^{-1}(f(U)) = U$ for each δ-open subset U of X. If X is sum δ-connected, then Y is also sum δ-connected.*

Proposition 3.12. *Every δ-continuous, δ-open image of a sum δ-connected proximity space is sum δ-connected.*

Proof. Let $f : (X, \delta) \longrightarrow (Y, \delta')$ be a δ-continuous, δ-open, surjective map and X be sum δ-connected. Let C_δ be a δ-component of Y and $x \in f^{-1}(C_\delta)$. Then there is a δ-component C_x in X containing x which is δ-open. Since f is δ-continuous and δ-open, $f(C_x) \subseteq C_\delta$ and $f(C_x)$ is δ-open. Therefore, $(f(x), Y \backslash f(C_x)) \notin \delta'$. Hence, $(f(x), Y \backslash C_\delta) \notin \delta'$. $\square$

Corollary 3.13. *If the product of proximity spaces is sum δ-connected, then each of its factor is also sum δ-connected.*

The product of sum δ-connected proximity spaces need not be sum δ-connected in general.

Example 3.14. Let $X = \{0, 1\}^\omega$ be infinite product of two point discrete proximity spaces. Then X is not discrete proximity space. Therefore, the δ-component $C_\delta(x)$ of x in X is $\{x\}$ itself, which is not δ-open. Hence, X is not sum δ-connected.

Theorem 3.15. *Let (X, δ) be a product of proximity spaces $(X_i, \delta_i)_{i \in \mathcal{I}}$, where $\mathcal{I}$ is an index set. Then $X = \prod_{i \in \mathcal{I}} X_i$ is sum δ-connected if and only if each X_i is sum δ-connected and all but finitely many X_i's are δ-connected.*

Proof. Let X be sum δ-connected. So, by Corollary 3.13, each X_i is sum δ-connected. Now, suppose that all but finitely many X_i's are not δ-connected. Then any δ-component of X is not δ-open in X, which is a contradiction.

Conversely, assume that each X_i is sum δ-connected and all but finitely many X_i's are δ-connected. Let C_δ be any δ-component of X and p_i be the i^{th} projection map. Then $p_i(C_\delta)$ is δ-connected for each $i \in \mathcal{I}$. Therefore, $\prod_{i \in \mathcal{I}} p_i(C_\delta)$ is also δ-connected. Thus, $C_\delta = \prod_{i \in \mathcal{I}} p_i(C_\delta)$. For each $i \in \mathcal{I}$, suppose C_{δ_i} be the δ_i-component of X_i containing $p_i(C_\delta)$. Put $C'_\delta = \prod_{i \in \mathcal{I}} C_{\delta_i}$. If $p_i(C_\delta) \subsetneq C_{\delta_i}$, then $C_\delta = C'_\delta$ as C_δ is δ-component of X. Thus, $p_i(C_\delta) = C_{\delta_i}$ for each $i \in \mathcal{I}$. Since all but finitely many X_i's are δ-connected, $p_i(C_\delta) = C_{\delta_i} = X_i$ for all but finitely many $i \in \mathcal{I}$. Hence, C_δ is δ-open set in X. $\square$

Theorem 3.16. *Every far proximity sum of sum δ-connected proximity spaces is sum δ-connected.*

It can be easily shown that a δ-closed subspace of sum δ-connected proximity space need not be sum δ-connected.

Corollary 3.17. *A proximity space X is locally δ-connected if and only if every δ-open subspace of X is sum δ-connected.*

Theorem 3.18. *Let X be a pseudocompact, separated, sum δ-connected proximity space. Then it has at most finitely many δ-components.*

Proof. Suppose X has infinitely many δ-components. Since collection of δ-components of X is locally finite and each δ-component of X is δ-open, we have a locally finite collection of non-empty δ-open sets which is not finite, a contradiction. □

Corollary 3.19. *If X is compact sum δ-connected proximity space, then it has at most finitely many δ-components.*

Corollary 3.20. *If X is Lindelof (or separable) sum δ-connected proximity space, then it has at most countably many δ-components.*

Theorem 3.21. *Every separated, zero-dimensional, sum δ-connected proximity space is discrete.*

Proof. Let X be any separated, zero-dimensional, sum δ-connected proximity space. Let S be a subset of X such that $x, y \in S$ with $x \neq y$. Therefore, $(\{x\}, \{y\}) \notin \delta$. Then, there exists $C \subset X$ such that $(C, X \backslash C) \notin \delta$, $(\{x\}, C) \notin \delta$ and $(X \backslash C, \{y\}) \notin \delta$. So, $(C, S \backslash C) \notin \delta$ which implies S is not δ-connected. Hence, every δ-component of X is singleton. As X is sum δ-connected, each singleton of X is δ-open. □

Next theorem shows that in a zero-dimensional proximity space, local δ-connectedness and sum δ-connectedness are equivalent.

Proposition 3.22. *A zero-dimensional proximity space X is locally δ-connected if and only if it is sum δ-connected.*

Proof. Necessity is obvious. For the sufficient part, let X be sum δ-connected. Let $x \in X$ and U be a δ-neighbourhood of x. Therefore, there exists $C \subset X$ such that $(C, X \backslash C) \notin \delta$, $(\{x\}, X \backslash C) \notin \delta$ and $(C, X \backslash U) \notin \delta$. Thus, C is δ-clopen and $x \in C \subset U$. So, by Theorem 3.7, there exists a δ-open δ-connected set W such that $x \in W \subset C \subset U$. Hence, X is locally δ-connected. □

Now, we find the relation of sum δ-connectedness of proximity space with its Stone-Čech compactification.

Theorem 3.23. *Let (X^*, δ^*) be the Stone-Čech compactification of the separated proximity space (X, δ). Then X^* is sum δ-connected if and only if X is sum δ-connected and has finitely many δ-components.*

Proof. Let X^* be sum δ-connected. Then, by Corollary 3.19, it has finitely many δ-components. So, $X^* = \bigcup_{i=1}^{n} C_\delta^i$, where C_δ^i is a δ-component of X^* for each $1 \leq i \leq n$. Therefore, $X = \bigcup_{i=1}^{n} (C_\delta^i \cap X)$. As each $C_\delta^i \cap X$ is δ-open in X and $(C_\delta^i \cap X, C_\delta^j \cap X) \notin \delta$ by using hypothesis, it suffices to show that each $C_\delta^i \cap X$ is δ-connected. Let $C_\delta^i \cap X = P \cup Q$ with $(P, Q) \notin \delta^*$. Note that

$Cl_{\delta^*}(C^i_\delta \cap X) = C^i_\delta$ because C^i_δ is δ-open in X^* and X is dense in X^*. Therefore, $C^i_\delta = Cl_{\delta^*}(C^i_\delta \cap X) = Cl_{\delta^*}(P) \cup Cl_{\delta^*}(Q)$ with $(Cl_{\delta^*}(P), Cl_{\delta^*}(Q)) \notin \delta^*$. Thus, C^i_δ is not δ-connected, a contradiction.

Conversely, assume X is sum δ-connected and has finitely many δ-components. Therefore, $X = \bigcup_{i=1}^n C^i_\delta$ where C^i_δ is a δ-component of X for each $1 \leq i \leq n$. Thus, $X^* = Cl_{\delta^*}(X) = \bigcup_{i=1}^n Cl_{\delta^*}(C^i_\delta)$. Since, $(C^i_\delta, C^j_\delta) \notin \delta$ for $i \neq j$, $(Cl_{\delta^*}(C^i_\delta), Cl_{\delta^*}(C^j_\delta)) \notin \delta^*$. Note that each $Cl_{\delta^*}(C^i_\delta)$ is δ-connected in X^*. Thus, each $Cl_{\delta^*}(C^i_\delta)$ is a δ-component in X^*. Since δ-components in X^* are finite, hence X^* is sum δ-connected. □

Corollary 3.24. *If X is pseudocompact, separated and sum δ-connected proximity space, then it's Stone-Čech compactification X^* is also sum δ-connected.*

Every sum connected proximity space is sum δ-connected. Following theorem gives the sufficient condition for a sum δ-connected proximity space to be sum connected.

Theorem 3.25. *Let $(X, \mathcal{T})$ be a Tychonoff space. If X is sum δ-connected and has finitely many δ-components with respect to any proximity δ compatible with $\mathcal{T}$, then X is sum connected. Moreover, it has at most finitely many components.*

Proof. Let $\mathcal{S}$ be the collection of all proximities which are compatible with $\mathcal{T}$. Let $\delta_0 = \sup \mathcal{S}$, then δ_0 is also compatible with $\mathcal{T}$. Therefore, by hypothesis, X is sum δ-connected and has finitely many δ-components with respect to δ_0. Since $\delta_0 = \sup \mathcal{S}$, the compactification (X^*, δ^*) corresponding to δ_0 is Stone-Čech compactification. So, by Theorem 3.23, X^* is sum δ-connected. Thus, X^* is sum connected. By Proposition 2.19, X is sum connected and has finitely many components. □

4. Weaker forms of sum δ-connectedness

In this section we give proximity versions of notions defined and considered in [6].

Definition 4.1. Let X be a proximity space which contains a point x. Then X is called :

- (i) weakly sum δ-connected at x if there exists a δ-connected δ-neighbourhood of x.
- (ii) quasi sum δ-connected at x if the δ-quasi component which contains x is a δ-neighbourhood of x.
- (iii) δ-padded at x if for every δ-neighbourhood W of x there exist δ-open sets U and V such that $x \in U \subseteq Cl_\delta(U) \subseteq V \subseteq W$ and $V \setminus Cl_\delta(U)$ has at most finitely many δ-components.

If a proximity space X is weakly sum δ-connected (or quasi sum δ-connected) at each of its points, then the space X is called weakly sum δ-connected (or quasi sum δ-connected). For a proximity space X, sum δ-connected $\Rightarrow$ weakly sum δ-connected $\Rightarrow$ quasi sum δ-connected

Example 4.2. In $\mathbb{R}^2$, let B_n be the infinite broom containing all the closed line segments joining the point $(\frac{1}{n}, 0)$ to the points $\{(\frac{1}{n+1}, \frac{1}{m}) : m = n, n+1, \cdots\}$, where $n = 1, 2, \cdots$. Let $B = \bigcup_{n=1}^{\infty} B_n$ and $A = \{(x, 0) : 0 \leq x \leq 2\} \cup \{(y, \frac{1}{n}) : 1 \leq y \leq 2 \text{ and } n = 1, 2, \cdots\}$. Let $X = A \cup B$. Then note that X is compact. Therefore, connectedness is equivalent to δ-connectedness. Hence, X is weak sum δ-connected but not sum δ-connected at $(0, 0)$.

Lemma 4.3. *Every δ-open δ-quasi component is a δ-component.*

Proof. Let U be a δ-open δ-quasi component of proximity space X and $x \in U$. Let V be the δ-component of x. Then $V \subset U$. Let $y \in U \backslash V$. So, $x \sim y$. Since V is δ-closed in X and $V \subset U$, V is δ-closed in U. So, $U \backslash V$ is δ-open in U. As U is δ-open in X, $U \backslash V$ is δ-open in X. Therefore, $(U \backslash V, X \backslash (U \backslash V)) \notin \delta$. Thus, $X = (U \backslash V) \cup (X \backslash (U \backslash V))$ with $(U \backslash V, X \backslash (U \backslash V)) \notin \delta$. Hence, $x \nsim y$ which is a contradiction. □

Proposition 4.4. *For a given proximity space X, the following statements are comparable:*

(i) *X is quasi sum δ-connected.*
(ii) *X is weakly sum δ-connected.*
(iii) *X is sum δ-connected.*
(iv) *δ-components of X are δ-open.*
(v) *δ-quasi components of X are δ-open.*

Proof. By Lemma 4.3, δ-open δ-quasi component is a δ-component. Therefore the statements (iv) and (v) are equivalent. The equivalence of (iv) with (i), (ii), (iii) follows from the fact that a set is δ-open if and only if it is a δ-neighbourhood of each of its points. □

Corollary 4.5. *A proximity space X is sum δ-connected if and only if it is the far proximity sum of its δ-components (δ-quasi components).*

Corollary 4.6. *Let X be a sum δ-connected proximity space. Then the map f on X is δ-continuous if and only if it is δ-continuous on each of its δ-component.*

Corollary 4.7. *Every locally δ-connected proximity space is the far proximity sum of its δ-components (δ-quasi components).*

Corollary 4.8. *If X is sum δ-connected proximity space and $U \subset X$, then U is a δ-component if and only if it is δ-quasi component. In particular, If Y is a locally δ-connected proximity space and $X \subset Y$ is δ-open, then $U \subset X$ is δ-component if and only if it is δ-quasi component.*

Proof. By Proposition 4.4 (iv), δ-components and δ-quasi components coincide in sum δ-connected proximity space. The last statement of corollary from the fact that every locally δ-connected proximity space is sum δ-connected;

and every δ-open subset of a locally δ-connected proximity space is locally δ-connected. $\square$

As in Example 3.5, sum δ-connected proximity space may not be locally δ-conneted. But, if sum δ-connected proximity space is δ-padded, then it is also locally δ-connected.

Proposition 4.9. *Let X be a sum δ-connected proximity space and $x \in X$. If X is δ-padded at x, then it is locally δ-connected at x.*

Proof. Let N be a δ-open δ-neighbourhood of x. As X is sum δ-connected, suppose that N is contained in δ-component C_δ. Since X is δ-padded at x, there are δ-open δ-neighbourhoods W and V of x such that $Cl_\delta(W) \subseteq V \subseteq N$ with $V \backslash Cl_\delta(W)$ has only finitely many δ-components $C_\delta^1, C_\delta^2, \cdots, C_\delta^n$. Now for each i, $1 \leq i \leq n$, there exist a δ-quasi component Q_δ^i such that $C_\delta^i \subseteq Q_\delta^i$. We show that each $v \in V$ is in some Q_δ^i. If there is some $v \in V$ such that $v \notin Q_\delta^i$ for each $1 \leq i \leq n$, then for each i we have $V = (V \backslash Q_\delta^i) \cup Q_\delta^i$ with $(V \backslash Q_\delta^i, Q_\delta^i) \notin \delta$. Let $W_i = V \backslash Q_\delta^i$ for each $1 \leq i \leq n$ and $M = \bigcap_i W_i$. Since $(V \backslash Q_\delta^i, Q_\delta^i) \notin \delta$ for each $1 \leq i \leq n$, $(M, Q_\delta^i) \notin \delta$. Note that $C_\delta \backslash M = \bigcup_i C_\delta \backslash W_i$ and for each i, $C_\delta \backslash W_i = (C_\delta \backslash V) \cup Q_\delta^i$. As V is δ-open in C_δ, $(V, C_\delta \backslash V) \notin \delta$ which implies $(M, C_\delta \backslash V) \notin \delta$. Thus, $(M, (C_\delta \backslash V) \cup Q_\delta^i) \notin \delta$, that is, $(M, C_\delta \backslash W_i) \notin \delta$ for each i. Therefore, $(M, C_\delta \backslash M) \notin \delta$. Therefore, C_δ is not δ-connected, a contradiction. Thus, each $v \in V$ is in some Q_δ^i. Therefore, V has only finitely many δ-quasi components and each of them is δ-open. Thus, each δ-quasi component is a δ-component. Hence, δ-component of x in V is δ-connected δ-open neighbourhood of x contained in N. $\square$

References

[1] G. Bezhanishvili, Zero-dimensional proximities and zero-dimensional compactifications, Topology Appl. 156 (2009), 1496–1504.
[2] R. Dimitrijević and Lj. Kočinac, On connectedness of proximity spaces, Matem. Vesnik 39 (1987), 27–35.
[3] R. Dimitrijević and Lj. Kočinac, On treelike proximity spaces, Matem. Vesnik 39, no. 3 (1987), 257–261.
[4] V. A. Efremovic, Infinitesimal spaces, Dokl. Akad. Nauk SSSR 76 (1951), 341–343 (in Russian).
[5] V. A. Efremovic, The geometry of proximity I, Mat. Sb. 31 (1952), 189–200 (in Russian).
[6] J. K. Kohli, A class of spaces containing all connected and all locally connected spaces, Math. Nachr. 82 (1978), 121–129.
[7] S. G. Mrówka and W. J. Pervin, On uniform connectedness, Proc. Amer. Math. Soc. 15 (1964), 446–449.
[8] S. Naimpally, Proximity Approach to Problems in Topology and Analysis, Oldenbourg Verlag, München, 2009.
[9] S. Naimpally and B. D. Warrack, Proximity Spaces, Cambridge Univ. Press, 1970.
[10] Y. M. Smirnov, On completeness of proximity spaces I, Amer. Math. Soc. Trans. 38 (1964), 37–73.
[11] Y. M. Smirnov, On proximity spaces, Amer. Math. Soc. Trans. 38 (1964), 5–35.
[12] S. Willard, General Topology, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.

