

## Intermediate rings of complex-valued continuous functions

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### ABSTRACT

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For a completely regular Hausdorff topological space  $X$ , let  $C(X, \mathbb{C})$  be the ring of complex-valued continuous functions on  $X$ , let  $C^*(X, \mathbb{C})$  be its subring of bounded functions, and let  $\Sigma(X, \mathbb{C})$  denote the collection of all the rings that lie between  $C^*(X, \mathbb{C})$  and  $C(X, \mathbb{C})$ . We show that there is a natural correlation between the absolutely convex ideals/prime ideals/maximal ideals/ $z$ -ideals/ $z^\circ$ -ideals in the rings  $P(X, \mathbb{C})$  in  $\Sigma(X, \mathbb{C})$  and in their real-valued counterparts  $P(X, \mathbb{C}) \cap C(X)$ . These correlations culminate to the fact that the structure space of any such  $P(X, \mathbb{C})$  is  $\beta X$ . For any ideal  $I$  in  $C(X, \mathbb{C})$ , we observe that  $C^*(X, \mathbb{C}) + I$  is a member of  $\Sigma(X, \mathbb{C})$ , which is further isomorphic to a ring of the type  $C(Y, \mathbb{C})$ . Incidentally these are the only  $C$ -type intermediate rings in  $\Sigma(X, \mathbb{C})$  if and only if  $X$  is pseudocompact. We show that for any maximal ideal  $M$  in  $C(X, \mathbb{C})$ ,  $C(X, \mathbb{C})/M$  is an algebraically closed field, which is furthermore the algebraic closure of  $C(X)/M \cap C(X)$ . We give a necessary and sufficient condition for the ideal  $C_{\mathcal{P}}(X, \mathbb{C})$  of  $C(X, \mathbb{C})$ , which consists of all those functions whose support lie on an ideal  $\mathcal{P}$  of closed sets in  $X$ , to be a prime ideal, and we examine a few special cases thereafter. At the end of the article, we find estimates for a few standard parameters concerning the zero-divisor graphs of a  $P(X, \mathbb{C})$  in  $\Sigma(X, \mathbb{C})$ .

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## 1. INTRODUCTION

In what follows,  $X$  stands for a completely regular Hausdorff topological space and  $C(X, \mathbb{C})$  denotes the ring of all complex-valued continuous functions on  $X$ .  $C^*(X, \mathbb{C})$  is the subring of  $C(X, \mathbb{C})$  containing those functions which are bounded over  $X$ . As usual  $C(X)$  designates the ring of all real-valued continuous functions on  $X$  and  $C^*(X)$  consists of those functions in  $C(X)$  which are bounded over  $X$ . An intermediate ring of real-valued continuous functions on  $X$  is a ring that lies between  $C^*(X)$  and  $C(X)$ . Let  $\Sigma(X)$  be the aggregate of all such rings. Likewise an intermediate ring of complex-valued continuous functions on  $X$  is a ring lying between  $C^*(X, \mathbb{C})$  and  $C(X, \mathbb{C})$ . Let  $\Sigma(X, \mathbb{C})$  be the family of all such intermediate rings. It turns out that each member  $P(X, \mathbb{C})$  of  $\Sigma(X, \mathbb{C})$  is absolutely convex in the sense that  $|f| \leq |g|, g \in P(X, \mathbb{C}), f \in C(X, \mathbb{C})$  implies  $f \in P(X, \mathbb{C})$ . It follows that each such  $P(X, \mathbb{C})$  is *conjugate-closed* in the sense that if whenever  $f + ig \in P(X, \mathbb{C})$  where  $f, g \in C(X)$ , then  $f - ig \in P(X, \mathbb{C})$ . It is realised that there is a natural correlation between the prime ideals/ maximal ideals/  $z$ -ideals/  $z^\circ$ -ideals in the rings  $P(X, \mathbb{C})$  and the prime ideals/ maximal ideals/  $z$ -ideals/  $z^\circ$ -ideals in the ring  $P(X, \mathbb{C}) \cap C(X)$ . In the second and third sections of this article, we examine these correlations in some detail. Incidentally an interconnection between prime ideals in the two rings  $C(X, \mathbb{C})$  and  $C(X)$  is already observed in Corollary 1.2[7]. As a follow up of our investigations on the ideals in these two rings, we establish that the structure spaces of the two rings  $P(X, \mathbb{C})$  and  $P(X, \mathbb{C}) \cap C(X)$  are homeomorphic. The structure space of a commutative ring  $R$  with unity stands for the set of all maximal ideals of  $R$  equipped with the well-known hull-kernel topology. It was established in [21] and [22], independently that the structure space of all the intermediate rings of real-valued continuous functions on  $X$  are one and the same viz the Stone-Ćech compactification  $\beta X$  of  $X$ . It follows therefore that the structure space of each intermediate ring of complex-valued continuous functions on  $X$  is also  $\beta X$ . This is one of the main technical results in our article. We like to mention in this context that a special case of this result telling that the structure space of  $C(X, \mathbb{C})$  is  $\beta X$  is quite well known, see [19]. We call a ring  $P(X, \mathbb{C})$  in the family  $\Sigma(X, \mathbb{C})$  a  $C$ -type ring if it is isomorphic to a ring of the form  $C(Y, \mathbb{C})$  for Tychonoff space  $Y$ . We establish that if  $I$  is any ideal of  $C(X, \mathbb{C})$ , then the linear sum  $C^*(X, \mathbb{C}) + I$  is a  $C$ -type ring. This is the complex analogue of the corresponding result in the intermediate rings of real-valued continuous functions on  $X$  as proved in [16]. We further realise that these are the only  $C$ -type intermediate rings in the family  $\Sigma(X, \mathbb{C})$  when and only when  $X$  is pseudocompact i.e.  $C(X, \mathbb{C}) = C^*(X, \mathbb{C})$ .

It is well-known that if  $M$  is a maximal ideal in  $C(X)$ , then the residue class field  $C(X)/M$  is real closed in the sense that every positive element in this field is a square and each odd degree polynomial over this field has a root in the same field [17, Theorem 13.4]. The complex analogue of this result as we realise

is that for a maximal ideal  $M$  in  $C(X, \mathbb{C})$ ,  $C(X, \mathbb{C})/M$  is an algebraically closed field and furthermore this field is the algebraic closure of  $C(X)/M \cap C(X)$ .

In section 4 of this article, we deal with a few special problems originating from an ideal  $\mathcal{P}$  of closed sets in  $X$  and a certain class of ideals in the ring  $C(X, \mathbb{C})$ . A family  $\mathcal{P}$  of closed sets in  $X$  is called an ideal of closed sets in  $X$  if for any two sets  $A, B$  in  $\mathcal{P}$ ,  $A \cup B \in \mathcal{P}$  and for any closed set  $C$  contained in  $A$ ,  $C$  is also a member of  $\mathcal{P}$ . We let  $C_{\mathcal{P}}(X, \mathbb{C})$  be the set of all those functions  $f$  in  $C(X, \mathbb{C})$  whose support  $cl_X(X \setminus Z(f))$  is a member of  $\mathcal{P}$ ; here  $Z(f) = \{x \in X : f(x) = 0\}$  is the zero set of  $f$  in  $X$ . We determine a necessary and sufficient condition for  $C_{\mathcal{P}}(X, \mathbb{C})$  to become a prime ideal in the ring  $C(X, \mathbb{C})$  and examine a few special cases corresponding to some specific choices of the ideal  $\mathcal{P}$ . The ring  $C_{\infty}(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : f \text{ vanishes at infinity in the sense that for each } n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\} \text{ is compact}\}$  is an ideal of  $C^*(X, \mathbb{C})$  but not necessarily an ideal of  $C(X, \mathbb{C})$ . On the assumption that  $X$  is locally compact, we determine a necessary and sufficient condition for  $C_{\infty}(X, \mathbb{C})$  to become an ideal of  $C(X, \mathbb{C})$ .

The fifth section of this article is devoted to finding out the estimates of a few standard parameters concerning zero divisor graphs of a few rings of complex-valued continuous functions on  $X$ . Thus for instance we have checked that if  $\Gamma(P(X, \mathbb{C}))$  is the zero divisor graph of an intermediate ring  $P(X, \mathbb{C})$  belonging to the family  $\Sigma(X, \mathbb{C})$ , then each cycle of this graph has length 3, 4 or 6 and each edge is an edge of a cycle with length 3 or 4. These are the complex analogues of the corresponding results in the zero divisor graph of  $C(X)$  as obtained in [9].

## 2. IDEALS IN INTERMEDIATE RINGS

Notation: For any subset  $A(X)$  of  $C(X)$  such that  $0 \in A(X)$ , we set  $[A(X)]_c = \{f + ig : f, g \in A(X)\}$  and call it the *extension* of  $A(X)$ . Then it is easy to see that  $[A(X)]_c \cap C(X) = A(X) = [A(X)]_c \cap A(X)$ . From now on, unless otherwise stated, we assume that  $A(X)$  is an intermediate ring of real-valued continuous functions on  $X$ , i.e.  $A(X)$  is a member of the family  $\Sigma(X)$ . It follows at once that  $[A(X)]_c$  is an intermediate ring of complex-valued continuous functions and it is not hard to verify that  $[A(X)]_c$  is the smallest intermediate ring in  $\Sigma(X, \mathbb{C})$  which contains  $A(X)$  and the constant function  $i$ . Furthermore  $[A(X)]_c$  is conjugate-closed meaning that if  $f + ig \in [A(X)]_c$  with  $f, g \in A(X)$ , then  $f - ig \in [A(X)]_c$ . The following result tells that intermediate rings in the family  $\Sigma(X, \mathbb{C})$  are the extensions of intermediate rings in  $\Sigma(X)$ .

**Theorem 2.1.** *Let  $P(X, \mathbb{C})$  be an intermediate ring of  $C(X, \mathbb{C})$ . Then  $P(X, \mathbb{C})$  is absolutely convex.*

*Proof.* Let  $|f| \leq |g|$ ,  $f \in C(X, \mathbb{C})$ ,  $g \in P(X, \mathbb{C})$ . Then  $f = \frac{f}{1+g^2}(1+g^2) \in P(X, \mathbb{C})$ . Hence  $P(X, \mathbb{C})$  is absolutely convex.  $\square$

**Theorem 2.2.** *An intermediate ring  $P(X, \mathbb{C})$  of  $C(X, \mathbb{C})$  is conjugate closed.*

*Proof.* Let  $f + ig \in P(X, \mathbb{C})$ . We have  $|f| \leq |f + ig|, |g| \leq |f + ig|$  and  $f + ig \in P(X, \mathbb{C})$ . Since  $P(X, \mathbb{C})$  is absolutely convex, then  $f, g \in P(X, \mathbb{C})$ . This implies  $f, ig \in P(X, \mathbb{C})$  as  $i \in P(X, \mathbb{C})$ . Thus  $f - ig \in P(X, \mathbb{C})$ . Hence  $P(X, \mathbb{C})$  is conjugate closed.  $\square$

**Theorem 2.3.** *A ring  $P(X, \mathbb{C})$  of complex valued continuous functions on  $X$  is a member of  $\Sigma(X, \mathbb{C})$  if and only if there exists a ring  $A(X)$  in the family  $\Sigma(X)$  such that  $P(X, \mathbb{C}) = [A(X)]_c$ .*

*Proof.* Assume that  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  and let  $A(X) = P(X, \mathbb{C}) \cap C(X)$ . Then it is clear that  $A(X) \in \Sigma(X)$  and  $[A(X)]_c \subseteq P(X, \mathbb{C})$ .

To prove the reverse containment, let  $f + ig \in P(X, \mathbb{C})$ . Here  $f, g \in C(X)$ . Since  $P(X, \mathbb{C})$  is conjugate closed,  $f - ig \in P(X, \mathbb{C})$ , and hence  $2f$  and  $2ig$  both belong to  $P(X, \mathbb{C})$ . Since constant functions are bounded and hence in  $P(X, \mathbb{C})$ , both the constant functions  $\frac{1}{2}$  and  $\frac{1}{2i}$  are in  $P(X, \mathbb{C})$ . It follows that both  $f$  and  $g$  are in  $P(X, \mathbb{C}) \cap C(X)$ , and hence in  $A(X)$ . Consequently,  $f + ig \in [A(X)]_c$ . Thus,  $P(X, \mathbb{C}) \subseteq [A(X)]_c$ .  $\square$

The following facts involving convex sets will be useful. A subset  $S$  of  $C(X)$  is called *absolutely convex* if whenever  $|f| \leq |g|$  with  $g \in S$  and  $f \in C(X)$ , then  $f \in S$ .

**Theorem 2.4.** *Let  $A(X) \in \Sigma(X)$ . Then*

- (a)  *$A(X)$  is an absolutely convex subring of  $C(X)$  (in the sense that if  $|f| \leq |g|$  with  $g \in A(X)$  and  $f \in C(X)$ , then  $f \in A(X)$ ) ([16, Proposition 3.3]).*
- (b) *A prime ideal  $P$  in  $A(X)$  is an absolutely convex subset of  $A(X)$  ([13, Theorem 2.5]).*

The following convenient formula for  $[A(X)]_c$  with  $A(X) \in \Sigma(X)$  will often be helpful to us.

**Theorem 2.5.** *For any  $A(X) \in \Sigma(X)$ ,  $[A(X)]_c = \{h \in C(X, \mathbb{C}) : |h| \in A(X)\}$ .*

*Proof.* First assume that  $h = f + ig \in [A(X)]_c$  with  $f, g \in A(X)$ . Then  $|h| \leq |f| + |g|$ . This implies, in view of Theorem 2.4(a), that  $h \in A(X)$  and also  $|h| \in A(X)$ . Conversely, let  $h = f + ig \in C(X, \mathbb{C})$  with  $f, g \in C(X)$ , be such that  $|h| \in A(X)$ . This means that  $(f^2 + g^2)^{\frac{1}{2}} \in A(X)$ . Since  $|f| \leq (f^2 + g^2)^{\frac{1}{2}}$ , this implies in view of Theorem 2.4(a) that  $f \in A(X)$ . Analogously  $g \in A(X)$ . Thus  $h \in [A(X)]_c$ .  $\square$

**Theorem 2.6.** *If  $I$  is an ideal in  $A(X) \in \Sigma(X)$ , then  $I_c = \{f + ig : f, g \in I\}$  is the smallest ideal in  $[A(X)]_c$  containing  $I$ . Furthermore  $I_c \cap A(X) = I = I_c \cap C(X)$ .*

*Proof.* It is easy to show that  $I_c$  is an ideal in  $[A(X)]_c$  containing  $I$ . Let  $K$  be an ideal of  $[A(X)]_c$  containing  $I$ . To show  $I_c \subseteq K$ . Let  $f + ig \in K$ , where  $f, g \in I$ . Since  $I \subseteq K$ , then  $f, g \in K$ . Now  $K$  is an ideal of  $[A(X)]_c$ ,  $f, g \in K$

implies  $f + ig \in K$ . Therefore  $I_c \subseteq K$ . Hence  $I_c$  is the smallest ideal of  $[A(X)]_c$  containing  $I$ .

Proof of the second part is trivial. □

**Theorem 2.7.** *If  $I$  and  $J$  are ideals in  $A(X) \in \Sigma(X)$ , then  $I \subseteq J$  if and only if  $I_c \subseteq J_c$ . Also  $I \subsetneq J$  when and only when  $I_c \subsetneq J_c$ .*

*Proof.* If  $I \subseteq J$ , then clearly  $I_c \subseteq J_c$ .

Conversely, let  $I_c \subseteq J_c$ . Let  $f \in I$ . Since  $I \subset I_c$ , we have  $f \in I_c \subseteq J_c$ . Now  $f = f + i0$  and  $J_c = \{g + ih : g, h \in J\}$ . Therefore  $f \in J$ . Hence  $I \subseteq J$ .

For the second part we consider  $I \subsetneq J$  and  $f \in J \setminus I$ . Then  $f \in J_c \setminus I_c$ . Thus  $I_c \subsetneq J_c$ .

Conversely, let  $I_c \subsetneq J_c$  and  $f + ig \in J_c \setminus I_c$ . Then either  $f$  or  $g$  is outside  $I$ . Let  $f \notin I$ . Then  $f \in J \setminus I$ . Hence  $I \subsetneq J$ . This completes the proof. □

We have the following convenient formula for  $I_c$  when  $I$  is an absolutely convex ideal of  $A(X)$ .

**Theorem 2.8.** *If  $I$  is an absolutely convex ideal of  $A(X)$  (in particular if  $I$  is a prime ideal or a maximal ideal of  $A(X)$ ), then  $I_c = \{h \in [A(X)]_c : |h| \in I\}$ .*

*Proof.* Let  $h = f + ig \in I_c$ . Then  $f, g \in I$ . Since  $|h| \leq |f| + |g|$ , the absolute convexity of  $I$  implies that  $|h| \in I$ . Conversely, let  $h = f + ig \in [A(X)]_c$  be such that  $|h| \in I$ . Here  $f, g \in A(X)$ . Since  $|f| \leq (f^2 + g^2)^{\frac{1}{2}} = |h|$ , it follows from the absolute convexity of  $I$  that  $f \in I$ . Analogously  $g \in I$ . Hence  $h \in I_c$ . □

The above theorem prompts us to define the notion of an absolutely convex ideal in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  as follows:

**Definition 2.9.** An ideal  $J$  in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is called absolutely convex if for  $g, h$  in  $C(X, \mathbb{C})$  with  $|g| \leq |h|$  and  $h \in J$ , it follows that  $g \in J$ .

The first part of the following proposition is immediate, while the second part follows from Theorem 2.3 and Theorem 2.8.

**Theorem 2.10.** *Let  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ .*

- (i) *If  $J$  is an absolutely convex ideal of  $P(X, \mathbb{C})$ , then  $J \cap C(X)$  is an absolutely convex ideal of the intermediate ring  $P(X, \mathbb{C}) \cap C(X) \in \Sigma(X)$ .*
- (ii) *An ideal  $I$  in  $P(X, \mathbb{C}) \cap C(X)$  is absolutely convex in this ring if and only if  $I_c$  is an absolutely convex ideal of  $P(X, \mathbb{C})$ .*
- (iii) *If  $J$  is an absolutely convex ideal of  $P(X, \mathbb{C})$ , then  $J = [J \cap C(X)]_c$ .*

*Proof.* (iii) It is trivial that  $[J \cap C(X)]_c \subseteq J$ . To prove the reverse implication relation let  $h = f + ig \in J$ , with  $f, g \in C(X)$ . The absolute convexity of  $J$  implies that  $|h| \in J$ . Consequently  $|h| \in J \cap C(X)$ . But since  $|f| \leq (f^2 + g^2)^{\frac{1}{2}} = |h|$ , it follows again due to the absolute convexity of  $P(X, \mathbb{C})$  as a subring of  $C(X, \mathbb{C})$  that  $f \in P(X, \mathbb{C})$ . We further use absolute convexity of  $J$  in  $P(X, \mathbb{C})$  to assert that  $f \in J$ . Analogously  $g \in J$ . Thus  $h = f + ig \in [J \cap C(X)]_c$ . Therefore  $J \subseteq [J \cap C(X)]_c$ . □

*Remark 2.11.* For any  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ , the assignment  $I \mapsto I_c$  provides a one-to-one correspondence between the absolutely convex ideals of  $P(X, \mathbb{C}) \cap C(X)$  and those of  $P(X, \mathbb{C})$ .

The following theorem gives a one-to-one correspondence between the prime ideals of  $P(X, \mathbb{C})$  and those of  $P(X, \mathbb{C}) \cap C(X)$ .

**Theorem 2.12.** *Let  $P(X, \mathbb{C})$  be member of  $\Sigma(X, \mathbb{C})$ . An ideal  $J$  of  $P(X, \mathbb{C})$  is prime if and only if there exists a prime ideal  $Q$  in  $P(X, \mathbb{C}) \cap C(X)$  such that  $J = Q_c$ .*

*Proof.* Let  $J$  be a prime ideal in  $P(X, \mathbb{C})$  and let  $Q = J \cap C(X)$  and  $A(X) = P(X, \mathbb{C}) \cap C(X)$ . Then  $Q$  is a prime ideal in the ring  $A(X)$ . It is easy to see that  $Q_c \subseteq J$ . To prove the reverse containment, let  $h = f + ig \in J$ , where  $f, g \in P(X, \mathbb{C})$ . Note that  $P(X, \mathbb{C}) = [A(X)]_c$  by Theorem 2.3. Hence  $f, g \in A(X)$  and therefore  $f - ig \in P(X, \mathbb{C})$ . As  $J$  is an ideal of  $P(X, \mathbb{C})$ , it follows that  $(f + ig)(f - ig) \in J$  i.e,  $f^2 + g^2 \in J \cap C(X) = Q$ . Since  $Q$  is a prime ideal in  $A(X)$ , we can apply Theorem 2.4(b), yielding  $f^2 \in Q$  and hence  $f \in Q$ . Analogously  $g \in Q$ . Thus  $h \in Q_c$ . Therefore  $J \subseteq Q_c$ .

To prove the converse of this theorem, let  $Q$  be a prime ideal in  $A(X)$ . It follows from Theorem 2.8 that  $Q_c = \{h \in P(X, \mathbb{C}) : |h| \in Q\}$  and therefore  $Q_c$  is a prime ideal in  $P(X, \mathbb{C})$ . Finally we note that  $Q_c \cap C(X) = Q$ .  $\square$

*Remark 2.13.* For any  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ , the collection of all prime ideals in  $P(X, \mathbb{C})$  is precisely  $\{Q_c : Q \text{ is a prime ideal in } P(X, \mathbb{C}) \cap C(X)\}$ .

*Remark 2.14.* The collection of all minimal prime ideals in  $P(X, \mathbb{C})$  is precisely  $\{Q_c : Q \text{ is a minimal prime ideal in } P(X, \mathbb{C}) \cap C(X)\}$ . [This follows from Remark 2.13 and Theorem 2.7].

**Theorem 2.15.** *For any  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ , the collection of all maximal ideals in  $P(X, \mathbb{C})$  is  $\{M_c : M \text{ is a maximal ideal of } P(X, \mathbb{C}) \cap C(X)\}$ .*

*Proof.* Let  $M$  be a maximal ideal in  $P(X, \mathbb{C}) \cap C(X) = A(X)$ . Then by Theorem 2.12,  $M_c$  is a prime ideal in  $P(X, \mathbb{C})$ . Suppose that  $M_c$  is not a maximal ideal in  $P(X, \mathbb{C})$ , then there exists a prime ideal  $T$  in  $P(X, \mathbb{C})$  such that  $M_c \subsetneq T$ . By Remark 2.11, there exists a prime ideal  $P$  in  $A(X)$  such that  $J = P_c$ . So  $M_c \subsetneq P_c$ . This implies in view of Theorem 2.5 that  $M \subsetneq P$ , a contradiction to the maximality of  $M$  in  $A(X)$ .

Conversely, let  $J$  be a maximal ideal of  $P(X, \mathbb{C})$ . In particular  $J$  is a prime ideal in this ring. By Remark 2.13,  $J = Q_c$  for some prime ideal  $Q$  in  $A(X)$ . We claim that  $Q$  is a maximal ideal in  $A(X)$ . Suppose not; then  $Q \subsetneq K$  for some proper ideal  $K$  in  $A(X)$ . Then by Theorem 2.7,  $Q_c \subsetneq K_c$  and  $K_c$  a proper ideal in  $P(X, \mathbb{C})$ ; this contradicts the maximality of  $J = Q_c$ .  $\square$

We next prove analogues of Remark 2.13 and Theorem 2.15 for two important classes of ideals viz  $z$ -ideals and  $z^\circ$ -ideals in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ . These ideals are defined as follows.

**Definition 2.16.** Let  $R$  be a commutative ring with unity. For each  $a \in R$ , let  $M_a$  (respectively  $P_a$ ) stand for the intersection of all maximal ideals (respectively all minimal prime ideals) which contain  $a$ . An ideal  $I$  in  $R$  is called a  $z$ -ideal (respectively  $z^\circ$ -ideal) if for each  $a \in I$ ,  $M_a \subseteq I$  (respectively  $P_a \subseteq I$ ).

This notion of  $z$ -ideals is consistent with the notion of  $z$ -ideal in  $C(X)$  (see [17, 4A5]). Since each prime ideal in an intermediate ring  $A(X) \in \Sigma(X)$  is absolutely convex (Theorem 2.4(b)), it follows from Theorem 2.10(ii) and Remark 2.13 that each prime ideal in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is absolutely convex. In particular each maximal ideal is absolutely convex. Now if  $I$  is a  $z$ -ideal in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  and  $|f| \leq |g|$ ,  $g \in I$ ,  $f \in P(X, \mathbb{C})$ , then  $M_g \subseteq I$ . Let  $M$  be a maximal ideal in  $P(X, \mathbb{C})$  containing  $g$ . It follows due to the absolute convexity of  $M$  that  $f \in M$ . Therefore  $f \in M_g \subseteq I$ . Thus each  $z$ -ideal in  $P(X, \mathbb{C})$  is absolutely convex. Analogously it can be proved that each  $z^\circ$ -ideal in  $P(X, \mathbb{C})$  is absolutely convex.

The following subsidiary result can be proved using routine arguments.

**Lemma 2.17.** For any family  $\{I_\alpha : \alpha \in \Lambda\}$  of ideals in an intermediate ring  $A(X) \in \Sigma(X)$ ,  $(\bigcap_{\alpha \in \Lambda} I_\alpha)_c = \bigcap_{\alpha \in \Lambda} (I_\alpha)_c$ .

**Theorem 2.18.** An ideal  $J$  in a ring  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is a  $z$ -ideal in  $P(X, \mathbb{C})$  if and only if there exists a  $z$ -ideal  $I$  in  $P(X, \mathbb{C}) \cap C(X)$  such that  $J = I_c$ .

*Proof.* First assume that  $J$  is a  $z$ -ideal in  $P(X, \mathbb{C})$ . Let  $I = J \cap C(X)$ . Since  $J$  is absolutely convex, it follows from Theorem 2.10(iii) that  $J = I_c$ . We show that  $I$  is a  $z$ -ideal in  $P(X, \mathbb{C}) \cap C(X)$ . Choose  $f \in I$ . Suppose  $\{M_\alpha : \alpha \in \Lambda\}$  is the set of all maximal ideals in the ring  $P(X, \mathbb{C}) \cap C(X)$  which contain  $f$ . It follows from Theorem 2.15 that  $\{(M_\alpha)_c : \alpha \in \Lambda\}$  is the set of all maximal ideals in  $P(X, \mathbb{C})$  containing  $f$ . Since  $f \in J$  and  $J$  is a  $z$ -ideal in  $P(X, \mathbb{C})$ , it follows that  $\bigcap_{\alpha \in \Lambda} (M_\alpha)_c \subseteq J$ . This implies in the view of Lemma 2.17 that  $(\bigcap_{\alpha \in \Lambda} M_\alpha)_c \cap C(X) \subseteq I$ , and hence  $f \in \bigcap_{\alpha \in \Lambda} M_\alpha \subseteq I$ . Thus it is proved that  $I$  is a  $z$ -ideal in  $P(X, \mathbb{C}) \cap C(X)$ .

Conversely, let  $I$  be a  $z$ -ideal in the ring  $P(X, \mathbb{C}) \cap C(X)$ . We shall prove that  $I_c$  is a  $z$ -ideal in  $P(X, \mathbb{C})$ . We recall from Theorem 2.3 that  $[P(X, \mathbb{C}) \cap C(X)]_c = P(X, \mathbb{C})$ . Choose  $f$  from  $I_c$ . From Theorem 2.8, it follows that (taking care of the fact that each  $z$ -ideal in  $P(X, \mathbb{C})$  is absolutely convex)  $|f| \in I$ . Let  $\{N_\beta : \beta \in \Lambda^*\}$  be the set of all maximal ideals in  $P(X, \mathbb{C}) \cap C(X)$  which contain the function  $|f|$ . The hypothesis that  $I$  is a  $z$ -ideal in  $P(X, \mathbb{C}) \cap C(X)$  therefore implies that  $\bigcap_{\beta \in \Lambda^*} N_\beta \subseteq I$ . This further implies in view of Lemma 2.17 that  $\bigcap_{\beta \in \Lambda^*} (N_\beta)_c \subseteq I_c$ . Again it follows from Theorem 2.8 that, for any maximal ideal  $M$  in  $P(X, \mathbb{C}) \cap C(X)$  and any  $g \in P(X, \mathbb{C})$ ,  $g \in M_c$  if and only if  $|g| \in M$ . Thus for any  $\beta \in \Lambda^*$ ,  $|f| \in N_\beta$  if and only if  $f \in (N_\beta)_c$ . This means that  $\{(N_\beta)_c\}_{\beta \in \Lambda^*}$  is the collection of maximal ideals in  $P(X, \mathbb{C})$  which contain  $f$ , and we have already observed that  $f \in \bigcap_{\beta \in \Lambda^*} (N_\beta)_c \subseteq I_c$ . Consequently  $I_c$  is a  $z$ -ideal in  $P(X, \mathbb{C})$ .  $\square$



If we use the result embodied in Remark 2.14 and take note of the fact that each minimal prime ideal in  $P(X, \mathbb{C})$  is absolutely convex and argue as in the proof of Theorem 2.18, we get the following proposition:

**Theorem 2.19.** *An ideal  $J$  in a ring  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is a  $z^\circ$ -ideal in  $P(X, \mathbb{C})$  if and only if there exists a  $z^\circ$ -ideal  $I$  in  $P(X, \mathbb{C}) \cap C(X)$  such that  $J = I_c$ .*

An ideal  $J$  in  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is called fixed if  $\bigcap_{f \in J} Z(f) \neq \emptyset$ . The following proposition is a straightforward consequence of Theorem 2.6.

**Theorem 2.20.** *An ideal  $J$  in a ring  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is a fixed ideal in  $P(X, \mathbb{C})$  if and only if  $J \cap C(X)$  is a fixed ideal in  $P(X, \mathbb{C}) \cap C(X)$ .*

We recall that a space  $X$  is called an *almost  $P$  space* if every non-empty  $G_\delta$  subset of  $X$  has non-empty interior. These spaces have been characterized via  $z$ -ideals and  $z^\circ$ -ideals in the ring  $C(X)$  in [8]. We would like to mention that the same class of spaces have witnessed a very recent characterization in terms of fixed maximal ideals in a given intermediate ring  $A(X) \in \Sigma(X)$ . We reproduce below these two results to make the paper self-contained.

**Theorem 2.21** ([8]).  *$X$  is an almost  $P$  space if and only if each maximal ideal in  $C(X)$  is a  $z^\circ$ -ideal if and only if each  $z$ -ideal in  $C(X)$  is a  $z^\circ$ -ideal.*

**Theorem 2.22** ([12]). *Let  $A(X) \in \Sigma(X)$  be an intermediate ring of real-valued continuous functions on  $X$ . Then  $X$  is an almost  $P$  space if and only if each fixed maximal ideal  $M_A^p = \{g \in A(X) : g(p) = 0\}$  of  $A(X)$  is a  $z^\circ$ -ideal.*

It is further realised in [12] that if  $X$  is an almost  $P$  space, then the statement of Theorem 2.21 cannot be improved by replacing  $C(X)$  by an intermediate ring  $A(X)$ , different from  $C(X)$ . Indeed it is shown in [12, Theorem 2.4] that if an intermediate ring  $A(X) \neq C(X)$ , then there exists a maximal ideal in  $A(X)$  (which is incidentally also a  $z$ -ideal in  $A(X)$ ), which is not a  $z^\circ$ -ideal in  $A(X)$ .

We record below the complex analogue of the above results.

**Theorem 2.23.**  *$X$  is an almost  $P$  space if and only if each maximal ideal of  $C(X, \mathbb{C})$  is a  $z^\circ$ -ideal if and only if each  $z$ -ideal in  $C(X, \mathbb{C})$  is a  $z^\circ$ -ideal.*

*Proof.* This follows from combining Theorems 2.15, 2.18, 2.19, and 2.21. □

**Theorem 2.24.** *Let  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ . Then  $X$  is almost  $P$  if and only if each fixed maximal ideal  $M_P^p = \{g \in P(X, \mathbb{C}) : g(p) = 0\}$  of  $P(X, \mathbb{C})$  is a  $z^\circ$ -ideal.*

*Proof.* This follows from combining Theorems 2.15, 2.20, and 2.22. □

**Theorem 2.25.** *Let  $X$  be an almost  $P$  space and let  $P(X, \mathbb{C})$  be a member of  $\Sigma(X, \mathbb{C})$  such that  $P(X, \mathbb{C}) \subsetneq C(X, \mathbb{C})$ . Then there exists a maximal ideal in  $P(X, \mathbb{C})$ , which is not a  $z^\circ$ -ideal in  $P(X, \mathbb{C})$ .*



Thus, within the class of almost  $P$ -spaces  $X$ ,  $C(X, \mathbb{C})$  is characterized amongst all the intermediate rings  $P(X, \mathbb{C})$  of  $\Sigma(X, \mathbb{C})$  by the requirement that  $z$ -ideals and  $z^\circ$ -ideals (equivalently maximal ideals and  $z^\circ$ -ideals) in  $P(X, \mathbb{C})$  are one and the same.

*Proof.* This follows from combining Theorems 2.15, 2.18, and 2.19 of this article together with [12, Theorem 2.4].  $\square$

We recall the classical result that  $X$  is a  $P$  space if and only if  $C(X)$  is a von-Neumann regular ring meaning that each prime ideal in  $C(X)$  is maximal. Incidentally the following fact was rather recently established:

**Theorem 2.26** ([3, 20, 12]). *If  $A(X) \in \Sigma(X)$  is different from  $C(X)$ , then  $A(X)$  is never a regular ring.*

Theorems 2.12, 2.15, and 2.26 yield in a straight forward manner the following result:

**Theorem 2.27.** *If  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is a proper subring of  $C(X, \mathbb{C})$ , then  $P(X, \mathbb{C})$  is not a von-Neumann regular ring.*

It is well-known that if  $P$  is a non maximal prime ideal in  $C(X)$  and  $M$  is the unique maximal ideal containing  $P$ , then the set of all prime ideals in  $C(X)$  that lie between  $P$  and  $M$  makes a Dedekind complete chain containing no fewer than  $2^{\aleph_1}$  many members (see [17, Theorem 14.19]). If we use this standard result and combine with Theorems 2.7, 2.12, and 2.15, we obtain the complex-version of this fact:

**Theorem 2.28.** *Suppose  $P$  is a non maximal prime ideal in the ring  $C(X, \mathbb{C})$ . Then there exists a unique maximal ideal  $M$  containing  $P$  in this ring. Furthermore, the collection of all prime ideals that are situated between  $P$  and  $M$  constitutes a Dedekind complete chain containing at least  $2^{\aleph_1}$  many members.*

Thus for all practical purposes (say for example when  $X$  is not a  $P$  space),  $C(X, \mathbb{C})$  is far from being a Noetherian ring. Incidentally we shall decide the Noetherianness condition of  $C(X, \mathbb{C})$  by deducing it from a result in Section 4; in particular, we show that  $C(X, \mathbb{C})$  is Noetherian if and only if  $X$  is a finite set.

### 3. STRUCTURE SPACES OF INTERMEDIATE RINGS

We need to recall a few technicalities associated with the hull-kernel topology on the set of all maximal ideals  $\mathcal{M}(A)$  of a commutative ring  $A$  with unity. If we set for any element  $a$  of  $A$ ,  $\mathcal{M}(A)_a = \{M \in \mathcal{M}(A) : a \in M\}$ , then the family  $\{\mathcal{M}(A)_a : a \in A\}$  constitutes a base for closed sets of the hull-kernel topology on  $\mathcal{M}(A)$ . We may write  $\mathcal{M}_a$  for  $\mathcal{M}(A)_a$  when context is clear. The set  $\mathcal{M}(A)$  equipped with this hull-kernel topology is called the *structure space* of the ring  $A$ .

For any subset  $\mathcal{M}_o$  of  $\mathcal{M}(A)$ , its closure  $\overline{\mathcal{M}_o}$  in this topology is given by:  $\overline{\mathcal{M}_o} = \{M \in \mathcal{M}(A) : M \supseteq \bigcap \mathcal{M}_o\}$ . For further information on this topology, see [17, 7M].

Following the terminology of [14], by a (Hausdorff) compactification of a Tychonoff space  $X$  we mean a pair  $(\alpha, \alpha X)$ , where  $\alpha X$  is a compact Hausdorff space and  $\alpha : X \rightarrow \alpha X$  a topological embedding with  $\alpha(X)$  dense in  $\alpha X$ . For simplicity, we often designate such a pair by the notation  $\alpha X$ . Two compactifications  $\alpha X$  and  $\gamma X$  of  $X$  are called *topologically equivalent* if there exists a homeomorphism  $\psi : \alpha X \rightarrow \gamma X$  with the property  $\psi \circ \alpha = \gamma$ . A compactification  $\alpha X$  of  $X$  is said to possess the *extension property* if given a compact Hausdorff space  $Y$  and a continuous map  $f : X \rightarrow Y$ , there exists a continuous map  $f^\alpha : \alpha X \rightarrow Y$  with the property  $f^\alpha \circ \alpha = f$ . It is well known that the Stone-Ćech compactification  $\beta X$  of  $X$  or more formally the pair  $(e, \beta X)$ , where  $e$  is the evaluation map on  $X$  induced by  $C^*(X)$  defined by the formula:  $e(x) = (f(x) : f \in C^*(X))$  such that  $e : X \mapsto \mathbb{R}^{C^*(X)}$ , enjoys the extension property. Furthermore this extension property characterizes  $\beta X$  amongst all the compactifications of  $X$  in the sense that whenever a compactification  $\alpha X$  of  $X$  has extension property, it is topologically equivalent to  $\beta X$ . For more information on these topic, see [14, Chapter 1].

The structure space  $\mathcal{M}(A(X))$  of an arbitrary intermediate ring  $A(X) \in \Sigma(X)$  has been proved to be homeomorphic to  $\beta X$ , independently by the authors in [21] and [22]. Nevertheless we offer yet another independent technique to establish a modified version of the same fact by using the above terminology of [14].

**Theorem 3.1.** *Let  $\eta_A : X \rightarrow \mathcal{M}(A(X))$  be the map defined by  $\eta_A(x) = M_A^x = \{g \in A(X) : g(x) = 0\}$  (a fixed maximal ideal in  $A(X)$ ). Then the pair  $(\eta_A, \mathcal{M}(A(X)))$  is a (Hausdorff) compactification of  $X$ , which further satisfies the extension property. Hence the pair  $(\eta_A, \mathcal{M}(A(X)))$  is topologically equivalent to the Stone-Ćech compactification  $\beta X$  of  $X$ .*

*Proof.* Since  $X$  is Tychonoff,  $\eta_A$  is one-to-one. Also  $cl_{\mathcal{M}(A(X))} \eta_A(X) = \{M \in \mathcal{M}(A(X)) : M \supseteq \bigcap_{x \in X} M_A^x\} = \{M \in \mathcal{M}(A(X)) : M \supseteq \{0\}\} = \mathcal{M}(A(X))$ . It follows from a result proved in Theorem 3.3 and Theorem 3.4 [23] that  $\mathcal{M}(A(X))$  is a compact Hausdorff space and  $\eta_A$  is an embedding. Thus  $(\eta_A, \mathcal{M}(A(X)))$  is a compactification of  $X$ . To prove that this compactification of  $X$  possesses the extension property we take a compact Hausdorff space  $Y$  and a continuous map  $f : X \rightarrow Y$ . It suffices to define a continuous map  $f^{\beta_A} : \mathcal{M}(A(X)) \rightarrow Y$  with the property that  $f^{\beta_A} \circ \eta_A = f$ . Let  $M$  be any member of  $\mathcal{M}(A(X))$  i.e.  $M$  is a maximal ideal of the ring  $A(X)$ . Define  $\hat{M} = \{g \in C(Y) : g \circ f \in M\}$ . Note that if  $g \in C(Y)$  then  $g \circ f \in C(X)$ . Further note that since  $Y$  is compact and  $g \in C(Y)$ ,  $g$  is bounded i.e.  $g(Y)$  is a bounded subset of  $\mathbb{R}$ . It follows that  $(g \circ f)(X)$  is a bounded subset of  $\mathbb{R}$  and hence  $g \circ f \in C^*(X)$ . Consequently  $g \circ f \in A(X)$ . Thus the definition of  $\hat{M}$  is without any ambiguity. It is easy to see that  $\hat{M}$  is an ideal of  $C(Y)$ . It follows, since  $M$  is a maximal ideal and therefore a prime ideal of  $A(X)$ , that

$\hat{M}$  is a prime ideal of  $C(Y)$ . Since  $C(Y)$  is a Gelfand ring,  $\hat{M}$  can be extended to a unique maximal ideal  $N$  in  $C(Y)$ . Since  $Y$  is compact,  $N$  is fixed (see [17, Theorem 4.11]). Thus we can write:  $N = N_y = \{g \in C(Y) : g(y) = 0\}$  for some  $y \in Y$ . We observe that  $y \in \bigcap_{g \in \hat{M}} Z(g)$ . Indeed  $\bigcap_{g \in \hat{M}} Z(g) = \{y\}$  for if  $y_1, y_2 \in \bigcap_{g \in \hat{M}} Z(g)$ , for  $y_1 \neq y_2$ , then  $\hat{M} \subseteq N_{y_1}$  and  $\hat{M} \subseteq N_{y_2}$  which is impossible as  $N_{y_1} \neq N_{y_2}$  and  $C(Y)$  is a Gelfand ring. We then set  $f^{\beta_A}(M) = y$ . Note that  $\{f^{\beta_A}(M)\} = \bigcap_{g \in \hat{M}} Z(g)$ . Thus  $f^{\beta_A} : \mathcal{M}(A(X)) \rightarrow Y$  is a well defined map. Now choose  $x \in X$  and then  $g \in \hat{M}_A^x$ ; then  $g \circ f \in M_A^x$ , which implies that  $(g \circ f)(x) = 0$ . Consequently  $f(x) \in Z(g)$  for each  $g \in \hat{M}_A^x$ . On the other hand  $\{f^{\beta_A}(M_A^x)\} = \bigcap_{g \in \hat{M}_A^x} Z(g)$ . This implies that  $f^{\beta_A}(M_A^x) = f(x)$ ; in other words  $(f^{\beta_A} \circ \eta_A)(x) = f(x)$  and this relation is true for each  $x \in X$ . Hence  $f^{\beta_A} \circ \eta_A = f$ .

Now towards the proof of the continuity of the map  $f^{\beta_A}$ , choose  $M \in \mathcal{M}(A(X))$  and a neighbourhood  $W$  of  $f^{\beta_A}(M)$  in the space  $Y$ . In a Tychonoff space every neighbourhood of a point  $x$  contains a zero set neighbourhood of  $x$ , which contains, a co-zero set neighbourhood of  $x$ . So there exist some  $g_1, g_2 \in C(Y)$ , such that  $f^{\beta_A}(M) \in Y \setminus Z(g_1) \subseteq Z(g_2) \subseteq W$ . It follows that  $g_1 g_2 = 0$  as  $Z(g_1) \cup Z(g_2) = Y$  which means that  $Z(g_1 g_2) = Y$ . Furthermore  $f^{\beta_A}(M) \notin Z(g_1)$ . Since  $\{f^{\beta_A}(M)\} = \bigcap_{g \in \hat{M}} Z(g)$ , as observed earlier, we then have  $g_1 \notin \hat{M}$ . This means that  $g_1 \circ f \notin M$ . In other words  $M \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}$ , which is an open neighbourhood of  $M$  in  $\mathcal{M}(A(X))$ . We shall check that  $f^{\beta_A}(\mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}) \subseteq W$  and that settles the continuity of  $f^{\beta_A}$  at  $M$ . Towards that end, choose a maximal ideal  $N \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}$ . This means that  $N \notin \mathcal{M}_{g_1 \circ f}$ , i.e.  $g_1 \circ f \notin N$ . Thus  $g_1 \notin \hat{N}$ . But as  $g_1 g_2 = 0$  and  $\hat{N}$  is prime ideal in  $C(Y)$ , it must be that  $g_2 \in \hat{N}$ . Since  $\{f^{\beta_A}(N)\} = \bigcap_{g \in \hat{N}} Z(g)$ , it follows that  $f^{\beta_A}(N) \in Z(g_2) \subseteq W$ .  $\square$

To achieve the complex analogue of the above mentioned theorem, we need to prove the following proposition, which is by itself a result of independent interest.

**Theorem 3.2.** *Let  $A(X) \in \Sigma(X)$ . Then the map  $\psi_A : \mathcal{M}([A(X)]_c) \rightarrow \mathcal{M}(A(X))$  mapping  $M \rightarrow M \cap A(X)$  is a homeomorphism from the structure space of  $[A(X)]_c$  onto the structure space of  $A(X)$ .*

*Proof.* That the above map  $\psi_A$  is a bijection between the structure spaces of  $[A(X)]_c$  and  $A(X)$  follows from Theorems 2.3, 2.6, 2.7, and 2.15. Recall (same notation as before) that  $\mathcal{M}([A(X)]_c)_f$  is the set of maximal ideals in the ring  $[A(X)]_c$  containing the function  $f \in [A(X)]_c$ . A typical basic closed set in the structure space  $\mathcal{M}([A(X)]_c)$  is given by  $\mathcal{M}([A(X)]_c)_h$  where  $h \in [A(X)]_c$ . Note that  $\mathcal{M}([A(X)]_c)_h = \{J \in \mathcal{M}([A(X)]_c) : h \in J\}$ . So for  $h \in [A(X)]_c$ ,  $J \in \mathcal{M}([A(X)]_c)_h$  if and only if  $h \in J$ , and this is true in view of Theorem 2.8 and the absolute convexity of maximal ideals (see Theorem 2.4(b) of the present article) if and only if  $|h| \in J \cap A(X)$ , and this holds when and only when  $J \cap A(X) \in \mathcal{M}(A(X))_{|h|}$ , which is a basic closed set in the structure space

$\mathcal{M}(A(X))$  of the ring  $A(X)$ . Thus

$$(3.1) \quad \psi_A[\mathcal{M}([A(X)]_c)_h] = \mathcal{M}(A(X))_{|h|}$$

Therefore  $\psi_A$  carries a basic closed set in the domain space onto a basic closed set in the range space. Now for a maximal ideal  $N$  in  $A(X)$  and a function  $g \in A(X)$ ,  $g$  belongs to  $N$  if and only if  $|g| \in N$ , because of the absolutely convexity of a maximal ideal in an intermediate ring. Consequently  $\mathcal{M}(A(X))_g = \mathcal{M}(A(X))_{|g|}$  for any  $g \in A(X)$ . Hence from relation (3.1), we get:  $\psi_A[\mathcal{M}([A(X)]_c)_g] = \mathcal{M}(A(X))_g$  which implies that  $\psi_A^{-1}[\mathcal{M}(A(X))_g] = \mathcal{M}([A(X)]_c)_g$ . Thus  $\psi_A^{-1}$  carries a basic closed set in the structure space  $\mathcal{M}(A(X))$  onto a basic closed in the structure space  $\mathcal{M}([A(X)]_c)$ . Altogether  $\psi_A$  becomes a homeomorphism.  $\square$

For any  $x \in X$  and  $A(X) \in \Sigma(X)$ , set  $M_{A[C]}^x = \{h \in [A(X)]_c : h(x) = 0\}$ . It is easy to check by using standard arguments, such as those employed to prove the textbook theorem [17, Theorem 4.1], that  $M_{A[C]}^x$  is a fixed maximal in  $[A(X)]_c$  and  $M_{A[C]}^x \cap A(X) = M_A^x = \{g \in A(X) : g(x) = 0\}$ . Let  $\zeta_A : X \mapsto \mathcal{M}([A(X)]_c)$  be the map defined by:  $\zeta_A(x) = M_{A[C]}^x$ . Then we have the following results.

**Theorem 3.3.**  *$(\zeta_A, \mathcal{M}([A(X)]_c))$  is a Hausdorff compactification of  $X$ . Furthermore  $(\psi_A \circ \zeta_A)(x) = \eta_A(x)$  for all  $x$  in  $X$ . Hence  $(\zeta_A, \mathcal{M}([A(X)]_c))$  is topologically equivalent to the Hausdorff compactification  $(\eta_A, \mathcal{M}(A(X)))$  as considered in Theorem 3.1. Consequently  $(\zeta_A, \mathcal{M}([A(X)]_c))$  turns out to be topologically equivalent to the Stone-Ćech compactification  $\beta X$  of  $X$ .*

*Proof.* Since  $\mathcal{M}(A(X))$  is Hausdorff [23], it follows from Theorem 3.2 that  $\mathcal{M}([A(X)]_c)$  is a Hausdorff space. Now by following closely the arguments made at the very beginning of the proof of Theorem 3.1, one can easily see that  $(\zeta_A, \mathcal{M}([A(X)]_c))$  is a Hausdorff compactification of  $X$ . The second part of the theorem is already realised in Theorem 3.2. The third part of the present theorem also follows from Theorem 3.2.  $\square$

**Definition 3.4.** An intermediate ring  $A(X) \in \Sigma(X)$  is called  $C$ -type in [16], if it is isomorphic to  $C(Y)$  for some Tychonoff space  $Y$ .

In [16], the authors have shown that if  $I$  is an ideal of the ring  $C(X)$ , then the linear sum  $C^*(X)+I$  is a  $C$ -type ring and of course  $C^*(X)+I \in \Sigma(X)$ . Recently the authors in [1] have realised that these are the only  $C$ -type intermediate rings of real-valued continuous functions on  $X$  if and only if  $X$  is pseudocompact. We now show that the complex analogues of all these results are also true. We reproduce the following result established in [15], which will be needed for this purpose.

**Theorem 3.5.** *A ring  $A(X) \in \Sigma(X)$  is  $C$ -type if and only if  $A(X)$  is isomorphic to the ring  $C(v_AX)$ , where  $v_AX = \{p \in \beta X : f^*(p) \in \mathbb{R} \text{ for each } f \in A(X)\}$  and  $f^* : \beta X \mapsto \mathbb{R} \cup \{\infty\}$  is the Stone extension of the function  $f$ .*

We extend the notion of  $C$ -type ring to rings of complex-valued continuous functions: a ring  $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$  is a  $C$ -type ring if it is isomorphic to a ring  $C(Y, \mathbb{C})$  for some Tychonoff space  $Y$ .

**Theorem 3.6.** *Suppose  $A(X) \in \Sigma(X)$  is a  $C$ -type intermediate ring of real-valued continuous functions on  $X$ . Then  $[A(X)]_c$  is a  $C$ -type intermediate ring of complex-valued continuous functions on  $X$ .*

*Proof.* Since  $A(X)$  is a  $C$ -type intermediate ring by Theorem 3.5, there exists an isomorphism  $\psi : A(X) \rightarrow C(v_A X)$ . Let  $\hat{\psi} : [A(X)]_c \rightarrow C(v_A X, \mathbb{C})$  be defined as follows:  $\hat{\psi}(f + ig) = \psi(f) + i\psi(g)$ , where  $f, g \in A(X)$ . It is not hard to check that  $\hat{\psi}$  is an isomorphism from  $[A(X)]_c$  onto  $C(v_A X, \mathbb{C})$ .  $\square$

**Theorem 3.7.** *Let  $I$  be a  $z$ -ideal in  $C(X, \mathbb{C})$ . Then  $C^*(X, \mathbb{C}) + I$  is a  $C$ -type intermediate ring of complex-valued continuous functions on  $X$ . Furthermore these are the only  $C$ -type rings lying between  $C^*(X, \mathbb{C})$  and  $C(X, \mathbb{C})$  if and only if  $X$  is pseudocompact.*

*Proof.* As mentioned above, it is proved in [16] that for any ideal  $J$  in  $C(X)$ ,  $C^*(X) + J$  is a  $C$ -type intermediate ring of real-valued continuous functions on  $X$ . In light of this and Theorem 3.6, it is sufficient to prove for the first part of this theorem that  $C^*(X, \mathbb{C}) + I = [C^*(X) + I \cap C(X)]_c$ . Towards proving that, let  $f, g \in C^*(X) + I \cap C(X)$ . We can write  $g = g_1 + g_2$  where  $g_1 \in C^*(X)$  and  $g_2 \in I \cap C(X)$ . It follows that  $ig_1 \in C^*(X, \mathbb{C})$  and  $ig_2 \in I$  and this implies that  $i(g_1 + g_2) \in C^*(X, \mathbb{C}) + I$ . Thus  $f + ig \in C^*(X) + I$ . Hence  $[C^*(X) + I \cap C(X)]_c \subseteq C^*(X, \mathbb{C}) + I$ . To prove the reverse inclusion relation, let  $h_1 + h_2 \in C^*(X, \mathbb{C}) + I$ , where  $h_1 \in C^*(X, \mathbb{C})$  and  $h_2 \in I$ . We can write  $h_1 = f_1 + ig_1, h_2 = f_2 + ig_2$ , where  $f_1, f_2, g_1, g_2 \in C(X)$ . Since  $h_1 \in C^*(X, \mathbb{C})$ , it follows that  $f_1, g_1 \in C^*(X)$ . Thus  $|f_2| \leq |h_2|$  and  $h_2 \in I$ . This implies, because of the absolute convexity of the  $z$ -ideal  $I$  in  $C(X, \mathbb{C})$ , that  $f_2 \in I$ . Analogously  $g_2 \in I$ . It is now clear that  $f_1 + f_2 \in C^*(X) + I \cap C(X)$  and  $g_1 + g_2 \in C^*(X) + I \cap C(X)$ . Thus  $h_1 + h_2 = (f_1 + f_2) + i(g_1 + g_2) \in [C^*(X) + I \cap C(X)]_c$ . Hence  $C^*(X, \mathbb{C}) + I \subseteq [C^*(X) + I \cap C(X)]_c$ .

To prove the second part of the theorem, we first observe that if  $X$  is pseudocompact, then there is practically nothing to prove. Assume therefore that  $X$  is not pseudocompact. Hence by [1], there exists an  $A(X) \in \Sigma(X)$  such that  $A(X)$  is a  $C$ -type ring but  $A(X) \neq C^*(X) + J$  for any ideal  $J$  in  $C(X)$ . It follows from Theorem 3.6 that  $[A(X)]_c$  is a  $C$ -type intermediate ring of complex-valued continuous functions belonging to the family  $\Sigma(X, \mathbb{C})$ . We assert that there does not exist any  $z$ -ideal  $I$  in  $C(X, \mathbb{C})$  with the relation:  $C^*(X, \mathbb{C}) + I = [A(X)]_c$  and that finishes the present theorem. Suppose towards a contradiction, there exists a  $z$ -ideal  $I$  in  $C(X, \mathbb{C})$  such that  $C^*(X, \mathbb{C}) + I = [A(X)]_c$ . Now from the proof of the first part of this theorem, we have already settled that  $C^*(X, \mathbb{C}) + I = [C^*(X) + I \cap C(X)]_c$ . Consequently  $[C^*(X) + I \cap C(X)]_c = [A(X)]_c$  which yields  $[C^*(X) + I \cap C(X)]_c \cap C(X) = [A(X)]_c \cap C(X)$ , and hence  $C^*(X) + I \cap C(X) = A(X)$ , a contradiction.  $\square$

We shall conclude this section after incorporating a purely algebraic result pertaining to the residue class field of  $C(X, \mathbb{C})$  modulo a maximal ideal in the same field.

For each  $a = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$  if  $\mathcal{P}_1a, \mathcal{P}_2a, \dots, \mathcal{P}_na$  are the zeroes of the polynomial  $P_a(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$ , ordered so that  $|\mathcal{P}_1a| \leq |\mathcal{P}_2a| \leq \dots \leq |\mathcal{P}_na|$ , then by following closely the arguments of [17, 13.3(a)], the following result can be obtained.

**Theorem 3.8.** *For each  $k$ , the function  $\mathcal{P}_k : \mathbb{C}^n \mapsto \mathbb{C}$ , described above, is continuous.*

By employing the main argument of [17, Theorem 13.4], we obtain the following proposition as a consequence of Theorem 3.8.

**Theorem 3.9.** *For any maximal ideal  $N$  in  $C(X, \mathbb{C})$ , the residue class field  $C(X, \mathbb{C})/N$  is algebraically closed.*

We recall from Theorem 2.15 that the assignment  $M \mapsto M_c$  establishes a one-to-one correspondence between maximal ideals in  $C(X)$  and those in  $C(X, \mathbb{C})$ . Let  $\phi : C(X)/M \mapsto C(X, \mathbb{C})/M_c$  be the induced assignment between the corresponding residue class fields, explicitly  $\phi(f + M) = f + M_c$  for each  $f \in C(X)$ . It is easy to check that  $\phi$  is a ring homomorphism and is one-to-one because if  $f + M_c = g + M_c$  with  $f, g \in C(X)$ , then  $f - g \in M_c \cap C(X) = M$  and hence  $f + M = g + M$ . Furthermore, if we choose an element  $f + ig + M_c$  from  $C(X, \mathbb{C})/M_c$ , with  $f, g \in C(X)$ , then one can verify easily that it is a root of the polynomial  $\lambda^2 - 2(f + M_c)\lambda + (f^2 + g^2 + M_c)$  over the field  $\phi(C(X)/M)$ . Identifying  $C(X)/M$  with  $\phi(C(X)/M)$ , and taking note of Theorem 3.9 we get the following result.

**Theorem 3.10.** *For any maximal ideal  $M$  in  $C(X)$ , the residue class field  $C(X, \mathbb{C})/M_c$  is the algebraic closure of  $C(X)/M$ .*

#### 4. IDEALS OF THE FORM $C_{\mathcal{P}}(X, \mathbb{C})$ AND $C_{\infty}^{\mathcal{P}}(X, \mathbb{C})$

Let  $\mathcal{P}$  be an ideal of closed sets in  $X$ . We set  $C_{\mathcal{P}}(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$  and  $C_{\infty}^{\mathcal{P}}(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : \text{for each } \epsilon > 0 \text{ in } \mathbb{R}, \{x \in X : |f(x)| \geq \epsilon\} \in \mathcal{P}\}$ . These are the complex analogues of the rings,  $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X \setminus Z(f)) \in \mathcal{P}\}$  and  $C_{\infty}^{\mathcal{P}}(X) = \{f \in C(X) : \text{for each } \epsilon > 0, \{x \in X : |f(x)| \geq \epsilon\} \in \mathcal{P}\}$  already introduced in [4] and investigated subsequently in [5], [12]. As in the real case, it is easy to check that  $C_{\mathcal{P}}(X, \mathbb{C})$  is a  $z$ -ideal in  $C(X, \mathbb{C})$  with  $C_{\infty}^{\mathcal{P}}(X, \mathbb{C})$  just a subring of  $C(X, \mathbb{C})$ . Plainly we have:  $C_{\mathcal{P}}(X, \mathbb{C}) \cap C(X) = C_{\mathcal{P}}(X)$  and  $C_{\infty}^{\mathcal{P}}(X, \mathbb{C}) \cap C(X) = C_{\infty}^{\mathcal{P}}(X)$ .

The following results need only routine verifications.

**Theorem 4.1.** *For any ideal  $\mathcal{P}$  of closed sets in  $X$ ,  $[C_{\mathcal{P}}(X)]_c = \{f + ig : f, g \in C_{\mathcal{P}}(X)\} = C_{\mathcal{P}}(X, \mathbb{C})$  and  $[C_{\infty}^{\mathcal{P}}(X)]_c = C_{\infty}^{\mathcal{P}}(X, \mathbb{C})$ .*

**Theorem 4.2.**

- a) *If  $I$  is an ideal of the ring  $C_{\mathcal{P}}(X)$ , then  $I_c = \{f + ig : f, g \in I\}$  is an ideal of  $C_{\mathcal{P}}(X, \mathbb{C})$  and  $I_c \cap C_{\mathcal{P}}(X) = I$ .*

- b) If  $I$  is an ideal of the ring  $C_\infty^{\mathcal{P}}(X)$ , then  $I_c$  is an ideal of  $C_\infty^{\mathcal{P}}(X, \mathbb{C})$  and  $I_c \cap C_\infty^{\mathcal{P}}(X) = I$ .

We record below the following consequence of the above theorem.

**Theorem 4.3.** *If  $I_1 \subsetneq I_2 \subsetneq \dots$  is a strictly ascending sequence of ideals in  $C_{\mathcal{P}}(X)$  (respectively  $C_\infty^{\mathcal{P}}(X)$ ), then  $I_{1c} \subsetneq I_{2c} \subsetneq \dots$  becomes a strictly ascending sequence of ideals in  $C_{\mathcal{P}}(X, \mathbb{C})$  (respectively  $C_\infty^{\mathcal{P}}(X, \mathbb{C})$ ).*

The analogous results for a strictly descending sequence of ideals in both the rings  $C_{\mathcal{P}}(X)$  and  $C_\infty^{\mathcal{P}}(X)$  are also valid.

**Definition 4.4.** A space  $X$  is called *locally  $\mathcal{P}$*  if each point of  $X$  has an open neighbourhood  $W$  such that  $cl_X W \in \mathcal{P}$ .

Observe that if  $\mathcal{P}$  is the ideal of all compact sets in  $X$ , then  $X$  is locally  $\mathcal{P}$  if and only if  $X$  is locally compact.

Towards finding a condition for which  $C_{\mathcal{P}}(X, \mathbb{C})$  and  $C_\infty^{\mathcal{P}}(X, \mathbb{C})$  are Noetherian ring/Artinian rings, we reproduce a special version of a fact proved in [6]:

**Theorem 4.5** (from [6, Theorem 1.1]). *Let  $\mathcal{P}$  be an ideal of closed sets in  $X$  and suppose  $X$  is locally  $\mathcal{P}$ . Then the following statements are equivalent:*

- 1)  $C_{\mathcal{P}}(X)$  is a Noetherian ring.
- 2)  $C_{\mathcal{P}}(X)$  is an Artinian ring.
- 3)  $C_\infty^{\mathcal{P}}(X)$  is a Noetherian ring.
- 4)  $C_\infty^{\mathcal{P}}(X)$  is an Artinian ring.
- 5)  $X$  is finite set.

We also note the following standard result of Algebra.

**Theorem 4.6.** *Let  $\{R_1, R_2, \dots, R_n\}$  be a finite family of commutative rings with identity. The ideals of the direct product  $R_1 \times R_2 \times \dots \times R_n$  are exactly of the form  $I_1 \times I_2 \times \dots \times I_n$ , where for  $k = 1, 2, \dots, n$ ,  $I_k$  is an ideal of  $R_k$ .*

Now if  $X$  is a finite set, with say  $n$  elements, then as it is Tychonoff, it is discrete space. Furthermore if  $X$  is locally  $\mathcal{P}$ , then clearly  $\mathcal{P}$  is the power set of  $X$ . Consequently  $C_{\mathcal{P}}(X, \mathbb{C}) = C_\infty^{\mathcal{P}}(X, \mathbb{C}) = C(X, \mathbb{C}) = \mathbb{C}^n$ , which is equal to the direct product of  $\mathbb{C}$  with itself ‘ $n$ ’ times. Since  $\mathbb{C}$  is a field, it has just 2 ideals, hence by Theorem 4.6 there are exactly  $2^n$  many ideals in the ring  $\mathbb{C}^n$ . Hence  $C_{\mathcal{P}}(X, \mathbb{C})$  and  $C_\infty^{\mathcal{P}}(X, \mathbb{C})$  are both Noetherian rings and Artinian rings. On the other hand if  $X$  is an infinite space and is locally  $\mathcal{P}$  space then it follows from the Theorem 4.3 and Theorem 4.5 that neither of the two rings  $C_{\mathcal{P}}(X, \mathbb{C})$  and  $C_\infty^{\mathcal{P}}(X, \mathbb{C})$  is either Noetherian or Artinian. This leads to the following proposition as the complex analogue of Theorem 4.5.

**Theorem 4.7.** *Let  $\mathcal{P}$  be an ideal of closed sets in  $X$  and suppose  $X$  is locally  $\mathcal{P}$ . Then the following statements are equivalent:*

- 1)  $C_{\mathcal{P}}(X, \mathbb{C})$  is a Noetherian ring.
- 2)  $C_{\mathcal{P}}(X, \mathbb{C})$  is an Artinian ring.



- 3)  $C_{\infty}^{\mathcal{P}}(X, \mathbb{C})$  is a Noetherian ring.
- 4)  $C_{\infty}^{\mathcal{P}}(X, \mathbb{C})$  is an Artinian ring.
- 5)  $X$  is finite set.

A special case of this theorem, choosing  $\mathcal{P}$  to be the ideal of all closed sets in  $X$  reads:  $C(X, \mathbb{C})$  is a Noetherian ring if and only if  $X$  is finite set.

The following gives a necessary and sufficient condition for the ideal  $C_{\mathcal{P}}(X, \mathbb{C})$  in  $C(X, \mathbb{C})$  to be prime.

**Theorem 4.8.** *Let  $\mathcal{P}$  be an ideal of closed sets in  $X$  and suppose  $X$  is locally  $\mathcal{P}$ . Then the following statements are equivalent:*

- (1)  $C_{\mathcal{P}}(X, \mathbb{C})$  is a prime ideal in  $C(X, \mathbb{C})$ .
- (2)  $C_{\mathcal{P}}(X)$  is a prime ideal in  $C(X)$ .
- (3)  $X \notin \mathcal{P}$  and for any two disjoint co-zero sets in  $X$ , one has its closure lying in  $\mathcal{P}$ .

*Proof.* The equivalence of (1) and (2) follows from Theorem 2.12 and Theorem 4.1. Towards the equivalence (2) and (3), assume that  $C_{\mathcal{P}}(X)$  is a prime ideal in  $C(X)$ . If  $X \in \mathcal{P}$ , then for each  $f \in C(X)$ ,  $cl_X(X \setminus Z(f)) \in \mathcal{P}$  meaning that  $f \in C_{\mathcal{P}}(X)$  and hence  $C_{\mathcal{P}}(X) = C(X)$ , a contradiction to the assumption that  $C_{\mathcal{P}}(X)$  is a prime ideal and in particular a proper ideal of  $C(X)$ . Thus  $X \notin \mathcal{P}$ . Now consider two disjoint co-zero sets  $X \setminus Z(f)$  and  $X \setminus Z(g)$  in  $X$ , with  $f, g \in C(X)$ . It follows that  $Z(f) \cup Z(g) = X$ , i.e.  $fg = 0$ . Since  $C_{\mathcal{P}}(X)$  is prime, this implies that  $f \in C_{\mathcal{P}}(X)$  or  $g \in C_{\mathcal{P}}(X)$ , i.e.  $cl_X(X \setminus Z(f)) \in \mathcal{P}$  or  $cl_X(X \setminus Z(g)) \in \mathcal{P}$ .

Conversely let the statement (3) be true. Since a  $z$ -ideal  $I$  in  $C(X)$  is prime if and only if for each  $f, g \in C(X)$ ,  $fg = 0$  implies  $f \in I$  or  $g \in I$  (see [17, Theorem 2.9]) and since  $C_{\mathcal{P}}(X)$  is a  $z$ -ideal in  $C(X)$ , it is sufficient to show that for each  $f, g \in C(X)$ , if  $fg = 0$  then  $f \in C_{\mathcal{P}}(X)$  or  $g \in C_{\mathcal{P}}(X)$ . Indeed  $fg = 0$  implies that  $X \setminus Z(f)$  and  $X \setminus Z(g)$  are disjoint co-zero sets in  $X$ . Hence by supposition (3), either  $cl_X(X \setminus Z(f)) \in \mathcal{P}$  or  $cl_X(X \setminus Z(g)) \in \mathcal{P}$  meaning that  $f \in C_{\mathcal{P}}(X)$  or  $g \in C_{\mathcal{P}}(X)$ . □

A special case of Theorem 4.8, with  $\mathcal{P}$  equal to the ideal of all compact sets in  $X$ , is proved in [10]. We examine a second special case of Theorem 4.8.

A subset  $Y$  of  $X$  is called a *bounded* subset of  $X$  if each  $f \in C(X)$  is bounded on  $Y$ . Let  $\beta$  denote the family of all closed bounded subsets of  $X$ . Then  $\beta$  is an ideal of closed sets in  $X$ . It is plain that a pseudocompact subset of  $X$  is bounded but a bounded subset of  $X$  may not be pseudocompact. Here is a counterexample: the open interval  $(0, 1)$  in  $\mathbb{R}$  is a bounded subset of  $\mathbb{R}$  without being a pseudocompact subset of  $\mathbb{R}$ . However for a certain class of subsets of  $X$ , the two notions of boundedness and pseudocompactness coincide. The following well-known proposition substantiates this fact:

**Theorem 4.9** (Mandelkar [18]). *A support of  $X$ , i.e. a subset of  $X$  of the form  $cl_X(X \setminus Z(f))$  for some  $f \in C(X)$ , is a bounded subset of  $X$  if and only if it is a pseudocompact subset of  $X$ .*

It is clear that the conclusion of Theorem 4.9 remains unchanged if we replace  $C(X)$  by  $C(X, \mathbb{C})$ .

Let  $C_\psi(X) = \{f \in C(X) : f \text{ has pseudocompact support}\}$  and recall that  $C_\beta(X) = \{f \in C(X) : f \text{ has bounded support}\}$ . We would like to mention here that the closed pseudocompact subsets of a pseudocompact space  $X$  might not constitute an ideal of closed sets in  $X$ . Indeed a closed subset of a pseudocompact space may not be pseudocompact. The celebrated example of a Tychonoff plank in [17, 8.20]:  $[0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$ , where  $\omega_1$  is the 1st uncountable ordinal and  $\omega$  is the first infinite ordinal, demonstrates this fact. Nevertheless  $C_\psi(X)$  is an ideal of the ring  $C(X)$ . Indeed it follows directly from Theorem 4.9 that  $C_\psi(X) = C_\beta(X)$ .

A Tychonoff space  $X$  is called *locally pseudocompact* if each point on  $X$  has an open neighbourhood with its closure pseudocompact. On the other hand,  $X$  is called *locally bounded* (or *locally  $\beta$* ) if each point in  $X$  has an open neighbourhood with its closure bounded. Since each open neighbourhood of a point  $x$  in a Tychonoff space  $X$  contains a co-zero set neighbourhood of  $x$ , it follows from Theorem 4.9 that  $X$  is locally bounded if and only if  $X$  is locally pseudocompact. This combined with Theorem 2.12 leads to the following special case of Theorem 4.8.

**Theorem 4.10.** *Let  $X$  be locally pseudocompact. Then the following statements are equivalent:*

- (1)  $C_\psi(X)$  is a prime ideal of  $C(X)$ .
- (2)  $C_\psi(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : f \text{ has pseudocompact support}\}$  is a prime ideal of  $C(X, \mathbb{C})$ .
- (3)  $X$  is not pseudocompact and for any two disjoint co-zero sets in  $X$ , the closure of one of them is pseudocompact.

Since for  $f \in C(X, \mathbb{C})$ ,  $f \in C_\infty(X, \mathbb{C})$  if and only if  $|f| \in C_\infty(X)$ , it follows that  $C_\infty(X, \mathbb{C})$  is an ideal of  $C(X, \mathbb{C})$  if and only if  $C_\infty(X)$  is an ideal of  $C(X)$ . In general however  $C_\infty(X)$  need not be an ideal of  $C(X)$ . If  $X$  is assumed to be locally compact, then it is proved in [2] and [11] that  $C_\infty(X)$  is an ideal of  $C(X)$  when and only when  $X$  is pseudocompact. Therefore the following theorem holds.

**Theorem 4.11.** *Let  $X$  be locally compact. Then the following three statements are equivalent:*

- 1)  $C_\infty(X, \mathbb{C})$  is an ideal of  $C(X, \mathbb{C})$ .
- 2)  $C_\infty(X)$  is an ideal of  $C(X)$ .
- 3)  $X$  is pseudocompact.

## 5. ZERO DIVISOR GRAPHS OF RINGS IN THE FAMILY $\Sigma(X, \mathbb{C})$

We fix any intermediate ring  $P(X, \mathbb{C})$  in the family  $\Sigma(X, \mathbb{C})$ . Suppose  $\mathcal{G} = \mathcal{G}(P(X, \mathbb{C}))$  designates the graph whose vertices are zero divisors of  $P(X, \mathbb{C})$  and there is an edge between vertices  $f$  and  $g$  if and only if  $fg = 0$ . For any two vertices  $f, g$  in  $\mathcal{G}$ , let  $d(f, g)$  be the length of the shortest path between  $f$  and

$g$  and  $\text{Diam } \mathcal{G} = \sup\{d(f, g) : f, g \in \mathcal{G}\}$ . Suppose  $\text{Gr } \mathcal{G}$  designates the length of the shortest cycle in  $\mathcal{G}$ , often called the girth of  $\mathcal{G}$ . It is easy to check that a vertex  $f$  in  $\mathcal{G}$  is a divisor of zero in  $P(X, \mathbb{C})$  if and only if  $\text{Int}_X Z(f) \neq \emptyset$ . This parallels the statement that a vertex  $f$  in the zero-divisor graph  $\Gamma C(X)$  of  $C(X)$  considered in [9] is a divisor of zero in  $C(X)$  if and only if  $\text{Int}_X Z(f) \neq \emptyset$ . We would like to point out in this connection that a close scrutiny into the proof of various results in [9] reveal that several facts related to the nature of the vertices and the length of the cycles related to  $\Gamma C(X)$  have been established in [9] by employing skillfully the last mentioned simple characterization of divisors of zero in  $C(X)$ . It is expected that the analogous facts pertaining to the various parameters of the graph  $\mathcal{G}(P(X, \mathbb{C})) = \mathcal{G}$  should also hold. We therefore just record the following results related to the graph  $\mathcal{G}$ , without any proof.

**Theorem 5.1.** *Let  $f, g$  be vertices of the graph  $\mathcal{G}$ . Then  $d(f, g) = 1$  if and only if  $Z(f) \cup Z(g) = X$ ;  $d(f, g) = 2$  if and only if  $Z(f) \cup Z(g) \subsetneq X$  and  $\text{Int}_X Z(f) \cap \text{Int}_X Z(g) \neq \emptyset$ ;  $d(f, g) = 3$  if and only if  $Z(f) \cup Z(g) \subsetneq X$  and  $\text{Int}_X Z(f) \cap \text{Int}_X Z(g) = \emptyset$ . Consequently on assuming that  $X$  contains at least 3 points,  $\text{Diam } \mathcal{G}$  and  $\text{Gr } \mathcal{G}$  are both equal to 3 (compare with [9, Corollary 1.3]).*

**Theorem 5.2.** *Each cycle in  $\mathcal{G}$  has length 3, 4 or 6. Furthermore every edge of  $\mathcal{G}$  is an edge of a cycle with length 3 or 4 (compare with [9, Corollary 2.3]).*

**Theorem 5.3.** *Suppose  $X$  contains at least 2 points. Then*

- a) *Each vertex of  $\mathcal{G}$  is a 4 cycle vertex.*
- b)  *$\mathcal{G}$  is a triangulated graph meaning that each vertex of  $\mathcal{G}$  is a vertex of a triangle if and only if  $X$  is devoid of any isolated point.*
- c)  *$\mathcal{G}$  is a hypertriangulated graph in the sense that each edge of  $\mathcal{G}$  is edge of a triangle if and only if  $X$  is a connected middle  $P$  space (compare with the analogous facts in [9, Proposition 2.1]).*

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