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Remarks on fixed point assertions in digital topology, 2

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Abstract

Several recent papers in digital topology have sought to obtain fixed point results by mimicking the use of tools from classical topology, such as complete metric spaces. We show that in many cases, researchers using these tools have derived conclusions that are incorrect, trivial, or limited.

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1. Introduction

This paper continues the work of [5] and quotes or paraphrases from it.

Recent papers have attempted to apply to digital images ideas from Euclidean topology and real analysis concerning metrics and fixed points. While the underlying motivation of digital topology comes from Euclidean topology and real analysis, some applications of fixed point theory recently featured in the literature of digital topology seem of doubtful worth. Although papers including [19, 4] have valid and interesting results for fixed points and for "almost" or "approximate" fixed points in digital topology, many other published assertions concerning fixed points in digital topology are incorrect, trivial (e.g.,

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applicable only to singletons, or only to constant functions), or limited, as discussed in [5]. After submitting [5], we learned of several additional publications with assertions characterized as above; these are discussed in the current paper.

The less-than-ideal papers we discuss have in common a definition of a digital metric space (X, d, κ), where X is a set of lattice points, d is a metric (typically, the Euclidean), and κ is an adjacency relation on X, making (X, κ) a graph; and then these papers never make use of κ. Functions considered usually are all continuous in the topological sense, since the metric d usually imposes a discrete topology on the digital image; but are often discontinuous in the digital sense of preserving graph connectedness.

Many of these papers' assertions are modifications of results known for the Euclidean topology of $\mathbb{R}^n$ that tell us little or nothing about digital images as graphs. Some of these papers' assertions are of interest if we regard the functions investigated as defined on subsets of $\mathbb{R}^n$. In many cases, we offer corrections, notes on their limitations, or improvements.

2. Preliminaries

We let $\mathbb{Z}$ denote the set of integers, and $\mathbb{R}$, the real line.

We consider a digital image as a graph (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation on X. We will often assume that X is a finite set, as in the "real world."

A digital metric space is [8] a triple (X, d, κ) where (X, κ) is a digital image and d is a metric for X. In [8], d was taken to be the Euclidean metric, as was the case in many subsequent papers, but we will not limit our discussion to the Euclidean metric. Often, however, we will assume d is an ℓ_p metric (see section 2.2).

The *diameter* of a metric space (X, d) is

$$diam\, X = \max\{d(x, y) \mid x, y \in X\}.$$

2.1. Adjacencies.

The most commonly used adjacencies for digital images are the c_u-adjacencies, defined as follows.

Definition 2.1. Let $p, q \in \mathbb{Z}^n$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, $p \neq q$. Let $1 \leq u \leq n$. We say p and q are c_u-adjacent, denoted $p \leftrightarrow_{c_u} q$ or $p \leftrightarrow q$ when the adjacency is understood, if

- for at most u distinct indices i, $|p_i - q_i| = 1$, and
- for all other indices j, $p_j = q_j$.

Often, a c_u-adjacency is denoted by the number of points in $\mathbb{Z}^n$ that are c_u-adjacent to a given point. E.g.,

- in $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 8-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

Other adjacencies for digital images are discussed in papers such as [9, 2, 3].

A *digital interval* is a digital image of the form $([a, b]_Z, 2)$, where $a < b$ and $[a, b]_Z = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

2.2. ℓ_p **metric.** Let $X \subset \mathbb{R}^n$ and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be points of X. Let $1 \leq p \leq \infty$. The ℓ_p metric d for X is defined by

$$d(x, y) = \begin{cases} \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} & \text{for } 1 \leq p < \infty; \\ \max\{|x_i - y_i|\}_{i=1}^{n} & \text{for } p = \infty. \end{cases}$$

For $p = 1$, this gives us the *Manhattan metric* $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$; for $p = 2$, we have the *Euclidean metric* $d(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{1/2}$.

The following are easily proved.

Proposition 2.2. *Let $x, y \in \mathbb{Z}^n$ and let d be any ℓ_p metric. Then*

- *if $d(x, y) < 1$, then $x = y$;*
- *if $1 \leq u \leq n$ and $x \leftrightarrow_{c_u} y$, then $d(x, y) \leq u^{1/p}$.*

2.3. **Digital continuity.**

Definition 2.3 ([19, 1])**.** A function $f : (X, \kappa) \to (Y, \lambda)$ between digital images is (κ, λ)-*continuous* (or just *continuous* when κ and λ are understood) if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.4 ([1])**.** *A function $f : (X, \kappa) \to (Y, \lambda)$ between digital images is* (κ, λ)-continuous *if and only if $x \leftrightarrow_\kappa x'$ in X implies either $f(x) = f(x')$ or $f(x) \leftrightarrow_\lambda f(x')$ in Y.*

2.4. **Cauchy sequences and complete metric spaces.** The papers [6, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22] apply to digital images the notions of Cauchy sequence and complete metric space. Since if the digital image X is finite or uses a common metric d such as an ℓ_p metric, the digital metric space (X, d, κ) is a discrete topological space, the digital versions of these notions are quite limited.

Recall that a sequence of points $\{x_n\}$ in a metric space (X, d) is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $d(x_m, x_n) < \varepsilon$. If every Cauchy sequence in X has a limit, then (X, d) is a *complete metric space.*

It has been shown that under a mild additional assumption, a digital Cauchy sequence is eventually constant.

Theorem 2.5 ([10, 5])**.** *Let $a > 0$. If d is a metric on a digital image (X, κ) such that for all distinct $x, y \in X$ we have $d(x, y) > a$, then for any Cauchy sequence $\{x_i\}_{i=1}^{\infty} \subset X$ there exists $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_m = x_n$.*

An immediate consequence of Theorem 2.5 is the following.

Corollary 2.6 ([10])**.** *Let (X, d, κ) be a digital metric space. If d is a metric on (X, κ) such that for all distinct $x, y \in X$ we have $d(x, y) > a$ for some constant $a > 0$, then any Cauchy sequence in X is eventually constant, and (X, d) is a complete metric space.*

Remarks 2.7 ([5]). It is easily seen that the hypotheses of Theorem 2.5 and Corollary 2.6 are satisfied for any finite digital metric space, or for a digital metric space (X, d, κ) for which the metric d is any ℓ_p metric. Thus, a Cauchy sequence that is not eventually constant can only occur in an infinite digital metric space with an unusual metric. Such an example is given below.

Example 2.8 ([5])**.** Let d be the metric on $(\mathbb{N}, c_1)$ defined by $d(i, j) = |1/i - 1/j|$. Then $\{i\}_{i=1}^{\infty}$ is a Cauchy sequence for this metric that does not have a limit.

2.5. **Function sets Ψ, Φ.** Below, we define sets of functions Ψ, Φ that will be used in the following.

Definition 2.9 ([15])**.** Let Ψ_0 be a set of functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi_0$ we have

- ψ is nondecreasing, and
- there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series $\sum_{k=1}^{\infty} v_k$ of non-negative terms such that $k \geq k_0$ implies $\psi^{k+1}(t) \leq a\psi^k(t) + v_k$ for all $t \in [0, \infty)$, where ψ^k represents the k-fold composition of ψ.

The following will be used later in the paper.

Example 2.10 ([5])**.** The constant function with value 0 is a member of Ψ_0.

Definition 2.11 ([13])**.** Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi$ we have

- ψ is nondecreasing, and
- $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n represents the n-fold composition of ψ.

Definition 2.12 ([21])**.** Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is increasing, $\phi(t) = 0$ if and only if $t = 0$, and $\phi(t) < t$ for $t > 0$.

Proposition 2.13. *Let $\psi \in \Psi$. Then for all $t > 0$, $\psi(t) < t$.*

Proof. Suppose there exists $t_0 > 0$ such that $\psi(t_0) \geq t_0$. Since ψ is nondecreasing, an easy induction yields that $\psi^{n+1}(t_0) \geq \psi^n(t_0)$ for all $n \in \mathbb{N}$. Therefore, $\sum_{n=1}^{\infty} \psi^n(t_0) = \infty$, contrary to Definition 2.11. The contradiction establishes the assertion. □

A notion often used with the set Ψ is given by the following.

Definition 2.14 ([20])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say T is *α-admissible* if $\alpha(x, y) \geq 1$ implies $\alpha(T(x), T(y)) \geq 1$.

2.6. **An example.**

Example 2.15 ([5])**.** Let

$$X = \{p_1 = (0, 0, 0, 0, 0), \quad p_2 = (2, 0, 0, 0, 0), \quad p_3 = (1, 1, 1, 1, 1)\} \subset \mathbb{Z}^5.$$

Let $f : X \to X$ be defined by $f(p_1) = f(p_2) = p_1$, $f(p_3) = p_2$. Then f is not (c_5, c_5)-continuous.

Despite its discontinuity, the function of Example 2.15 was shown in [5] to exemplify many different types of functions studied in [8, 10, 11, 12, 14, 15, 16, 17, 18]. In the current paper, this example is also used to show that functions of the type studied need not be digitally continuous.

3. Compatible functions

3.1. **Equivalence of compatibilities.** In this section, we examine the equivalence of several different types of "compatible" functions discussed in [6].

Definition 3.1 ([6]). Suppose S and T are self-maps on a digital metric space (X, d, κ). Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$lim_{n\to\infty} S(x_n) = lim_{n\to\infty} T(x_n) = t \text{ for some } t \in X. \tag{3.1}$$

We have the following.

- S and T are called *compatible* if $lim_{n\to\infty} d(S(T(x_n)), T(S(x_n))) = 0$ for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (3.1).
- S and T are called *compatible of type (A)* if

$$lim_{n\to\infty} d(S(T(x_n)), T(T(x_n))) = 0 =$$

$$lim_{n\to\infty} d(T(S(x_n)), S(S(x_n)))$$

  for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (3.1).
- S and T are called *compatible of type (P)* if

$$lim_{n\to\infty} d(S(S(x_n)), T(T(x_n))) = 0$$

  for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy statement (3.1).

Because digital metric spaces are typically discrete, these properties can be simplified as shown in the remainder of this section.

Proposition 3.2 ([6]). *Let S and T be compatible maps of type (A) on a digital metric space (X, d, κ). If one of S and T is continuous, then S and T are compatible.*

The proof given for Proposition 3.2 clarifies that the continuity expected of S or T is in the sense of the classical "$\varepsilon - \delta$ definition", not in the sense of digital continuity. However, it turns out that this assumption is usually unnecessary, as shown below.

Theorem 3.3. *Let (X, d, κ) be a digital metric space, where either X is finite or d is an ℓ_p metric. Let S and T be self-maps on X. Then the following are equivalent.*

- *S and T are compatible.*
- *S and T are compatible of type (A).*
- *S and T are compatible of type (P).*

Proof. Throughout this proof, let $\{x_n\}_{n=1}^{\infty} \subset X$ satisfy statement (3.1).

Suppose S and T are compatible. Then, by the Triangle Inequality,

$$d(S(S(x_n)), T(T(x_n))) \leq \tag{3.2}$$

$$d(S(S(x_n)), S(T(x_n))) + d(S(T(x_n)), T(S(x_n))) + d(T(S(x_n)), T(T(x_n))).$$

By (3.1) and Corollary 2.6, the first term on the right side of (3.2) is, for sufficiently large n,

$$d(S(S(x_n)), S(T(x_n))) = d(S(t), S(t)) = 0.$$

Similarly, the third term on the right side of (3.2) is, for sufficiently large n,

$$d(T(S(x_n)), T(T(x_n))) = d(T(t), T(t)) = 0.$$

The middle term on the right side of (3.2) is $d(S(T(x_n)), T(S(x_n)))$, which, by compatibility, tends to 0 as $n \to \infty$. Since all the terms on the right side of (3.2) tend to 0, S and T are compatible of type (P).

Suppose S and T are compatible of type (P). Then, using Corollary 2.6,

$$lim_{n\to\infty} d(S(T(x_n)), T(T(x_n))) = lim_{n\to\infty} d(S(t), T(T(x_n))) =$$

$$lim_{n\to\infty} d(S(S(x_n)), T(T(x_n))) = \text{ (because compatible of type (P)) } 0.$$

Similarly, $lim_{n\to\infty} d(T(S(x_n)), S(S(x_n))) = 0$. Therefore, S and T are compatible of type (A).

If S and T are compatible of type (A), then, using the triangle inequality and Definition 3.1,

$$d(S(T(x_n)), T(S(x_n))) \leq d(S(T(x_n)), T(T(x_n))) + d(T(T(x_n)), T(S(x_n)))$$

$$\to_{n\to\infty} 0 + 0 = 0.$$

Hence, S and T are compatible. □

3.2. **Compatible functions' common fixed points.** The assertions stated as Theorem 23 and Theorem 24 of [6] are concerned with the existence of a common fixed point of four self-maps of a digital metric space. However, these assertions are incorrect. We give a counterexample below.

Stated as Theorem 23 of [6] is the following.

> *Let $A, B, S,$ and T be self-maps on a complete digital metric space (X, d, κ) such that*
> *(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$;*
> *(b) the pairs (A, S) and (B, T) are compatible;*
> *(c) one of $S, T, A,$ and B is continuous; and*
> *(d) we have*
> $$F[d(A(x), B(y)), d(S(x), T(y)), d(A(x), S(x)), d(B(y), T(y)),$$
> $$d(A(x), T(y)), d(B(y), S(x))] \leq 0$$
> *for all $x, y \in X$.*
> *Then $A, B, S,$ and T have a unique common fixed point.*

Condition (d) above gives the only assumption stated in [6] about the function F. Perhaps the authors intended to say more, but their statement permits F to be any function that satisfies (d).

Stated as Theorem 24 of [6] is the following.

> *Let A, B, S, and T be self-maps on a complete digital metric space (X, d, κ) satisfying conditions (a), (c), and (d). If the pairs (A, S) and (B, T) are compatible of type (A) or of type (P) then A, B, S, and T have a unique common fixed point.*

By Theorem 3.3, if d is an ℓ_p metric then these assertions are equivalent. A counterexample to these assertions:

Example 3.4. Let $X = \mathbb{Z}$, $d(x, y) = |x - y|$, $\kappa = c_1$, $A(x) = B(x) = x - 1$, $S(x) = T(x) = x + 1$, $F(x_1, x_2, x_3, x_4, x_5, x_6) = 0$.

Proof. That this is a counterexample to the assertion stated as Theorem 23 of [6] is shown as follows. Clearly $S(X) = B(X) = \mathbb{Z} = T(X) = A(X)$. The pair (A, S) is compatible, since $A(S(x)) = x = S(A(x))$ for all $x \in X$; similarly, (B, T) is a compatible pair. All of S, T, A, and B are continuous in both the "$\varepsilon - \delta$" sense and in the digital sense with respect to the c_1 adjacency. Trivially,

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \leq 0,$$

for all $x, y \in X$. However, none of S, T, A, and B has a fixed point. □

4. Expansive mappings

The paper [13] is concerned with fixed points for expansive maps on digital metric spaces.

Definition 4.1 ([13])**.** Let T be a self map on a complete metric space (X, d) such that T is onto, and for some $k \geq 1$ and all $x, y \in X$,

$$d(T(x), T(y)) \geq k\, d(x, y). \tag{4.1}$$

Then T is called an *expansive map*.

Remarks 4.2. In [15], the constant k of (4.1) was restricted to $k > 1$. Theorem 4.3 below shows there is no such map if X is finite.

The following shows a limitation on the application of expansive maps in digital topology. The result seems contrary to the spirit of Definition 4.1.

Theorem 4.3. *If T is an expansive map on a finite digital image (X, d, κ), then for all $x, y \in X$, $d(T(x), T(y)) = d(x, y)$.*

Proof. It is shown at Theorem 4.9 of [5] that T cannot satisfy (4.1) for $k > 1$. Thus, we have $k = 1$, so

$$d(T(x), T(y)) \geq d(x, y) \text{ for all } x, y \in X. \tag{4.2}$$

Since X is finite, there exists a maximal finite set $\{d_i\}_{i=1}^m \in (0, \infty)$ such that $0 < d_1 < d_2 < \ldots < d_m$ and sets

$$S_i = \{(u, v) \in X^2 \mid d(u, v) = d_i\} \neq \varnothing.$$

Suppose that there exist $x, y \in X$ such that $d(T(x), T(y)) > d(x, y)$. Then there exists j such that

$$j = \min\{i \in \{1, \ldots, m\} \mid d(T(x_0), T(y_0)) > d(x_0, y_0) \text{ for some } \{x_0, y_0\} \in S_i\}.$$

Thus $\{\{T(x), T(y)\} \mid \{x, y\} \in S_j\} \not\subset S_j$. But T is onto and X is finite, so there exist $x_1, y_1 \in X$ such that $\{x_1, y_1\} \notin S_j$ and $\{T(x_1), T(y_1)\} \in S_j$. By our choice of j, there exists an index $k > j$ such that $\{x_1, y_1\} \in S_k$. Therefore,

$$d(T(x_1), T(y_1)) = d_j < d_k = d(x_1, y_1),$$

a contradiction of (4.2). This establishes the assertion. □

Even being an isomorphism need not make a self-map expansive, as shown in the following.

Example 4.4. Let $X = \{p_0 = (0,0), p_1 = (1,0), p_2 = (1,1)\} \subset \mathbb{Z}^2$. Let $f : (X, c_2) \to (X, c_2)$ be the rotation defined by

$$f(p_i) = p_{(i+1) \mod 3}.$$

Then it is easily seen that f is a (c_2, c_2)-isomorphism. However, if we let d be any ℓ_p metric, then by Theorem 4.3, f is not an expansive mapping, since

$$d(p_1, p_2) = 1 < 2^{1/p} = d(p_2, p_0) = d(f(p_1), f(p_2)).$$

Definition 4.5 ([13])**.** Let (X, d, κ) be a digital metric space and let $T : X \to X$ be a mapping. We say that T is a *generalised α-ψ-expansive mapping* if there exist functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ we have

$$\psi(d(T(x), T(y))) \geq \alpha(x, y) M(x, y) \tag{4.3}$$

where

$$M(x, y) = \max\{d(x, y), \frac{d(x, T(x)) + d(y, T(y))}{2}, \frac{d(x, T(y)) + d(y, T(x))}{2}\}.$$

Remarks 4.6. Any function $T : X \to X$ on a digital metric space is a generalized α-ψ-expansive mapping if α is the constant function with value 0. Therefore, a generalised α-ψ-expansive mapping need not be digitally continuous.

If one desires an example of a discontinuous generalised α-ψ-expansive mapping for which α is not the constant function with value 0, consider the following. This example also shows that the status of a map as an expansive map depends on the metric used, and that an expansive map need not be digitally continuous.

Example 4.7. Let $X = \{p_0, p_1, p_2\} \subset \mathbb{Z}^2$, where $p_0 = (0,0)$, $p_1 = (1,1)$, $p_2 = (2,0)$. Let $T : X \to X$ be the circular rotation $T(p_i) = p_{(i+1) \mod 3}$. Then

- T is an expansive map with respect to the Manhattan metric, but not with respect to the Euclidean metric;
- T is not (c_2, c_2)-continuous;

- T is a generalised α-ψ-expansive mapping for $\psi(t) = t/2$ and $\alpha(x, y) = 1/3$.

Proof. With respect to the Manhattan metric,

$$i \neq j \text{ implies } d(p_i, p_j) = 2. \tag{4.4}$$

Since T is a bijection, we have $d(T(p_i), T(p_j)) = d(p_i, p_j)$, so T is an expansive map.

With respect to the Euclidean metric,

$$d(p_0, p_2) = 2 > \sqrt{2} = d(p_1, p_0) = d(T(p_0), T(p_2)),$$

so T is not an expansive map.

Since $p_1 \leftrightarrow_{c_2} p_2$ but $T(p_1) = p_2$ and $T(p_2) = p_0$ are neither equal nor c_2-adjacent, T is not c_2-continuous.

Using the Manhattan metric, we have from (4.4) that $i \neq j$ implies

$$d(f(x_i), f(x_j)) = 2 = d(x_i, x_j).$$

Therefore, from Definition 4.5 we have $M(x_i, x_i) = 0$ and $i \neq j$ implies $M(x_i, x_j) = 2$. Then one sees easily that T is a generalised α-ψ-expansive mapping for $\psi(t) = t/2$ and $\alpha(x, y) = 1/3$. □

The assertion that appears as Theorem 3.4 of [13] can be corrected and improved as discussed below. The assertion is

> *Let (X, d, κ) be a complete digital metric space and let $T : X \to X$ be a bijective and generalised α-ψ-expansive mapping that satisfies the following conditions:*
> (1) *T^{-1} is α-admissible;*
> (2) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$; and*
> (3) *T is digitally continuous.*
> *Then T has a fixed point.*

Remarks 4.8. There are several errors in the argument given in [13] as a proof of the assertion above, including confusion of digital continuity with "$\varepsilon - \delta$ continuity." The following is a corrected, somewhat modified, version of the assertion above. Note we do not require T to be continuous.

Theorem 4.9. *Let (X, d, κ) be a complete digital metric space and let $T : X \to X$ be a bijective and generalized α-ψ-expansive mapping that satisfies the following conditions:*

(1) *T^{-1} is α-admissible;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$.*

Assume also that either X is finite or d is an ℓ_p metric. Then T has a fixed point.

Proof. Our argument is based on its analog in [13].

We have hypothesized the existence of $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$. Define the sequence $\{x_n\}_{n=0}^{\infty} \in X$ by $x_{n+1} = T^{-1}(x_n)$ for $n > 0$. If $x_{m+1} = x_m$

for some m, then x_m is a fixed point of T^{-1}, hence of T. Otherwise, $x_{n+1} \neq x_n$ for all n.

Since T^{-1} is α-admissible, we have that $\alpha(x_0, x_1) = \alpha(x_0, T^{-1}(x_0)) \geq 1$ implies $\alpha(x_1, x_2) = \alpha(T^{-1}(x_0), T^{-1}(x_1)) \geq 1$, and, by induction,

$$\alpha(x_n, x_{n+1}) = \alpha(T^{-1}(x_{n-1}), T^{-1}(x_n)) \geq 1 \text{ for all } n.$$

From Definition 4.5, we have

$$\begin{aligned} \psi(d(x_{n-1}, x_n)) = \psi(d(T(x_n), T(x_{n+1})) &\geq \alpha(x_n, x_{n+1}) M(x_n, x_{n+1}) \\ &\geq M(x_n, x_{n+1}) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), \frac{d(x_n, T(x_n)) + d(x_{n+1}, T(x_{n+1}))}{2}, \\ &\qquad \frac{d(x_n, T(x_{n+1})) + d(T(x_n), x_{n+1})}{2}\} \\ &= \max\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}, \\ &\qquad \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2}\} \\ &= \max\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\}. \end{aligned}$$

Since by the triangle inequality, we have

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2},$$

it follows that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}\}. \tag{4.6}$$

It follows from Proposition 2.13 and inequalities (4.5) and (4.6) that

$$d(x_{n-1}, x_n) > \psi(d(x_{n-1}, x_n)) \geq$$
$$\max\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n-1}) + d(x_{n+1}, x_n)}{2}\},$$

so

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{4.7}$$

By Theorem 2.5, it follows that $lim_{n \to \infty} d(x_n, x_{n+1}) = 0$. From Corollary 2.6, we conclude that for large n we have $T(x_{n+1}) = x_n = x_{n+1}$. Thus, x_{n+1} is a fixed point of T. □

Theorem 3.5 of [13] is concerned with the existence of a fixed point for a generalised α-ψ-expansive mapping satisfying conditions somewhat like those of Theorem 4.9. The assertion is incorrectly stated (corrections noted below) as follows.

> *Let (X, d, κ) be a complete digital metric space. Suppose $T : X \to X$ is a generalized α-ψ expansive mapping such that*

(1) T^{-1} *is* α*-admissible;*
(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T^{-1}(x_0)) \geq 1$*; and*
(3) *if* $\{x_n\}_{n=0}^{\infty} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $\{x_n\}_{n=0}^{\infty}$ *is digitally convergent to* $x' \in X$*, then*

$$\alpha(T^{-1}(x_n), T^{-1}(x)) \geq 1 \text{ for all } n.$$

Then T *has a fixed point.*

Remarks 4.10. Theorem 3.5 of [13] can be improved as follows.

- Since the existence of the function T^{-1} is assumed, it should be stated that T is assumed to be a bijection.
- The term "digitally convergent" is undefined. What the proof actually uses is metric convergence.
- If we add the hypothesis that X is finite or d is an ℓ_p metric, then the assertion, as amended by these observations, follows from our Theorem 4.9.

Remarks 4.11. Examples 3.6 and 3.7 of [13] claim $[0, \infty)$ as a digital metric space. Clearly, it is not.

Remarks 4.12. In the proof of Theorem 3.8 of [13], the inequality

$$d(u, T^n v) \leq \psi(d(u, v)) \text{ for all } n = 1, 2, 3, \ldots$$

should be

$$d(u, T^n v) \leq \psi^n(d(u, v)) \text{ for all } n = 1, 2, 3, \ldots$$

5. Common fixed points

5.1. **Commuting maps.** The paper [18] is concerned with common fixed points of pairs of commuting self-maps on digital metric spaces.

Theorem 5.1 ([18]). *Let* T *be a continuous mapping of a complete digital metric space* (X, d, κ) *into itself. Then* T *has a fixed point if and only if there exists* $\alpha \in (0, 1)$ *and a function* $S : X \to X$ *that commutes with* T *such that*

$$S(X) \subset T(X) \text{ and } d(S(x), S(y)) \leq \alpha d(T(x), T(y)) \text{ for all } x, y \in X. \tag{5.1}$$

Remarks 5.2. Let S and T be as in Theorem 5.1, where "continuous" is interpreted as (c_u, c_u)-continuous, X is c_u-connected, d is an ℓ_p metric, and $0 < \alpha < \frac{1}{u^{1/p}}$. Then S must be a constant function.

Proof. Let $x \leftrightarrow_{c_u} x'$ in X. Since T is (c_u, c_u)-continuous, either $T(x) = T(x')$ or $T(x) \leftrightarrow_{c_u} T(x')$, so $d(T(x), T(x')) \leq u^{1/p}$. By the inequality (5.1), $d(S(x), S(x')) \leq \alpha d(T(x), T(x')) < 1$. Since d is an ℓ_p metric, $d(S(x), S(x')) = 0$, so $S(x) = S(x')$. It follows from the c_u-connectedness of X that S is constant. □

Below, we show that if we assume common conditions, the requirement that T be continuous in Theorem 5.1 is unnecessary. Our proof is similar to its analog in [18].

Theorem 5.3. *Let T be a mapping of a digital metric space (X, d, κ) into itself, where X is finite or d is an ℓ_p metric. Then T has a fixed point if and only if there exists $\alpha \in (0,1)$ and a function $S : X \to X$ that commutes with T such that*

$$S(X) \subset T(X) \text{ and } d(S(x), S(y)) \leq \alpha d(T(x), T(y)) \text{ for all } x, y \in X. \tag{5.2}$$

Proof. Suppose T has a fixed point, say, $T(a) = a$ for some $a \in X$. Let $S : X \to X$ be the constant function $S(x) = a$. Then clearly $S \circ T = T \circ S$ and the condition (5.2) is satisfied.

Suppose there exist a function $S : X \to X$ and $\alpha \in (0,1)$ such that $S \circ T = T \circ S$ and the condition (5.2) is satisfied. Let $x_0 \in X$. Since $S(X) \subset T(X)$, there exists $x_1 \in X$ such that $T(x_1) = S(x_0)$, and, inductively, for all $n \in \mathbb{N}$ there exists $x_n \in X$ such that $T(x_n) = S(x_{n-1})$. It follows from (5.2) that

$$d(T(x_{n+1}), T(x_n)) = d(S(x_n), S(x_{n-1})) \leq \alpha d(T(x_n), T(x_{n-1})).$$

An easy induction yields that

$$d(T(x_{n+1}), T(x_n)) \leq \alpha^n d(T(x_1), T(x_0)).$$

Since the right side of the latter inequality tends to 0 as $n \to \infty$, it follows from Theorem 2.5 that for sufficiently large n, $T(x_{n+1}) = T(x_n) = t$ for some $t \in X$. But $T(x_{n+1}) = S(x_n)$, so for sufficiently large n, $S(x_n) = t$, and since S and T commute,

$$T(t) = T(T(x_n)) = T(S(x_n)) = S(T(x_n)) = S(t). \tag{5.3}$$

Thus,

$$T(T(t)) = T(S(t)) = S(T(t)),$$

so

$$d(S(t), S(S(t))) \leq \alpha d(T(t), T(S(t))) = \alpha d(S(t), S(S(t))), \text{ or}$$

$0 \leq (\alpha - 1) d(S(t), S(S(t)))$. Since the factor $\alpha - 1 < 0$, we must have $d(S(t), S(S(t))) = 0$, so $S(t) = S(S(t))$. From (5.3) it follows that $S(t) = S(S(t)) = S(T(t)) = T(S(t))$, so $S(t)$ is a common fixed point of S and T.

Suppose x and y are common fixed points of S and T. Then

$$d(x, y) = d(S(x), S(y)) \leq \alpha d(T(x), T(y)) = \alpha d(x, y).$$

Since $0 < \alpha < 1$, we must have $d(x, y) = 0$, so $x = y$. □

Similarly, under common circumstances we can omit the assumption of continuity that is in the version of the following that appears in [18] .

Corollary 5.4. *Let T and S be commuting mappings of a complete digital metric space (X, d, κ) into itself, where d is an ℓ_p metric. Suppose that $S(X) \subset T(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that $d(S^k(x), S^k(y)) \leq \alpha d(T(x), T(y))$ for all $x, y \in X$, then T and S have a common fixed point.*

Proof. Our proof is much like that of its analog in [18]. Since S and T commute, S^k and T commute. Further, $S^k(X) \subset S(X) \subset T(X)$. Therefore, we can apply Theorem 5.3 to the maps S^k and T to conclude that there is a unique $a \in X$ that is a common fixed point of S^k and T, i.e., $S^k(a) = T(a) = a$. Therefore,

$$T(S(a)) = S(T(a)) = S(a) = S(S^k(a)) = S^k(S(a)),$$

so $S(a)$ is a common fixed point of T and S^k. But a is the unique common fixed point of T and S^k, so $a = S(a)$. □

Remarks 5.5. The assertion given as Corollary 3.2.5 of [18] has errors in its statement and in the argument given as its proof. In order for the assertion to be a corollary of the preceding Theorem 3.2.4, the former should be stated as

> Corollary. Let n be a positive integer and let $0 < K < 1$. If S is a self-map of a digital metric space (X, d, κ) such that $d(S^n(x), S^n(y)) \leq K d(x, y)$ for all $x, y \in X$, then S has a unique fixed point.

Remarks 5.6. The assertion at Example 3.3.8 of [18] considers the maps $S, T : (\mathbb{Z}, d, c_1) \to \mathbb{Z}$ given by $S(x) = 2 - x^2$, $T(x) = x^2$ for all $x \in \mathbb{Z}$, where $d(x, y) = |x - y|$. It is claimed that $d(S(x), S(y)) \leq \frac{1}{2} d(T(x), T(y))$ for all $x, y \in \mathbb{Z}$, but this is clearly incorrect, since $d(S(x), S(y)) = d(T(x), T(y))$.

5.2. **Other common fixed point assertions.** The paper [22] is another that is concerned with common fixed points of pairs of self-maps on digital metric spaces. We have the following.

Definition 5.7 ([22])**.** Let (X, d, κ) be a digital metric space. Let $S, T : X \to X$. Then S and T are *weakly compatible* if $S(x) = T(x)$ implies $S(T(x)) = T(S(x))$.

I.e., S and T are weakly compatible if they commute at all coincidence points. Note this definition does not require that coincidence points exist.

Example 5.8. The maps $S, T : [0, 1]_{\mathbb{Z}} \to [0, 1]_{\mathbb{Z}}$ given by $S(x) = 0$, $T(x) = 1$, are weakly compatible, despite having no coincidence points, as they vacuously satisfy the requirement of commuting at all coincidence points.

Theorem 3.6 and Corollaries 3.7 and 3.8 of [22] are concerned with common fixed points of self-maps S and T on a digital metric space (X, d, κ) for which $d(S(x), S(y)) < d(T(x), T(y))$ for all $x, y \in X$ such that $x \neq y$. But such results may be quite limited under common conditions, as in the following Propositions 5.9 and 5.10.

Proposition 5.9. *Let (X, d, κ) be a digital metric space with $|X| > 1$. Let $S, T : X \to X$ be such that $x \neq y$ implies $d(S(x), S(y)) < d(T(x), T(y))$. Then T is one-to-one. If, further, X is finite, then T is a bijection and S is neither one-to-one nor onto.*

Proof. Since $x \neq y$ implies $0 \leq d(S(x), S(y)) < d(T(x), T(y))$, we have that $x \neq y$ implies $T(x) \neq T(y)$. Therefore, T is one-to-one.

Suppose X is finite. Since T is one-to-one, it follows that T is a bijection. Further, there exist $x_0, y_0 \in X$ such that $d(x_0, y_0) = diam X$. If S were onto, there would exist $x', y' \in X$ such that $S(x') = x_0$ and $S(y') = y_0$. By hypothesis, we would then have

$$d(x_0, y_0) = d(S(x'), S(y')) < d(T(x'), T(y')),$$

contrary to our choice of x_0, y_0. Therefore, S is not onto. Since X is finite, it follows that S is not one-to-one. □

Proposition 5.10. *Let (X, d, c_1) be a connected digital metric space, where d is an ℓ_p metric. Let $S, T : \mathbb{Z} \to \mathbb{Z}$ be such that $x \neq y$ implies $d(S(x), S(y)) < d(T(x), T(y))$. If T is c_1-continuous, then S is a constant function.*

Proof. Since T is c_1-continuous, for $x \leftrightarrow_{c_1} x'$, we have $T(x) \leftrightarrow_{c_1} T(x')$ or $T(x) = T(x')$, so, since d is an ℓ_p metric, $d(T(x), T(x')) \in \{0, 1\}$. Since $d(S(x), S(y)) < d(T(x), T(y))$, we must have $S(x) = S(x')$. Since X is c_1-connected, it follows that S is a constant function. □

6. Contractive mappings

6.1. ϕ-contractive, contraction, α-ϕ-ψ-contraction maps.

The paper [21] discusses fixed point assertions for contractive-type mappings on digital metric spaces.

Definition 6.1 ([21])**.** Suppose (X, d, κ) is a digital metric space, $T : X \to X$, and $\phi \in \Phi$. If

$$d(T(x), T(y)) \leq \phi(d(x, y)) \text{ for all } x, y \in X,$$

then T is called a *digital ϕ-contraction*.

Remarks 6.2. The function of Example 2.15 is a digital ϕ-contraction for $\phi(t) = t/2$. This shows that a digital ϕ-contraction need not be digitally continuous.

A limitation of such functions is given in the following.

Proposition 6.3. *Let (X, d, κ) be a digital metric space and let $T : X \to X$ be a digital ϕ-contraction for some $\phi \in \Phi$. Suppose d is an ℓ_p metric and $\phi(t) < 1$ for all $t \in \mathbb{R}$. Then T is a constant function.*

Proof. Let $x, x' \in X$. Then

$$d(T(x), T(x')) \leq \phi(d(x, x')) < 1.$$

Therefore $T(x) = T(x')$. It follows that T is a constant function. □

Definition 6.4. Let (X, d, κ) be a digital metric space, $T : X \to X$, and $\phi \in \Phi$. We say that

- T is *ϕ-contractive* [21] if $\phi(d(T(x), T(y))) < \phi(d(x, y))$ for all $x, y \in X$, $x \neq y$.
- T is a *digital contraction map* [8] if for some $\alpha \in (0, 1)$, $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $t \in \mathbb{R}$.

The similarity of these definitions yields the following.

Proposition 6.5. *Let (X, d, κ) be a digital metric space, $|X| > 1$, and let $T : X \to X$. If T is a non-constant digital contraction map then for some $\phi \in \Phi$, T is ϕ-contractive. The converse is true when X is finite.*

Proof. Let T be a digital contraction map. Then for some $\alpha \in (0,1)$, $d(T(x), T(y)) < \alpha d(x, y)$ for all $x, y \in X$. Therefore, $\alpha d(T(x), T(y)) < \alpha^2 d(x, y)$. Note the function $\phi(t) = \alpha t$ is a member of Φ. Then $x \neq y$ implies

$$\phi(d(T(x), T(y))) = \alpha d(T(x), T(y)) \leq \alpha^2 d(x, y) < \alpha d(x, y) = \phi(d(x, y)),$$

so T is ϕ-contractive.

Suppose X is finite and T is ϕ-contractive for some $\phi \in \Phi$. Then $x \neq y$ implies

$$\phi(d(T(x), T(y))) < \phi(d(x, y)).$$

Since ϕ is monotone increasing, $x \neq y$ implies

$$d(T(x), T(y)) < d(x, y).$$

Since X is finite and T is non-constant, we can have $\alpha \in (0, 1)$ well defined by

$$\alpha = \max\left\{ \frac{d(T(x), T(y))}{d(x, y)} \,|\, x, y \in X, x \neq y \right\}.$$

Since X is finite, $x \neq y$ implies

$$\frac{d(T(x), T(y))}{d(x, y)} \leq \alpha, \text{ or } d(T(x), T(y)) \leq \alpha d(x, y).$$

Since the latter inequality also holds when $x = y$, it follows that T is a digital contraction map. □

Limitations on digital contraction maps are discussed in [5]. Fixed point results for such maps appear in [8, 21].

Definition 6.6 ([21])**.** Let (X, d, κ) be a digital metric space. The function $T : X \to X$ is an *α-ψ-ϕ-contractive type mapping* if there exist three functions $\alpha : X \times X \to [0, \infty)$ and $\psi, \phi \in \Phi$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \text{ for all } x, y \in X.$$

Remarks 6.7. One sees easily that every map $T : X \to X$ is an α-ψ-ϕ-contractive type mapping, for $\alpha(x, y) = 0$ and $\psi(t) \geq \phi(t)$. Hence such a function T need not be digitally continuous.

The assertion stated as Theorem 3.12 of [21] is (after correction) as follows.

> *Let (X, d, κ) be a digital metric space, $T : X \to X$, and $\alpha : X^2 \to [0, \infty)$. Suppose*
> (1) *T is α-admissible;*
> (2) *there exists $x_0 \in X$ such that $\alpha(x_0, T(x_0)) \geq 1$;*
> (3) *T is digitally continuous; and*

(4) $\alpha(x,y)\psi(d(T(x),T(y))) \leq \psi(M(x,y)) - \phi(M(x,y))$, *where*

$$M(x,y) =$$

$$\max\{d(x,y),\, d(x,T(x)),\, d(y,T(y)),\, [d(x,T(y)) + d(y,T(x))]/2\}$$

for all $x, y \in X$.

Then T *has a fixed point. Further, if* u *and* v *are fixed points of* T *such that* $\alpha(u,v) \geq 1$, *then* $u = v$.

This assertion is incorrect without additional hypotheses. For example, if we take $X = [0,1]_{\mathbb{Z}}$, $T(x) = 1 - x$, $\alpha(x,y) = 1$, $\psi(x) = x$, $\phi(x) = -1$, then all the hypotheses above are satisfied, but T has no fixed points.

If X is finite or d is an ℓ_p metric, then the argument given as a proof for this assertion in [21] does not require the continuity assumption, but does require that ψ and ϕ have properties of Φ. Thus, a correct, somewhat modified version is as follows.

Theorem 6.8. *Let* (X, d, κ) *be a digital metric space where* X *is finite or* d *is an* ℓ_p *metric. Let* $T : X \to X$, *and* $\alpha : X^2 \to [0, \infty)$. *Suppose*

(1) T *is* α*-admissible;*
(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, T(x_0)) \geq 1$*; and*
(3) *There exist* $\psi, \phi \in \Phi$ *such that*

$$\alpha(x,y)\psi(d(T(x),T(y))) \leq \psi(M(x,y)) - \phi(M(x,y)),$$

where for all $x, y \in X$, $M(x,y) =$

$$\max\{d(x,y),\, d(x,T(x)),\, d(y,T(y)),\, [d(x,T(y)) + d(y,T(x))]/2\}.$$

Then T *has a fixed point. Further, if* u *and* v *are fixed points of* T *such that* $\alpha(u,v) \geq 1$, *then* $u = v$.

Proof. Our argument is similar to its analog in [21].

Let $x_0 \in X$ be such that $\alpha(x_0, T(x_0)) \geq 1$. Let x_n be defined inductively by $x_{n+1} = T(x_n)$ for $n \geq 0$. Therefore, $\alpha(x_0, x_1) = \alpha(x_0, T(x_0)) \geq 1$, so since T is α-admissible, we have

$$\alpha(x_1, x_2) = \alpha(T(x_0), T(x_1)) \geq 1,$$

and, inductively,

$$\alpha(x_n, x_{n+1}) = \alpha(T(x_{n-1}), T(x_n)) \geq 1 \text{ for all } n.$$

Then

$$\psi(d(x_n, x_{n+1})) = \psi(d(T(x_{n-1}), T(x_n))) \leq \alpha(x_{n-1}, x_n)\psi(d(T(x_{n-1}), T(x_n))$$
$$\leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)) < \psi(M(x_{n-1}, x_n)).$$

Since ψ is increasing,

$$d(x_n, x_{n+1}) < M(x_{n-1}, x_n) =$$
$$\max\{d(x_{n-1}, x_n), d(x_{n-1}, T(x_{n-1})), d(x_n, T(x_n)),$$
$$[d(x_{n-1}, T(x_n)) + d(x_n, T(x_{n-1}))]/2\} =$$
$$\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]/2\} =$$

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), [d(x_{n-1}, x_{n+1}) + 0]/2\} =$$

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})/2\} \tag{6.1}$$

By the triangle inequality,

$$d(x_{n-1}, x_{n+1})/2 \leq [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]/2$$

$$\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \tag{6.2}$$

Combining (6.1) and (6.2), $d(x_n, x_{n+1}) < \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$, so

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Since X is finite or d is an ℓ_p metric, it follows that for sufficiently large n, $d(x_n, x_{n+1}) = 0$, or $x_n = x_{n+1} = T(x_n)$, so x_n is a fixed point of T.

Suppose u and v are fixed points of T with $\alpha(u, v) \geq 1$. We have

$$\psi(d(u, v)) = \psi(d(T(u), T(v)) \leq \alpha(u, v)\psi(d(T(u), T(v))) \leq$$

$$\psi(M(u, v)) - \phi(M(u, v)) = \psi(d(u, v)) - \phi(d(u, v)),$$

or $0 \leq -\phi(d(u, v))$, which implies $d(u, v) = 0$, or $u = v$. □

6.2. **Weakly uniformly strict contractions.** In [5], we discussed the paper [7], including mention of the fact that the author of the current work was identified as a reviewer of [7] and that the latter work was published without correction of several flaws mentioned in the review. Here, we present additional discussion of [7].

Definition 6.9 ([7])**.** Let (X, d, κ) be a digital metric space. A self map $T : X \to X$ is a *weakly uniformly strict digital contraction* if given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(T(x), T(y)) < \varepsilon$ for all $x, y \in X$.

Lemma 6.10. *Let (X, d, κ) be a digital metric space. Let $T : X \to X$ be a weakly uniformly strict digital contraction. Then for all $x, y \in X$ such that $x \neq y$, $d(T(x), T(y)) < d(x, y)$.*

Proof. This is shown in the first paragraph of the argument given as a proof of Theorem 3.3 in [7]. □

Corollary 6.11. *Let (X, d, κ) be a digital metric space, such that X is finite. Let $T : X \to X$ be a weakly uniformly strict digital contraction. Then T is a digital contraction map.*

Proof. Without loss of generality, T is not constant. Since X is finite,

$$\alpha = \max\left\{\frac{d(T(x), T(y))}{d(x, y)} \mid x, y \in X, x \neq y\right\}$$

is well defined. Since T is not constant, $\alpha > 0$; and, since X is finite, by Lemma 6.10, $\alpha < 1$. Then for $x, y \in X$ such that $x \neq y$,

$$d(T(x), T(y)) = \frac{d(T(x), T(y))}{d(x, y)} d(x, y) \leq \alpha d(x, y).$$

By Definition 6.1, T is a digital contraction map. □

Proposition 6.12. *Let (X, d, κ) be a digital metric space such that $1 < |X| < \infty$. Let $T : X \to X$ be a weakly uniformly strict digital contraction. Then T is neither one-to-one nor onto.*

Proof. Since $1 < |X| < \infty$, $m = \min\{d(x, y) \,|\, x, y \in X, x \neq y\}$ and $M = \max\{d(x, y) \,|\, x, y \in X, x \neq y\}$ are well defined positive numbers, and there exist $x_0, y_0 \in X$ such that $x_0 \neq y_0$ and $d(x_0, y_0) = m$, and $x_1, y_1 \in X$ such that $x_1 \neq y_1$ and $d(x_1, y_1) = M$.

By Lemma 6.10, $d(T(x_0), T(y_0)) < d(x_0, y_0) = m$. By definition of m, this implies $T(x_0) = T(y_0)$, so T is not one-to-one.

If T is onto, then there exist $u, v \in X$ such that $T(u) = x_1$ and $T(v) = y_1$. Thus, by Lemma 6.10 we conclude that

$$M = d(x_1, y_1) = d(T(u), T(v)) < d(u, v),$$

which contradicts our choice of M. Therefore, T is not onto. □

A limitation on weakly uniformly strict digital contractions is shown in the following.

Proposition 6.13. *Let (X, d, c_u) be a digital metric space, where d is an ℓ_p metric. Let $T : X \to X$ be a self map such that for some $\alpha > 0$, $1 \leq d(x, y) < u^{1/p} + \alpha$ implies $d(T(x), T(y)) < 1$. If X is c_u-connected, then T is constant.*

Proof. Let $x \leftrightarrow_{c_u} y$ in X. Then for every $\alpha > 0$, $1 \leq d(x, y) < u^{1/p} + \alpha$, so $d(T(x), T(y)) < 1$. Therefore, $T(x) = T(y)$. Since X is c_u-connected, it follows that T is constant. □

As an immediate consequence, we have the following.

Corollary 6.14. *Let (X, d, c_1) be a digital metric space, where d is an ℓ_p metric. Let $T : X \to X$ be a weakly uniformly strict digital contraction. If X is c_1-connected, then T is constant.*

Proof. Since T is a weakly uniformly strict digital contraction, for some $\alpha > 0$, $1 \leq d(x, y) < 1 + \alpha$ implies $d(T(x), T(y)) < 1$. Since d is an ℓ_p metric, c_1-adjacent $x, y \in X$ satisfy $d(x, y) = 1$, hence $d(T(x), T(y)) < 1$. Since X is c_1-connected, it follows from Theorem 6.13 that T is constant. □

The arguments given as proof for Theorems 3.1 and 3.3 in [7] are marred by confusion of digital and metric continuity. Below, we make minor modifications of the stated assumptions - in both theorems, we replace the assumption of a complete metric space with the assumption of an ℓ_p metric - and give corrected proofs that are much shorter than the arguments given in [7], in part because discussion of continuity is unnecessary.

The following is Theorem 3.1 of [7], modified as discussed above.

Theorem 6.15. *Let (X, d, κ) be a digital metric space, where d is an ℓ_p metric, and suppose that $T : X \to X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, where $\psi \in \Phi$. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$. Inductively, let $x_{n+1} = T(x_n)$ for $n \geq 0$. Then

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq \psi(d(x_{n-1}, x_n))$$

and a simple induction argument leads to the conclusion that for $n \geq 0$,

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \to_{n\to\infty} 0.$$

By Corollary 2.6, for sufficiently large n we have $x_n = x_{n+1} = T(x_n)$, so x_n is a fixed point.

Suppose x and x' are fixed points of T. Then

$$d(x, x') = d(T(x), T(x')) \leq \psi(d(x, x')),$$

since $\psi \in \Phi$, where equality occurs only for $d(x, x') = 0$, i.e., $x = x'$. Thus T has a unique fixed point. □

Remarks 6.16. If, in Theorem 6.15, d is an ℓ_p metric and X is c_1-connected, then T is a constant function.

Proof. Given $x \leftrightarrow_{c_1} x'$ in X, we have, since d is an ℓ_p metric, $d(x, x') = 1$, and

$$d(T(x), T(x')) \leq \psi(d(x, x')) = \psi(1) < 1,$$

so $T(x) = T(x')$. Since X is c_1-connected, it follows that T is constant. □

The following is Theorem 3.3 of [7], modified as discussed above.

Theorem 6.17. *Let (X, d, κ) be a complete digital metric space, where d is an ℓ_p metric, and let $T : X \to X$ be a weakly uniformly strict digital contraction mapping. Then T has a unique fixed point z. Moreover, for any $x \in X$, $lim_{n\to\infty} T^n(x) = z$.*

Proof. By Lemma 6.10,

$$\text{for all } x, y \in X,\ d(T(x), T(y)) < d(x, y).$$

Let $x_0 \in X$ and, for $n \geq 0$, inductively define $x_{n+1} = T(x_n)$. We may assume $x_1 \neq x_0$, since, otherwise, x_0 is a fixed point of T. We have, for $n > 0$, $d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) < d(x_{n-1}, x_n)$. Since d is an ℓ_p metric, for sufficiently large n, $x_n = x_{n+1} = T(x_n)$; thus, x_n is a fixed point of T.

Suppose x and y are fixed points of T. If $x \neq y$ then, by Lemma 6.10,

$$d(x, y) = d(T(x), T(y)) < d(x, y),$$

which is impossible. Therefore $x = y$. □

7. Concluding remarks

We have corrected or noted limitations of published assertions concerning fixed points for self-maps on digital metric spaces. This continues the work of [5]. In many cases, flaws we have noted were so obvious that reviewers and/or editors of these papers share responsibility with the authors. We have also offered improvements to some of the assertions discussed.

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