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A class of ideals in intermediate rings of continuous functions

Sagarmoy Bag, Sudip Kumar Acharyya and Dhananjoy Mandal

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata 700019, West Bengal, India (sagarmoy.bag01@gmail.com, sdpacharyya@gmail.com, dmandal.cu@gmail.com)

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Abstract

For any completely regular Hausdorff topological space X, an intermediate ring $A(X)$ of continuous functions stands for any ring lying between $C^*(X)$ and $C(X)$. It is a rather recently established fact that if $A(X) \neq C(X)$, then there exist non maximal prime ideals in $A(X)$. We offer an alternative proof of it on using the notion of z°-ideals. It is realized that a P-space X is discrete if and only if $C(X)$ is identical to the ring of real valued measurable functions defined on the σ-algebra $\beta(X)$ of all Borel sets in X. Interrelation between z-ideals, z°-ideal and $\mathfrak{z}_A$-ideals in $A(X)$ are examined. It is proved that within the family of almost P-spaces X, each $\mathfrak{z}_A$-ideal in $A(X)$ is a z°-ideal if and only if each z-ideal in $A(X)$ is a z°-ideal if and only if $A(X) = C(X)$.

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1. Introduction

Let $C(X)$ be the ring of all real valued continuous functions on a completely regular Hausdorff topological space X. $C^*(X)$ is the subring of $C(X)$ consisting of those functions which are bounded over X. A ring $A(X)$ lying between $C^*(X)$ and $C(X)$ is called an intermediate ring. These intermediate rings have an important common property which says that the structure space of all these rings are one and the same and is the Stone-Čech compactification βX of X

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(see [7]). The structure space of a commutative ring R with unity stands for the set of all maximal ideals of R equipped with hull kernel topology. In the present paper our purpose is to point out a few dissimilarities existing between the ambient ring $C(X)$ and its proper intermediate subrings. To achieve that we have chosen three special classes of ideals z-ideals, z°-ideals and $\mathfrak{Z}_A$-ideals in a typical intermediate ring $A(X)$. An ideal I unmodified in a commutative ring R with unity will also designate a proper ideal of R.

For each a in R, let $M_a(P_a)$ be the intersection of all maximal ideals (minimal prime ideals) of R containing a. An ideal I in R is called a z-ideal (respectively a z°-ideal) if for each a in I, $M_a \subseteq I$(respectively $P_a \subseteq I$). It is well known that if R is a reduced ring meaning that $'0'$ is the only nilpotent element of R, then each z°-ideal of R is a z-ideal also (see [4]). In particular therefore each z°-ideal in an intermediate ring $A(X)$ is a z-ideal. The notion z-ideals and z°-ideals in commutative rings are quite well known and are being investigated since around 1970's. However the concept of $\mathfrak{Z}_A$-ideals in an intermediate ring is rather recent and is initiated in [15]. Given $E \subseteq X$ an $f \in A(X)$ is called E-regular if there exist $g \in A(X)$ such that $f(x)g(x) = 1$ for each $x \in E$. For any non-invertible $f \in A(X), \mathcal{Z}_A(f) = \{E \in Z[X] : f \text{ is } X \setminus E\text{-regular}\}$ and $\mathfrak{Z}_A(f) = \{E \in Z[X] : f \text{ is } H\text{-regular for each zero set } H \subseteq X \setminus E\}$ are z-filters on X, here $Z[X]$ stands for the family of all zero sets in X. For any ideal I in $A(X)$, $\mathcal{Z}_A[I] = \cup_{f\in I}\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A[I] = \cup_{f\in I}\mathfrak{Z}_A(f)$ are z-filters on X. For any z-filter $\mathfrak{F}$ on X, $\mathcal{Z}_A^{-1}[\mathfrak{F}] = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathfrak{F}\}$ and $\mathfrak{Z}_A^{-1}[\mathfrak{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathfrak{F}\}$ are easily seen to be ideals in $A(X)$. It is easily verified that for any ideal I in $A(X)$, $\mathfrak{Z}_A^{-1}\mathfrak{Z}_A[I] \supseteq I$. I is called $\mathfrak{Z}_A$-ideal if $\mathfrak{Z}_A^{-1}\mathfrak{Z}_A[I] = I$. It is plain that $\mathfrak{Z}_C$-ideals and z-ideals in $C(X)$ are same. For an arbitrary $A(X)$, we check that every $\mathfrak{Z}_A$-ideal is a z-ideal (Theorem 3.4). We establish a partial converse of this theorem that, within the class of P spaces X if every z-ideal is $\mathfrak{Z}_A$-ideal in $A(X)$ then $A(X) = C(X)$ (Theorem 3.5). A $\mathfrak{Z}_A$-ideal in $A(X)$ need not be a z°-ideal. Indeed a $\mathfrak{Z}_C$-ideal in $C(X)$ is not necessarily a z°-ideal, a fact which is not hard to realize on choosing $X = \mathbb{R}$. We prove that such things do not happen if and only if X is an almost P-space (Theorem 3.6). This further yields that if X is almost P and $A(X) \neq C(X)$, then there does exist a $\mathfrak{Z}_A$ ideal in $A(X)$ which is not a z°-ideal (Theorem 3.7). Relations between maximal ideals and z°-ideals in $A(X)$ have also been investigated by the present authors, indeed an improved version of such interrelations have already been pondered upon in a recently communicated paper [5] However to make the present article self contained and also for the benefit of the readers we reproduce a few of these relevant facts from this paper in the technical section §2.

It is a rather recently established fact that if an intermediate ring $A(X)$ is different from $C(X)$, then there exist non maximal prime ideals in $A(X)$ [1], [11]. We give an alternative proof of it in the concluding section §4 of this article. In this last section we find out two conditions each necessary and sufficient for a P-space X to be discrete (Theorem 4.3). On using certain facts about z°-ideals we prove a second special result which tells that certain important subspaces of an UMP-space are also UMP-spaces. A space X is called an UMP-space if every maximal ideal in $C(X)$ is a union of minimal prime ideals contained in it (see [6] for results about these spaces).

2. z°-ideals in intermediate rings versus almost Pspaces

We start with the following characterization of minimal prime ideals in a commutative ring R with unity ([12]), which is also recorded in ([10], Lemma 1.1).

Theorem 2.1. *A prime ideal P in R is a minimal prime ideal if and only if given $a \in P$, there is a $b \in R \setminus P$ such that $a.b$ is a nilpotent element of R, in particular $ab = 0$, if R is assumed to be a reduced ring.*

It follows from this theorem that each non zero element of a minimal prime ideal in a reduced ring R is a divisor of zero in R. Therefore non zero elements of a z°-ideal in such a ring R are all divisors of zero. We shall use this singularly important fact in the proof of several theorems that follow. Incidentally it is not hard to prove on using Theorem 3.1 that if $Ann(a)$ is the annihilator of an element a in a reduced ring R, then $P_a = \{b \in R : Ann(a) \subseteq Ann(b)\}$. This formula together with the fact that each bounded function in $C(X)$ belongs to each intermediate ring $A(X)$ yields the following theorem:

Theorem 2.2. *For $f \in A(X), P_f = \{g \in A(X) : int_X Z(f) \subseteq int_X Z(g)\}$, here $Z(f)$ stands for the zero set of f in X.*

As recorded in [9], P-spaces are fairly rare. Interesting examples of non discrete P-spaces are rather pathological (see Examples 7.3,7.4,7.5,7.6 in [9]). A larger family of spaces, the so-called almost P-spaces X viz those for which non empty zero sets in X have non empty interior (equivalently non empty G_δ sets in X have non empty interior) have been introduced in [13]. It turns out that almost P-spaces are far more abundant than P-spaces. Those spaces have already been characterized viz z°-ideals and z-ideals in $C(X)$ in [3]. We have offered a some what improved characterization of those spaces via z°-ideals in intermediate rings. The following proposition attests to this fact:

Theorem 2.3. *Suppose $A(X)$ is an intermediate ring. Then X is an almost P-space if and only if each fixed maximal ideal $M_A^p = \{f \in A(X) : f(p) = 0\}, p \in X$ of $A(X)$ is a z°-ideal.*

Proof. Let X be almost P-space and $p \in X$. Choose $f \in M_A^p$ and $g \in P_f \equiv$ the intersection of all minimal prime ideals of $A(X)$ which contain f. Then from Theorem 3.2, $int_X Z(f) \subseteq int_X Z(g)$. Therefore the hypothesis that X is almost P implies that $Z(f) = cl_X int_X Z(f) \subseteq cl_X int_X Z(g) = Z(g)$ and therefore $g \in M_A^p$. Thus $P_f \subseteq M_A^p$ and hence M_A^p is a z°-ideal of $A(X)$. To prove the other containment le X be not almost P. So there exists $f \in C^*(X)$ such that $Z(f) \neq \phi$ but $int_X Z(f) = \phi$. Hence $f \in M_A^p$ yet f is not a divisor of zero in $A(X)$. As members of z°-ideals are necessarily divisors of zero in the ambient ring, if follows that M_A^p is not a z°-ideal in $A(X)$. □

We would like to mention in this context that in the paper [3] a space X was realized as almost P when and only when each maximal ideal of $C(X)$ is a z°ideal. We show in the next result that, the last characterization can not be improved to say that X is almost P if and only if each maximal ideal of an intermediate ring $A(X)$ is a z°-ideal.

Theorem 2.4. *Let X be almost P. Then for an intermediate ring $A(X)$, each maximal ideal is a z°-ideal in $A(X)$ if and only if each z-ideal in $A(X)$ is a z°-ideal if and only if $A(X) = C(X)$.*

Proof. If $A(X) = C(X)$, then it follows from [3], that each z ideal of $A(X)$, in particular each maximal ideal of $A(X)$ is a z°-ideal. To prove the converse let $A(X) \neq C(X)$. Then there exists $f \in C(X)$ such that $f \notin A(X)$. Since $A(X)$ is an absolutely convex subring of $C(X)$ (see [7]), it follows that $|f| \notin A(X)$. It follows that $g = \frac{1}{1+|f|}$ is an element of $A(X)$ which is not invertible in this ring. Accordingly there exists a maximal ideal M of $A(X)$ (which is incidentally a z ideal of $A(X)$ also) such that $g \in M$. As $Z(g) = \phi, g$ is not a divisor of zero, hence M can not be a z°-ideal of $A(X)$. □

Remark 2.5. Since the converse part of Theorem 2.4 does not use the almost P hypothesis on X, we can say that for any space X (not necessarily almost P), if $A(X)$ is an intermediate subring of $C(X)$ properly contained in $C(X)$, then there exists a free maximal ideal of $A(X)$, which is not a z°-ideal.

3. $\mathfrak{Z}_A$-ideals in $A(X)$ versus P-spaces/ almost P-spaces X

For any z-filter $\mathfrak{F}$ on X, the hull $h\mathfrak{F}$ of $\mathfrak{F}$ is the set of all z-ultrafilters containing $\mathfrak{F}$ and for every set $\mathcal{U}$ of z-ultrafilters on X, the kernel $k\mathcal{U}$ of $\mathcal{U}$ is the intersection of all z-ultrafilters belonging to $\mathcal{U}$. We reproduce the following theorem established in [16].

Theorem 3.1. *For any f in $A(X), \mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$.*

This further yields the following result:

Theorem 3.2. *For $f, g \in A(X), h\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(g)$ if and only if $\mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f)$.*

Proof. Let $h\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(g)$. then $kh\mathcal{Z}_A(g) \subseteq kh\mathcal{Z}_A(f)$ and hence from Theorem 3.1, $\mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f)$. To prove the other containment let $\mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f)$. Choose p from the set $h\mathcal{Z}_A(f)$, then $\mathcal{Z}_A(f) \subseteq \mathcal{U}^p$ (here we are identifying the point p in βX with the z-ultrafilter $\mathcal{U}^p$ on X associated with p). This means that $\mathcal{U}^p \in h\mathcal{Z}_A(f)$ and therefore $\mathcal{Z}_A(g) \subseteq kh\mathcal{Z}_A(g) = \mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f) \subseteq \mathcal{U}^p$.

Consequently, $\mathcal{U}^p \in h\mathcal{Z}_A(g)$. Thus $h\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(g)$. □

The next proposition furnishes us with a convenient formula for the intersection of all maximal ideals containing a function f in $A(X)$.

Theorem 3.3. *Let $f \in A(X)$. Then the intersection of all maximal ideals of $A(X)$ containing f is given by: $M_f = \{g \in A(X) : h\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(g)\} = \{g \in A(X) : \mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f)\}$.*

Proof. It follows from Theorem 3.3 of [7] that the maximal ideal in $A(X)$ corresponding to a point $p \in \beta X$ is given by: $M_A^p = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{U}^p\} \equiv \mathcal{Z}_A^{-1}[\mathcal{U}^p] = \{f \in A(X) : p \in h\mathcal{Z}_A(f)\}$. The desired result therefore follows in view of Theorem 3.2. □

The following result comes out as a consequence of the last one:

Theorem 3.4. *Every $\mathfrak{Z}_A$ ideal in $A(X)$ is a z-ideal.*

Proof. Let I be a $\mathfrak{Z}_A$-ideal in $A(X)$ and $f \in I$. Let $g \in M_f$. Then from Theorem 3.3, we have $\mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A(f)$. As $f \in I$, it is plain that $\mathfrak{Z}_A(f) \subseteq \mathfrak{Z}_A[I]$ and hence $\mathfrak{Z}_A(g) \subseteq \mathfrak{Z}_A[I]$. Since I is a $\mathfrak{Z}_A$-ideal in $A(X)$, it follows that $g \in I$. thus $M_f \subseteq I$ and hence I is a z-ideal in $A(X)$. □

It is recently established in [11], Theorem 3.7 that if X is a P-space and each ideal in $A(X)$ is a $\mathfrak{Z}_A$ ideal then $A(X) = C(X)$.

The following theorem is a somewhat improved version of this fact.

Theorem 3.5. *Let X be a P-space. Then $A(X) = C(X)$ if and only if every z-ideal in $A(X)$ is a $\mathfrak{Z}_A$-ideal.*

Proof. If $A(X) = C(X)$, then z-ideals and $\mathfrak{Z}_C$-ideals in $C(X)$ are the same. To prove the converse let $A(X) \subsetneq C(X)$. Then by Theorem 3.10 in [11], it follows that there is a point $p \in \beta X$ for which $O_A^p = \{f \in A(X) : p \in int_{\beta X} h\mathcal{Z}_A(f)\} \subsetneq M_A^p = \{f \in A(X) : p \in h\mathcal{Z}_A(f)\}$. It is not hard to verify that O_A^p is a z-ideal in $A(X)$ indeed let $f \in O_A^p$ and $g \in M_f$. Then from Theorem 3.3, we have $h\mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(g)$. Hence $p \in int_{\beta X} h\mathcal{Z}_A(f) \subseteq int_{\beta X} h\mathcal{Z}_A(g)$, this implies that $g \in O_A^p$. We assert that O_A^p is not a $\mathfrak{Z}_A$ ideal in $A(X)$. We argue by contradiction and let O_A^p be a $\mathfrak{Z}_A$-ideal. Since X is a P-space it follows from Corollary 2.4 of [11] that for any ideal I in $A(X)$, we have $\mathfrak{Z}_A[I] = \mathcal{Z}_A[I]$. Again from [7], Theorem 4.1 it follows that $\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$. Thus we can write $\mathfrak{Z}_A[O_A^p] = \mathfrak{Z}_A[M_A^p]$, which implies in view of the assumption that O_A^p is a $\mathfrak{Z}_A$ ideal in $A(X)$ that $M_A^p = O_A^p$, a contradiction. □

A $\mathfrak{Z}_A$-ideal in $A(X)$ need not be a z°-ideal, indeed a maximal ideal in $C(X)$ and therefore a $\mathfrak{Z}_C$-ideal in $C(X)$ is not necessarily a z°-ideal. An easy example is produced by $M_0 = \{f \in C(\mathbb{R}) : f(0) = 0\}$, we only note that the function $i \in C(\mathbb{R})$ defined by $i(r) = r, r \in \mathbb{R}$ is a member of M_0 without being a divisor of zero in the ring $C(\mathbb{R})$. The following theorem settles the exact class of spaces X for which $\mathfrak{Z}_C$-ideals in $C(X)$ are z°-ideals.

Theorem 3.6. *X is an almost P-space if and only if every $\mathfrak{Z}_C$-ideal in $C(X)$ is a z°-ideal.*

Proof. Let X be almost P and I a $\mathfrak{Z}_C$-ideal of $C(X)$. Then from Theorem 3.4, I is a z ideal of $C(X)$. The hypothesis, X is almost P implies in view of Theorem 2.14 in [3] that I is a z°-ideal of $C(X)$.

To prove the converse let every $\mathfrak{z}_C$-ideal in $C(X)$ be a z°-ideal. Since maximal ideals in $C(X)$ are always $\mathfrak{z}_C$-ideals (see [15]), it follows from the Theorem 2.3 that, X is almost P-space. □

The next proposition shows that, the last result characterizes $C(X)$ amongst all the intermediate rings within the class of almost P-spaces.

Theorem 3.7. *Let X be an almost P-space. then $A(X) = C(X)$ if and only if each $\mathfrak{z}_A$ ideal in $A(X)$ is a z°-ideal.*

Proof. Let $A(X) = C(X)$. then from Theorem 3.6, it follows that every $\mathfrak{z}_A$-ideal in $A(X)$ is a z°-ideal. To prove the converse let $A(X) \subsetneq C(X)$. then as in the proof of the converse part of Theorem 2.4, we can ensure the existence of a maximal ideal M of $A(X)$, which is not a z°-ideal. surely M is a $\mathfrak{z}_A$-ideal in $A(X)$. □

4. Two special results

It has been established recently by the authors in [1] and [11], independently that if $A(X)$ is an intermediate ring, properly contained in $C(X)$, then $A(X)$ is never regular in the sense of Von-Neumann, which means that there exist non maximal prime ideals in $A(X)$. We offer yet another proof of the above mentioned fact by using the notion of z°-ideals. We will need the following general result for commutative rings.

Theorem 4.1. *Let R be a commutative reduced regular ring with unity. Then each ideal in R is a z°-ideal.*

Proof. Let I be an ideal in R. Let $a \in I$ and $b \in P_a$, then $Ann(a) \subseteq Ann(b)$. Since R is regular there exists $x \in R$ such that $a = a^2x$. Therefore $a(1-ax) = 0$ and hence $1 - ax \in Ann(a) \subseteq Ann(b)$. This implies that $(1 - ax)b = 0$, hence $b = abx$ and therefore $b \in I$ as $a \in I$. Thus $P_a \subseteq I$. Hence I is a z°-ideal in R. □

Theorem 4.2. *Let $A(X) \neq C(X)$. then $A(X)$ is not Von-Neumann regular, equivalently if $A(X)$ is a regular ring, then $A(X) = C(X)$.*

Proof. Assume that $A(X)$ is a regular ring and choose $f \in C(X)$. To show that f lies in $A(X)$ it is sufficient to show in view of the absolute convexity of $A(X)$ that $g = \frac{1}{1+|f|}$ is a multiplicative unit of the ring $A(X)$. If possible let g be not a unit in $A(X)$. Then there exists a maximal ideal M in $A(X)$ such that $g \in M$. Surely g is not a divisor of zero in $A(X)$ and therefore M cannot be a z°-ideal in $A(X)$. On the other hand it follows from Theorem 4.1 that each ideal of $A(X)$ is a z°-ideal, a contradiction. □

Before using the above theorem, we let $\mathcal{B}(X)$ be the set of all Borel sets in the space X. Thus $\mathcal{B}(X)$ is the smallest σ-algebra on X containing all the open sets in X. We call a function $f : X \mapsto \mathbb{R}, \mathcal{B}$ measurable if for any open set V in $\mathbb{R}, f^{-1}(V)$ is a member of $\mathcal{B}(X)$. It is quite well known that the family $B(X)$ of all $\mathcal{B}$ measurable functions on X constitutes a commutative lattice ordered ring with unity if the relevant operations are defined point wise on X and of course $C(X) \subseteq B(X)$ (see [2]).

The following theorem gives three conditions involving $C(X)$ and $B(X)$ for a P-space X to become a discrete one.

Theorem 4.3. *Let X be a P-space. then the following three statements are equivalent:*

(1) *X is discrete.*
(2) *$Z[X] = \mathcal{B}(X)$, we recall that $Z[X]$ is the family of all zero sets in x.*
(3) *$C(X) = B(X)$.*

Proof. It is trivial that the truth of the statement (1) implies the truth of each of the statements $(2), (3)$. $(2) \Rightarrow (1)$: Let X be not discrete, then there exists $x \in X$ such that $\{x\}$ is not open in X. But $\{x\}$ is closed in X implies that $\{x\} \in \mathcal{B}(X)$. On the other hand the fact that each zero set in a P-space X is open implies that $\{x\} \notin Z(X)$. Thus $Z[X] \neq \mathcal{B}(X)$. Hence the statements (1) and (2) are equivalent. We make the further observation that the characteristic function $\chi_{\{x\}} : X \mapsto \mathbb{R}$ defined by

$$\chi_{\{x\}}(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

is not continuous as $\{x\}$ is not open in X. But $\chi_{\{x\}} \in B(X)$ because $\{x\}$ is a Borel set in X. Thus $C(X) \neq B(X)$. So $(3) \Rightarrow (1)$ is also proved. □

We shall now prove the last principal theorem of this paper.

Theorem 4.4. *Every dense C^*-embedded subspace Y of an UMP-space X is an UMP-space.*

Proof. Define a map $\phi : C(X) \mapsto C(Y)$ as follows $\phi(f) = f|_Y$. Then ϕ is an injective homomorphism. Since Y is C^*-embedded in X, it follows that $\phi(C^*(X)) = C^*(Y)$. Consequently $\phi(C(X))$ becomes an intermediate subring of $C(Y)$, say $\phi(C(X)) = A(Y)$. The hypothesis X is an UMP-space, therefore ensures that $A(Y)$ is an UMP-ring meaning that each maximal ideal is union of minimal prime ideals contained in it. In particular each maximal ideal of $A(X)$ consists of divisor of zero. But as we have observed in the proof of the second part of Theorem 2.4 and also in Remark 2.5 that if $A(Y) \subsetneq C(Y)$, then there exist a maximal ideal M of $A(Y)$ and $g \in M$ such that g is not a divisor of zero. Hence we should necessarily have $A(Y) = C(Y)$. thus $\phi(C(X)) = C(Y)$. Hence Y is an UMP-space. □

Corollary 4.5. *If X is UMP-space, then every subspace of υX containing X is UMP-space.*

In this context we record the following result proved in [6], Corollary 1.11.

Theorem 4.6. *No dense C^*-embedded proper realcompact subspace of a compact UMP-space is a UMP-space.*

This above two theorems 4.4 and 4.6 shows that a compact UMP-space does not contains any proper dense C^*-embedded realcompact subspace.

We conclude this article after raising the following open questions:

Question 4.7. *Does an isomorphism between the rings $C(X)$ and $B(X)$ of a P-space X imply that $C(X) = B(X)$?*

Question 4.8. *Is $O^p_A, p \in \beta X$ necessarily a z°-ideal of $A(X)$?*

Question 4.9. *If $A(X) \subsetneq C(X)$, then what is the least cardinal number of the set of all free maximal ideals of $A(X)$, which are not z°-ideal?*

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