The Extension of a Trivial Problem

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INTRODUCTION

This research was suggested at a District Teachers' Meeting held in the spring of 1968 at East Central State College, Ada, Oklahoma. The problem under consideration is to find the set of all numbers x, y, $y \neq 0$ such that $xy = x + y = x/y$ in: (a) the set of integers Z; and (b) the set of rational numbers Q'.

The study contains considerable manipulation of algebraic principles, with results brought into focus by solving the above problem and extending it to involve other sets of numbers.

Procedures—Throughout this paper roots and powers will be defined as follows: (a) $(x^a)^{1/b} = x^{a/b}$, $b \neq 0$; and (b) $[(x^a)^{1/b}]^{c/d} = (x^{a/b})^{c/d} = x^{ac/bd}$, $bd \neq 0$. In the conclusion to problem C (see below), the three cube roots of -1 are indicated. Similarly there exist three cube roots of ½. However, since these three roots are unimportant to the pattern being developed, they are left in the form $(1/2)^{1/3}$ (see summary).

No literature pertaining to the problem was found. My sponsor suggested the problem and left the development to me.

DATA AND RESULTS

Problem A—Find all numbers x, y, $y \neq 0$ such that $xy = x + y = x/y$ in (a) the set Z and (b) the set Q.

Solution—(I) Set $xy = x/y$, $y \neq 0$ implies $xy^2 = x$ implies $y^2 = 1$, if $x \neq 0$ implies $y = \pm 1$. If $x = 0$, consider $xy = x/y$ for all y in Z, $y \neq 0$. Therefore, the only cases when $xy = x/y$, $y \neq 0$ in Z are: (1) $y = \pm 1$, x in Z, $x \neq 0$, or (2) $x = 0$, y in Z, $y \neq 0$.

(II) Set $x + y = x/y$, $y \neq 0$ implies $xy + y^2 = x$ implies $y^2 = x - xy$ implies $y^2/(1-y) = x$, if $y \neq 1$. Consider if $y = 1$, $x + y = x/y$ implies $x + 1 = x$, which is not possible. Obviously $x = 0$ cannot be a valid value since $y \neq 0$.

Conclusion—The intersection of the solutions of parts I and II would be the only solutions for x, y in Z such that $xy = x + y = x/y$, $y \neq 0$. If this is considered, one finds $y = -1$ as the only result occurring in both parts I and II and, therefore, this is the only possible solution for y. Substituting this value in $y^2/(1-y) = x$, one finds that $x = 1/2$. Therefore, the only set of rational numbers satisfying the equation in question is $x = 1/2$ when $y = -1$. A set x, y in Z, that satisfies this equation, does not exist.

Problem B—Find the set of all rational numbers x, y, $y \neq 0$ such that $x^2y^2 = x^2 + y^2 = x^2/y^2$.

Solution— (I) Set $x^2y^2 = x^2/y^2$, $y \neq 0$ implies $x^2y^4 = x^2$ implies $y^4 = 1$, if $x \neq 0$ implies $(y - 1)(y + 1)(y^2 + 1) = 0$ implies $y = \pm 1$ or $y = \pm i$. If $x = 0$, consider $x^2y^2 = x^2/y^2$ for all y in Z, $y \neq 0$. Therefore, the only cases when $x^2y^2 = x^2/y^2$, $y \neq 0$ are: (1) $y = \pm 1$, x any element of Q, $x \neq 0$, or (2) $y = \pm i$, x any element of Q, $x \neq 0$, or (3) $x = 0$, y any element of Q, $y \neq 0$.

(II) Set $x^2 + y^2 = x^2/y^2$, $y \neq 0$ implies $x^2y^2 + y^4 = x^2$ implies $y^4 = x^2(1 - y^2)$ implies $y^4/(1 - y^2) = x^2$, *if* $y \neq \pm 1$. Consider if $y = 1$ or

$y = -1$, $x^2 + y^2 = x^2/y^2$ implies $x^2 + 1 = x^2$, which is not possible. Once again it is obvious that $x = 0$ is not a solution.

Conclusion—The intersection of the solutions of parts I and II would be the only solution for x, y in Q such that $x^2y^2 = x^2 + y^2 = x^2/y^2$, $y \neq 0$. Since $y = \pm i$ are the only values occurring in both parts I and II, the only possible solutions for y are these. Substituting these values in $y^4/(1 - y^2) = x^2$, one finds that $x = \pm(1/2)^{1/2}$. Note that $y = \pm i$ and $x = \pm(1/2)^{1/2}$ are possible solutions, but neither is in Q. Hence, these solutions are not valid under the given set Q. Therefore, the only solutions to this problem would consist of the following complex sets: $x = \pm(1/2)^{1/2}$ when $y = i$ or $x = \pm(1/2)^{1/2}$ when $y = -i$.

Problem C—Find the set of all complex numbers x, y, $y \neq 0$ such that $x^3y^3 = x^3 + y^3 = x^3/y^3$.

Solution—(I) Set $x^3y^3 = x^3/y^3$, $y \neq 0$ implies $x^3y^6 = x^3$ implies $y^6 = 1$, if $x \neq 0$ implies $(y^3 - 1)(y^3 + 1) = 0$ implies $(y - 1)(y^2 + y + 1)(y + 1)(y^2 - y + 1) = 0$ implies $y = 1$, $y = [-1 \pm (-3^{1/2})]/2$, $y = -1$, or $y = [+1 \pm (-3)^{1/2}]/2$. Consider if $x = 0$, $x^3y^3 = x^3/y^3$ holds for any y in the complex numbers, $y \neq 0$. Therefore, the only cases when $x^3y^3 = x^3/y^3$, $y \neq 0$ are the following: (1) $y = \pm 1$, x any complex number, $x \neq 0$, (2) $y = [\pm 1 \pm (3)^{1/2}]/2$, x any complex number, $x \neq 0$, or (3) $x = 0$, y any complex number, $y \neq 0$.

(II) Set $x^3 + y^3 = x^3/y^3$, $y \neq 0$ implies $x^3y^3 + y^6 = x^3$ implies $y^6 = x^3(1 - y^3)$ implies $y^6/(1 - y^3 — = x^3$, if $y^3 \neq 1$. Consider $y^3 - 1 \neq 0$ implies $(y - 1)(y^2 + y + 1) \neq 0$ implies $y \neq 1$ or $y \neq [-1 \pm (-3)^{1/2}]/2$. Obviously $x = 0$ is not a solution once again.

Conclusion—The intersection of the solutions of parts I and II would be the only solution for x, y in the set of complex numbers such that $x^3y^3 = x^3 + y^3 = x^3/y^3$, $y \neq 0$. If this is considered, one finds $y = -1$ or $y = [1 \pm (-3^{1/2})]/2$. If $y = [1 \pm (-3^{1/2})]/2$ or $y = -1$, consider substituting in $y^6/(1 - y^3) = x^3$ implies $1/2 = x^3$ implies $(1/2)^{1/3} = x$. Therefore, the possible solutions in the set of complex numbers for the equation $x^3y^3 = x^3 + y^3 = x^3/y^3$, $y \neq 0$ is $y = [1 \pm (-3^{1/2})]/2$ and $x = (1/2)^{1/3}$ or $y = -1$ and $x = (1/2)^{1/3}$ (see procedures in introduction).

CONCLUSION

A pattern now appeared to be developing. In problem A, involving variables to the first power, the solution set was $x = 1/2$ when $y = -1$. In problem B, concerning variables to power of 2, the solution set, when simplified, was $x = \pm(1/2)^{1/2}$ when $y = \pm(-1)^{1/2}$. Then in problem C, concerning variables to the third power, the solution set when simplified was $x = (1/2)^{1/3}$ when $y = (-1)^{1/3}$. These problems seemed to imply a possible formula concerning a solution to the problem of finding all x, y in the set of complex numbers such that $x^ny^n = x^n + y^n = x^n/y^n$, $y \neq 0$, n is an element of Z'. Consider the solution $x = \pm(1/2)^{1/n}$ when $y = \pm(-1)^{1/n}$.

The following cases were studied concerning the exponent n and the solution to the above problem.

Case I—Consider n such that $n = 0$. The equation $x^0y^0 = x^0 + y^0 = x^0/y^0$ obviously is not true. Therefore, rule out $n = 0$.

Case II—Consider n such that n is an even integer, $n \neq 0$. Let $x = \pm(1/2)^{1/n}$ and $y = \pm(-1)^{1/n}$ and substitute into $x^ny^n = x^n + y^n = x^n/y^n$, $y \neq 0$. $x^ny^n = [\pm(1/2)^{1/n}]^n[\pm(-1)^{1/n}]^n = (1/2)(-1) = -1/2$. $x^n + y^n = [\pm(1/2)^{1/n}]^n + [\pm(-1)^{1/n}]^n = (1/2) + (-1) = -1/2$. $x^n/y^n = [\pm(1/2)^{1/n}/\pm(-1)^{1/n}]^n = 1/2/-1 = -1/2$. Therefore, the proposed solution holds true in this case.

Case III—Consider n such that n is an odd integer. If procedures as in Case II are used, one finds that when $x = (\frac{1}{2})^{1/n}$ and $y = -1^{1/n}$, $x^n y^n = x^n + y^n = x^n/y^n$.

Case IV—Consider n such that n is a rational number. If similar procedures as above are used, one finds that when $x = (\frac{1}{2})^{1/n}$ and $y = -1^{1/n}$, $x^n y^n = x^n + y^n = x^n/y^n$. The possible solution set found in problem A, $x = \frac{1}{2}$ and $y = -1$, is in the set of rational numbers. Then when the original problem is extended in problem B, the possible solutions sets $x = \pm (\frac{1}{2})^{\frac{1}{2}}$ when $y = i$ or $x = \pm (\frac{1}{2})^{\frac{1}{2}}$ when $y = -i$, are extended into the set of complex numbers. In problem C, the possible solution set $y = -1^{\frac{1}{3}}$ and $x = (\frac{1}{2})^{\frac{1}{3}}$ is also contained in the set of complex numbers. The results of extending the original problem yields one solution set for $x^n y^n = x^n + y^n = x^n/y^n$, x any complex number, y any complex number, $y \neq 0$, n any rational number, $n \neq 0$. The solution set would be $x = (\frac{1}{2})^{1/n}$ when $y = (-1)^{1/n}$. Note that this might not be the only solution set.

SUMMARY

Original Problem—Find all x, y such that $xy = x + y = x/y$, $y \neq 0$ in (a) the set of integers Z and in ((b) the set of rationals Q.

Solution— $x = \frac{1}{2}$ when $y = -1$. No solution in Z.

Extension I—Find all x, y such that $x^2y^2 = x^2 + y^2 = x^2/y^2$, $y \neq 0$ in the set of rationals.

Solution—$x = \pm (\frac{1}{2})^{\frac{1}{2}}$ when $y = i$ or $x = \pm (\frac{1}{2})^{\frac{1}{2}}$ when $y = -i$; no solution in Q.

Extension II—Find all x, y such that $x^3y^3 = x^3 + y^3 = x^3/y^3$, $y \neq 0$ in the set of complex numbers.

Solution—$x = (\frac{1}{2})^{\frac{1}{3}}$ when $y = [1 \pm (-3)^{\frac{1}{2}}]/2$ or $x = (\frac{1}{2})^{\frac{1}{3}}$ when $y = -1$.

Extension III—Find all x, y such that $x^n y^n = x^n + y^n = x^n/y^n$, $y \neq 0$ in the set of complex numbers when (a) n is in Z and (b) n is in Q.

Solution—$x = (\frac{1}{2})^{1/n}$ when $y = -1^{1/n}$, $n \neq 0$.

Conclusion—The extension of the original problem led to the following general result: If $x = (\frac{1}{2})^{1/n}$, $y = -1^{1/n}$, n in Q, $n \neq 0$, then $x^n y^n = x^n + y^n = x^n/y^n$.