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Original Research Paper

An Extended Algebraic Method to the Fractional Diffusive Predator-Prey Model

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Abstract. In this paper, we use an algebraic method to obtain analytical solutions for conformable time fractional differential equations. This method has been successfully applied to study and obtain analytical solutions for the predator-prey model. In this method, a suitable fractional transform along with the property of fractional calculus has been employed to reduce fractional partial differential equations to ordinary differential equations.

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1 Introduction

In recent decades, fractional differential equations(FDEs) have been used widely to model physical phenomena, especially uncommon phenomena and complex natural processes that can not be efficiently described by classical calculus[14, 17]. These equations have applica-

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tions in various fields, such as mathematical biology, fluid mechanics, nonlinear optics, image processing, plasma physics and so on. We need to find solutions for these equations in order research and describe these phenomena. Many researchers have developed and implemented numerical and analytical techniques to solve these equations in recent years. Some of these techniques are such as, finite difference method[3], first integral method[4], Adomian decomposition method[5], exp-expansion method[6], variational iteration method[7, 8], Lie group method[9], $\frac{G'}{G^2}$−expansion method[10], Fan-sub equation method[11], simplest equation method[12, 20], differential transform method[16] and so on. In this paper, we study the fractional diffusive predator-prey model(PPM) in the following form[2]:

$$\begin{cases} D_t^{\alpha} r - r_{xx} + \beta r - (1+\beta) r^2 + r^3 + rs = 0, \\ D_t^{\alpha} s - s_{xx} + \beta s + \delta s^3 + krs = 0. \end{cases}$$

Where $r = r(x,t)$ and $s = s(x,t)$ are the predator–prey functions. The parameters β, δ, k are positive constants and $0 < \alpha < 1$. Also, D_t^{α} is the time-fractional derivative, as described in section 2. Many researchers have been utilized some techniques to study the diffusive predator–prey model for $\alpha = 1$ in some works, such as the first-integral method [18], $\exp(-\varphi(\xi))$-expansion method [1], modified simple equation method[21] and $\frac{G'}{G}$-expansion method[15] and so on. Our aim in this paper is to find analytical solutions for the fractional PPM using an algebraic method, as described in section 3.

The rest of our work is organized as follows. In section 2, we present definition of the conformable derivative with its properties. Description of method and its applications to the time fractional differential equations are described in section 3. Then the mentioned method is applied to the fractional PPM in section 4. Discussion and conclusions are presented in section 5.

2 Definition of the Conformable Derivative with its Properties

Now, let us describe the definition and some its important properties of the conformable fractional derivative(CFD) of order γ as follows[13].

Definition 2.1. *For a function $h : [0, \infty] \to \mathbb{R}$, the CFD of h of order γ is defined by*

$$D^{\gamma}\{h(\zeta)\} = \lim_{\eta \to 0} \frac{h(\zeta + \eta\zeta^{1-\gamma}) - h(\zeta)}{\eta}.$$

Some well-known properties to this newly defined fractional derivative are as follows.
If g and $h \neq 0$ be two functions γ-differentiable, $\gamma \in (0, 1]$ and $a, b \in \mathbb{R}$. Then, we have

$$\begin{aligned}
&(1) \quad D^{\gamma}\{ag(\zeta) + b\,h(\zeta)\} = a\,D^{\gamma}g(\zeta) + b\,D^{\gamma}h(\zeta),\\
&(2) \quad D^{\gamma}\{g(\zeta)\,h(\zeta)\} = g(\zeta)\,D^{\gamma}h(\zeta) + h(\zeta)\,D^{\gamma}g(\zeta),\\
&(3) \quad D^{\gamma}\{\frac{g(\zeta)}{h(\zeta)}\} = \frac{h(\zeta)\,D^{\gamma}g(\zeta) - g(\zeta) \quad D^{\gamma}h(\zeta)}{h^2(\zeta)},\\
&(4) \quad D^{\gamma}C = 0, \quad for\,all\,constant\,functions\,f(z) = C,\\
&(5) \quad D^{\gamma}(g)(\zeta) = \zeta^{1-\gamma}\frac{dg}{d\zeta}.
\end{aligned}$$

For some special functions, we have[13]

$$\begin{aligned}
&(a) \quad D^{\gamma}(\zeta^r) = r\zeta^{r-\gamma}\,for\,all\,r \in \mathbb{R},\\
&(b) \quad D^{\gamma}(1) = 0,\\
&(c) \quad D^{\gamma}(e^{c\zeta}) = c\zeta^{1-\gamma}e^{c\zeta},\ c \in \mathbb{R},\\
&(d) \quad D^{\gamma}(\sin b\zeta) = b\zeta^{1-\gamma}\cos b\zeta,\ b \in \mathbb{R},\\
&(e) \quad D^{\gamma}(\cos b\zeta) = -b\zeta^{1-\gamma}\sin b\zeta,\ b \in \mathbb{R},\\
&(f) \quad D^{\gamma}(\frac{1}{\gamma}\zeta^{\gamma}) = 1.
\end{aligned}$$

Definition 2.2. *Let $\gamma \in (n, n+1]$, and h be an γ-differentiable at $t > 0$. Then the CFD of h of order γ is defined as*

$$D^{\gamma}(h(t)) = \lim_{\eta \to 0} \frac{h^{(\lceil\gamma\rceil - 1)}(t + \eta t^{(\lceil\gamma\rceil - \gamma)}) - h^{(\lceil\gamma\rceil - 1)}(t)}{\eta}.$$

Where $\lceil\gamma\rceil$ is the smallest integer greater than or equal to γ.

3 Description of Method and its Applications to the Time FDEs

In this section, we outline the main steps of this method for solving FDEs. For a given FDE in two variables x and t we have

$$Q\left(u, u_x, u_t, D_t^{\alpha} u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{1}$$

where $D_t^{\alpha} u$ is the CFD of u, $u = u(x,t)$ is an unknown function and Q is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.
We take the travelling wave transformation

$$\eta = x - \lambda \frac{t^{\alpha}}{\alpha}, \tag{2}$$

where λ is a nonzero constant to be determined later. Substituting (2) into (1), we reduce (1) to the following ODE

$$\tilde{N}(U, U', U'', U''', \ldots) = 0. \tag{3}$$

Here $U^{(n)} = \frac{d^n U}{d\eta^n}$. Exact solutions for this equation can be constructed as a finite series

$$U(\eta) = \sum_{m=0}^{n} A_m \psi^m, \tag{4}$$

where $A_m \, (A_n \neq 0)$ are constants to be determined later, and the positive integer n can be determined by considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of $u(\eta)$ in equation (3). Here $\psi = \psi(\eta)$ satisfies the following ODE

$$\psi'(\eta) = \ln(\sigma)(a + b\psi(\eta) + c\psi^2(\eta)), \;\; \sigma \neq 0, 1, \tag{5}$$

where a, b and c are constants and which has the following special solutions with $\Delta = b^2 - 4ac$ [19].

Case(1). For $\Delta < 0$ and $c \neq 0$,

$$\psi_1(\eta) = -\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}\tan_\sigma(\frac{\sqrt{-\Delta}}{2}\eta),$$
$$\psi_2(\eta) = -\frac{b}{2c} - \frac{\sqrt{-\Delta}}{2c}\cot_\sigma(\frac{\sqrt{-\Delta}}{2}\eta),$$
$$\psi_3(\eta) = -\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}[\tan_\sigma(\sqrt{-\Delta}\ \eta) \pm \sqrt{mn}\sec_\sigma(\sqrt{-\Delta}\ \eta)];\ mn \geq 0,$$
$$\psi_4(\eta) = -\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}[\cot_\sigma(\sqrt{-\Delta}\ \eta) \pm \sqrt{mn}\csc_\sigma(\sqrt{-\Delta}\ \eta)];\ mn \geq 0,$$
$$\psi_5(\eta) = -\frac{b}{2c} + \frac{\sqrt{-\Delta}}{4c}[\tan_\sigma(\frac{\sqrt{-\Delta}}{4}\eta) - \cot_\sigma(\frac{\sqrt{-\Delta}}{4}\eta)].$$

Case(2). For $\Delta > 0$ and $c \neq 0$,

$$\psi_6(\eta) = -\frac{b}{2c} - \frac{\sqrt{\Delta}}{2c}\tanh_\sigma(\frac{\sqrt{\Delta}}{2}\eta),$$
$$\psi_7(\eta) = -\frac{b}{2c} - \frac{\sqrt{\Delta}}{2c}\coth_\sigma(\frac{\sqrt{\Delta}}{2}\eta),$$
$$\psi_8(\eta) = -\frac{b}{2c} + \frac{\sqrt{\Delta}}{2c}[-\tanh_\sigma(\sqrt{\Delta}\ \eta) \pm \sqrt{-mn}\ sech_\sigma(\sqrt{\Delta}\ \eta)];\ mn \leq 0,$$
$$\psi_9(\eta) = -\frac{b}{2c} + \frac{\sqrt{\Delta}}{2c}[-\coth_\sigma(\sqrt{\Delta}\ \eta) \pm \sqrt{mn}\ csch_\sigma(\sqrt{\Delta}\ \eta)];\ mn \geq 0,$$
$$\psi_{10}(\eta) = -\frac{b}{2c} - \frac{\sqrt{\Delta}}{4c}[\tanh_\sigma(\frac{\sqrt{\Delta}}{4}\eta) + \coth_\sigma(\frac{\sqrt{\Delta}}{4}\eta)].$$

Case(3). For $ac > 0$ and $b = 0$,

$$\psi_{11}(\eta) = \sqrt{\frac{a}{c}}\tan_\sigma(\sqrt{ac}\,\eta),$$
$$\psi_{12}(\eta) = -\sqrt{\frac{a}{c}}\cot_\sigma(\sqrt{ac}\,\eta),$$
$$\psi_{13}(\eta) = \sqrt{\frac{a}{c}}[\tan_\sigma(2\sqrt{ac}\ \eta) \pm \sqrt{mn}\sec_\sigma(2\sqrt{ac}\ \eta)];\ mn \geq 0,$$
$$\psi_{14}(\eta) = \sqrt{\frac{a}{c}}[-\cot_\sigma(2\sqrt{ac}\ \eta) \pm \sqrt{mn}\csc_\sigma(2\sqrt{ac}\ \eta)];\ mn \geq 0,$$
$$\psi_{15}(\eta) = \frac{1}{2}\sqrt{\frac{a}{c}}[\tan_\sigma(\frac{\sqrt{ac}}{2}\eta) - \cot_\sigma(\frac{\sqrt{ac}}{2}\eta)].$$

Case(4). For $ac < 0$ and $b = 0$,

$$\psi_{16}(\eta) = -\sqrt{-\frac{a}{c}}\tanh_\sigma(\sqrt{-ac}\,\eta),$$
$$\psi_{17}(\eta) = -\sqrt{-\frac{a}{c}}\coth_\sigma(\sqrt{-ac}\,\eta),$$
$$\psi_{18}(\eta) = \sqrt{-\frac{a}{c}}[-\tanh_\sigma(2\sqrt{-ac}\ \eta) \pm \sqrt{mn}\,sech_\sigma(2\sqrt{-ac}\ \eta)];\ mn \leq 0,$$
$$\psi_{19}(\eta) = \sqrt{-\frac{a}{c}}[-\coth_\sigma(2\sqrt{-ac}\ \eta) \pm \sqrt{mn}\,csch_\sigma(2\sqrt{-ac}\ \eta)];\ mn \geq 0,$$
$$\psi_{20}(\eta) = -\frac{1}{2}\sqrt{-\frac{a}{c}}[\tanh_\sigma(\frac{\sqrt{-ac}}{2}\eta) + \coth_\sigma(\frac{\sqrt{-ac}}{2}\eta)].$$

Case(5). For $a = c$ and $b = 0$,

$$\psi_{21}(\eta) = \tan_\sigma(a\eta),$$
$$\psi_{22}(\eta) = -\cot_\sigma(a\eta),$$
$$\psi_{23}(\eta) = \tan_\sigma(2a\eta) \pm \sqrt{mn}\sec_\sigma(2a\eta);\ mn \geq 0,$$
$$\psi_{24}(\eta) = -\cot_\sigma(2a\eta) \pm \sqrt{mn}\csc_\sigma(2a\eta);\ mn \geq 0,$$
$$\psi_{25}(\eta) = \frac{1}{2}[\tan_\sigma(\frac{a}{2}\eta) - \cot_\sigma(\frac{a}{2}\eta)].$$

Case(6). For $a=-c$ and $b=0$,

$$\begin{aligned}
\psi_{26}(\eta) &= -\tanh_\sigma(a\eta),\\
\psi_{27}(\eta) &= -\coth_\sigma(a\eta),\\
\psi_{28}(\eta) &= -\tanh_\sigma(2a\eta) \pm \sqrt{-mn}\, sech_\sigma(2a\eta);\ mn \le 0,\\
\psi_{29}(\eta) &= -\coth_\sigma(2a\eta) \pm \sqrt{mn}\, csch_\sigma(2a\eta);\ mn \ge 0,\\
\psi_{30}(\eta) &= -\frac{1}{2}[\tanh_\sigma(\frac{a}{2}\eta) + \coth_\sigma(\frac{a}{2}\eta)].
\end{aligned}$$

Case(7). For $b^2 = 4ac$,

$$\psi_{31}(\eta) = \frac{-2a(b\ln(\sigma)\eta + 2)}{b^2\eta\ln\sigma}.$$

Case(8). For $b=p$, $a=pq$, $(q \neq 0)$ and $c=0$.

$$\psi_{32}(\eta) = \sigma^{p\eta} - q.$$

Case(9). For $b=c=0$.

$$\psi_{33}(\eta) = a\ln(\sigma)\eta.$$

Case(10). For $a=b=0$.

$$\psi_{34}(\eta) = -\frac{1}{c\ln(\sigma)\eta}.$$

Case(11). For $a=0$ and $b \neq 0$.

$$\psi_{35}(\eta) = -\frac{mb}{c(\cosh_\sigma(b\eta) - \sinh_\sigma(b\eta) + m)},$$

$$\psi_{36}(\eta) = -\frac{b(\cosh_\sigma(b\eta) + \sinh_\sigma(b\eta))}{c(\cosh_\sigma(b\eta) + \sinh_\sigma(b\eta) + n)}.$$

Case(12). For $b=p$, $c=pq$, $(q \neq 0)$ and $a=0$.

$$\psi_{37}(\eta) = -\frac{m\sigma^{p\eta}}{m - qn\sigma^{p\eta}}.$$

We know that

$$
\begin{aligned}
\sinh_\sigma(\eta) &= \frac{m\sigma^{\eta} - n\sigma^{-\eta}}{2}, & \cosh_\sigma(\eta) &= \frac{m\sigma^{\eta} + n\sigma^{-\eta}}{2},\\
\tanh_\sigma(\eta) &= \frac{m\sigma^{\eta} - n\sigma^{-\eta}}{m\sigma^{\eta} + n\sigma^{-\eta}}, & \coth_\sigma(\eta) &= \frac{m\sigma^{\eta} + n\sigma^{-\eta}}{m\sigma^{\eta} - n\sigma^{-\eta}},\\
sech_\sigma(\eta) &= \frac{2}{m\sigma^{\eta} + n\sigma^{-\eta}}, & csch_\sigma(\eta) &= \frac{2}{m\sigma^{\eta} - n\sigma^{-\eta}},\\
\sin_\sigma(\eta) &= \frac{m\sigma^{i\eta} - n\sigma^{-i\eta}}{2i}, & \cos_\sigma(\eta) &= \frac{m\sigma^{i\eta} + n\sigma^{-i\eta}}{2},\\
\tan_\sigma(\eta) &= -i\frac{m\sigma^{i\eta} - n\sigma^{-i\eta}}{m\sigma^{i\eta} + n\sigma^{-i\eta}}, & \cot_\sigma(\eta) &= i\frac{m\sigma^{i\eta} + n\sigma^{-i\eta}}{m\sigma^{i\eta} - n\sigma^{-i\eta}},\\
sec_\sigma(\eta) &= \frac{2}{m\sigma^{i\eta} + n\sigma^{-i\eta}}, & csc_\sigma(\eta) &= \frac{2i}{m\sigma^{i\eta} - n\sigma^{-i\eta}}.
\end{aligned}
$$

Here m and n are arbitrary constants and known as deformation parameters.
Now, this method for obtaining exact solutions of FDEs consists from the following two main steps:

- Step (1). By substituting (4) with Eq.(5) into (3) and collecting all terms with the same powers of ψ together, the left hand side of Eq.(3) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of A_m $(m = 0, 1, 2, ..., n)$, a, b, c.

- Step (2). Solving the system of algebraic equations and then substituting the case(1)-case(12) into (4), it gives travelling wave solutions of (3).

4 Application

In this section, we consider the time fractional predator-prey model as follows[2]

$$D_t^{\alpha} r - r_{xx} + \beta r - (1+\beta) r^2 + r^3 + rs = 0, \tag{6}$$
$$D_t^{\alpha} s - s_{xx} + \beta s + \delta s^3 + krs = 0. \tag{7}$$

Where r and s are functions of space variable x and time variable t and $0 < \alpha < 1$.
For obtaining exact solutions of (6) and (7), We take the traveling wave transformation

$$\begin{aligned} & r(x,t) = r(\eta), \quad s(x,t) = s(\eta), \\ & \eta = x - \frac{\lambda}{\alpha} t^{\alpha}, \end{aligned}$$

where λ is a constant which should to be determined later. Then equations (6) and (7) are reduced into two nonlinear ODEs

$$\lambda r' + r'' - \beta r + (1+\beta) r^2 - r^3 - rs = 0, \tag{8}$$
$$\lambda s' + s'' - \beta s - \delta s^3 + krs = 0. \tag{9}$$

For solving this system, we consider the following transformation

$$r = \sqrt{\delta} s, \tag{10}$$

substituting (10) into equations (8) and (9), we obtain

$$r'' + \lambda r' - \beta r + kr^2 - r^3 = 0, \tag{11}$$

Balancing r'' with r^3 in (11) gives n=1, therefore, according to (4), solution of (11) can be expressed by a polynomial in ψ as follows:

$$U(\eta) = A_0 + A_1 \psi, \quad A_1 \neq 0, \tag{12}$$

where ψ is the solution of equation (5). Substituting (12) into (11) and making use of equation (5) and equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for A_0, A_1, a, b, c. Solving obtained system using *Mathematica*, we obtain

$$\begin{aligned} \bullet Set\,1: \quad & A_0 = \frac{k}{3}, \; A_1 = \frac{4k(b^2 - ac)\ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}, \quad \lambda = -3b\ln(\sigma), \\ & k = \sqrt{3(\beta + 2(b^2 - ac)\ln^2(\sigma))}, \; ab \neq 0. \end{aligned} \tag{13}$$

By using of the (12), (13) and case(1) respectively, we get

$$r_1(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}(-\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}\tan_\sigma(\frac{\sqrt{-\Delta}}{2}(x - \frac{\lambda t^\alpha}{\alpha}))),$$

$$r_2(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}(-\frac{b}{2c} - \frac{\sqrt{-\Delta}}{2c}\cot_\sigma(\frac{\sqrt{-\Delta}}{2}(x - \frac{\lambda t^\alpha}{\alpha}))),$$

$$r_3(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}[(-\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}(\tan_\sigma(\sqrt{-\Delta}(x - \frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{mn}\sec_\sigma(\sqrt{-\Delta}(x - \frac{\lambda t^\alpha}{\alpha})))]; \quad mn \geq 0,$$

$$r_4(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}[(-\frac{b}{2c} + \frac{\sqrt{-\Delta}}{2c}(\cot_\sigma(\sqrt{-\Delta}(x - \frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{mn}\csc_\sigma(\sqrt{-\Delta}(x - \frac{\lambda t^\alpha}{\alpha})))]; \quad mn \geq 0,$$

$$r_5(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}[-\frac{b}{2c} + \frac{\sqrt{-\Delta}}{4c}(\tan_\sigma(\frac{\sqrt{-\Delta}}{4}(x - \frac{\lambda t^\alpha}{\alpha})) - \cot_\sigma(\frac{\sqrt{-\Delta}}{4}(x - \frac{\lambda t^\alpha}{\alpha})))],$$

By using of the (12), (13) and case(2) respectively, we get

$$r_6(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}(-\frac{b}{2c} - \frac{\sqrt{\Delta}}{2c}\tanh_\sigma(\frac{\sqrt{\Delta}}{2}(x - \frac{\lambda t^\alpha}{\alpha}))),$$

$$r_7(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}(-\frac{b}{2c} - \frac{\sqrt{\Delta}}{2c}\coth_\sigma(\frac{\sqrt{\Delta}}{2}(x - \frac{\lambda t^\alpha}{\alpha}))),$$

$$r_8(x,t) = \frac{k}{3} + \frac{4k(b^2-ac)\ \ln^2(\sigma) - k\beta}{18ab\ln^2(\sigma)}[(-\frac{b}{2c} + \frac{\sqrt{\Delta}}{2c}(-tanh_\sigma(\sqrt{\Delta}(x - \frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{-mn}\,sech_\sigma(\sqrt{\Delta}(x - \frac{\lambda t^\alpha}{\alpha})))]; \quad mn \leq 0,$$

$$r_9(x,t)=\frac{k}{3}+\frac{4k(b^2-ac)\,\ln^2(\sigma)-k\beta}{18ab\ln^2(\sigma)}[(-\frac{b}{2c}+\frac{\sqrt{\Delta}}{2c}(-coth_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{mn}\,csch_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha})))];\quad mn\geq 0,$$

$$r_{10}(x,t)=\frac{k}{3}+\frac{4k(b^2-ac)\,\ln^2(\sigma)-k\beta}{18ab\ln^2(\sigma)}[-\frac{b}{2c}-\frac{\sqrt{\Delta}}{4c}(\tanh_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha})) +\coth_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha})))],$$

By using of the (12), (13) and case(8), we have

$$r_{11}(x,t)=\frac{k}{3}+\frac{4k(b^2-ac)\,\ln^2(\sigma)-k\beta}{18ab\ln^2(\sigma)}(\sigma^{p(x-\frac{\lambda t^\alpha}{\alpha})}-q),$$

$$\begin{aligned}\bullet Set\,2:\qquad & A_0=\frac{k}{3},\ A_1=\pm\sqrt{2}\,c\ln(\sigma),\quad \lambda=-3b\ln(\sigma),\\ & k=3\sqrt{2}\,b\ln(\sigma),\ \ a=0,\ \ bc\neq 0.\end{aligned}\tag{14}$$

By using of the (12), (14) and case(2) respectively, we get

$$r_{12}(x,t)=\frac{k}{3}\pm\sqrt{2}\ln(\sigma)(-\frac{b}{2}-\frac{\sqrt{\Delta}}{2}\tanh_\sigma(\frac{\sqrt{\Delta}}{2}(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{13}(x,t)=\frac{k}{3}\pm\sqrt{2}\ln(\sigma)(-\frac{b}{2}-\frac{\sqrt{\Delta}}{2}\coth_\sigma(\frac{\sqrt{\Delta}}{2}(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{14}(x,t)=\frac{k}{3}\pm\sqrt{2}\ln(\sigma)[(-\frac{b}{2}+\frac{\sqrt{\Delta}}{2}(-tanh_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{-mn}\,sech_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha})))];\quad mn\leq 0,$$

$$r_{15}(x,t)=\frac{k}{3}\pm\sqrt{2}\ln(\sigma)[(-\frac{b}{2}+\frac{\sqrt{\Delta}}{2}(-coth_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha}) \pm\sqrt{mn}\,csch_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha})))];\quad mn\geq 0,$$

$$r_{16}(x,t)=\frac{k}{3}\pm\sqrt{2}\ln(\sigma)[-\frac{b}{2}-\frac{\sqrt{\Delta}}{4}(\tanh_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha}))+\coth_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha})))],$$

By using of the (12), (14) and case(11) respectively, we have

$$r_{17}(x,t)=\frac{k}{3}\pm[\frac{\sqrt{2}mb\ln(\sigma)}{\cosh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha}))-\sinh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha}))+m}],$$

$$r_{18}(x,t)=\frac{k}{3}\mp\sqrt{2}\ln(\sigma)[\frac{b(\sinh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha}))+\cosh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha})))}{\sinh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha}))+\cosh_\sigma(b(x-\frac{\lambda t^\alpha}{\alpha}))+n}],$$

By using of the (12), (14) and case(12), we have

$$r_{19}(x,t)=\frac{k}{3}\mp\frac{\sqrt{2}mc\ln(\sigma)\sigma^{p(x-\frac{\lambda t^\alpha}{\alpha})}}{m-qn\sigma^{p(x-\frac{\lambda t^\alpha}{\alpha})}},$$

$$\bullet Set\,3:\quad A_0=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)}),\ A_1=\frac{2\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda},$$
$$\lambda=\pm6\sqrt{-ac\ln^2(\sigma)},k=6\sqrt{2}\sqrt{-ac\ln^2(\sigma)},\ \ b=0,\ ac<0. \tag{15}$$

By using of the (12), (15) and case(2) respectively, we have

$$r_{20}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})+\frac{\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}(-\frac{b}{c}-\frac{\sqrt{\Delta}}{c}\tanh_\sigma(\frac{\sqrt{\Delta}}{2}(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{21}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})+\frac{\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}\\(-\frac{b}{c}-\frac{\sqrt{\Delta}}{c}\coth_\sigma(\frac{\sqrt{\Delta}}{2}(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{22}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})+\frac{\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}(-\frac{b}{c}+\frac{\sqrt{\Delta}}{c}\\(-\,tanh_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha}))\pm\sqrt{-mn}\,sech_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha})))));\quad mn\leq 0,$$

$$r_{23}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})+\frac{\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}(-\frac{b}{c}+\frac{\sqrt{\Delta}}{c}\\(-\,coth_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha}))\pm\sqrt{mn}\,csch_\sigma(\sqrt{\Delta}(x-\frac{\lambda t^\alpha}{\alpha})))));\quad mn\geq 0,$$

$$r_{24}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})+\frac{\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}[-\frac{b}{c}-\frac{\sqrt{\Delta}}{2c}\\(\tanh_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha}))+\coth_\sigma(\frac{\sqrt{\Delta}}{4}(x-\frac{\lambda t^\alpha}{\alpha})))],$$

By using of the (12), (15) and case(4) respectively, we have

$$r_{25}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})-\frac{2\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}\\(\sqrt{-\frac{a}{c}}\tanh_\sigma(\sqrt{-ac}\,(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{26}(x,t)=\frac{1}{3}(k\pm\sqrt{k^2+54ac\ln^2(\sigma)})-\frac{2\sqrt{k^2+54ac\ln^2(\sigma)}}{\lambda}\\(\sqrt{-\frac{a}{c}}\coth_\sigma(\sqrt{-ac}\,(x-\frac{\lambda t^\alpha}{\alpha}))),$$

$$r_{27}(x,t) = \frac{1}{3}(k \pm \sqrt{k^2 + 54ac\ln^2(\sigma)}) + \frac{2\sqrt{k^2 + 54ac\ln^2(\sigma)}}{\lambda}(\sqrt{-\frac{a}{c}}$$
$$(-\tanh_\sigma(2\sqrt{-ac}\,(x - \frac{\lambda t^\alpha}{\alpha})) \pm \sqrt{mn}\,sech_\sigma(2\sqrt{-ac}\,(x - \frac{\lambda t^\alpha}{\alpha})))); \; mn \leq 0,$$

$$r_{28}(x,t) = \frac{1}{3}(k \pm \sqrt{k^2 + 54ac\ln^2(\sigma)}) + \frac{2\sqrt{k^2 + 54ac\ln^2(\sigma)}}{\lambda}(\sqrt{-\frac{a}{c}}$$
$$(-\coth_\sigma(2\sqrt{-ac}\,(x - \frac{\lambda t^\alpha}{\alpha})) \pm \sqrt{mn}\,csch_\sigma(2\sqrt{-ac}\,(x - \frac{\lambda t^\alpha}{\alpha})))); \; mn \geq 0,$$

$$r_{29}(x,t) = \frac{1}{3}(k \pm \sqrt{k^2 + 54ac\ln^2(\sigma)}) - \frac{\sqrt{k^2 + 54ac\ln^2(\sigma)}}{\lambda}(\sqrt{-\frac{a}{c}}$$
$$(\tanh_\sigma(\frac{\sqrt{-ac}}{2}(x - \frac{\lambda t^\alpha}{\alpha})) + \coth_\sigma(\frac{\sqrt{-ac}}{2}(x - \frac{\lambda t^\alpha}{\alpha})))).$$

Using the relation which is depicted in (10), we have the following solution of the predator-prey model

$$s = \frac{1}{\sqrt{\delta}}\, r.$$

5 Concluding Remarks

In this paper, an extended algebraic method has successfully been applied to study a system of fractional differential equations, namely, the predator-prey model. It has been shown that the applied method is effective, and many other nonlinear evolution equations can be solved by this method. *Mathematica* has been used for computations and programming in this paper.

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