Int. J. Anal. Appl. (2025), 23:316

International Journal of Analysis and Applications

Computational and Periodic Wave Solutions for the Time-Fractional Calogero-Bogoyavlenskii-Schiff Equation

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Abstract. The time-fractional differential equation of Calogero-Bogoyavlenskii-Schiff (CBS) has an important role in plasma waves and shallow water and ocean waves. By using symbolic computation, we applied the generalized Kudryashov method (GKM) and the generalized He's Exp-function method (GHEFM) For the purpose of creating novel time-fractional CBS equation solutions. Utilizing suitable fractional transformation the governing equation reduced to ordinary differential equation. New different wave solutions are obtained using the methodology of both GKM and GHEFM. Kink wave, single wave, and solitary wave solutions are represented by these solutions, which also include trigonometric and hyperbolic functions.

1. Introduction

Partial differential equations have an important role in the fields of science and engineering. The obtaining of exact solutions for partial differential equations are explaining many scientific phenomena such as optical fibers, plasma physics, electrochemistry and many others. The use of traveling wave techniques is crucial for obtaining precise solutions for PDE. The most well-known of these efficient techniques are Exp-function method $[1-3]$, Generalization He's Exp-Function Method $[4,5]$, (G^l/G)-expansion method $[6-9]$, improved $\tan(\phi/2)$-expansion method [10], Direct algebraic method [11], Extended auxiliary equation mapping method [12], Modified Kudryashov method $[13,14]$ and Generalization Kudryashov method $[15-17]$.

In this paper, we studied the time-fractional CBS equations [18]

$$\partial_t^\alpha w + \Psi(w)\, w_y + \Phi(w) w_z = 0,$$
$$\Psi(w) = \partial_x^2 + aw + bw_x\partial_x^{-1},$$

Received: Sep. 29, 2025.

2020 *Mathematics Subject Classification.* 35K05, 35L05, 37C79.

Key words and phrases. Fractional derivative; the generalized Kudryashov method; the generalized He's Exp-function method.

https://doi.org/10.28924/2291-8639-23-2025-316

ISSN: 2291-8639

$$\Phi(w) = \partial_x^2 + cw + dw_x\partial_x^{-1}, \tag{1}$$

where $\partial_t^\alpha = \frac{\partial^\alpha}{\partial t}$ and $\partial_x^{-1} f = \int f dx$, since g_1, g_2, g_3 and g_4 are constants. Eq.(1) can be taken the form

$$\partial_t^\alpha w + g_1 w w_y + g_3 w w_z + g_2 w_x \partial_x^{-1} w_y + g_4 w_x \partial_x^{-1} w_z + w_{xxy} + w_{xxz} = 0. \tag{2}$$

Substituting potential $w = u_x$ then Eq.(1) can be write in the form

$$\partial_t^\alpha u_x + g_1 u_x u_{xy} + g_3 u_x u_{xz} + g_2 u_{xx} u_y + g_4 u_{xx} u_z + u_{xxxy} + u_{xxxz} = 0. \tag{3}$$

The CBS equation was developed by Schiff and Bogoyavlenskii. The CBS equation was obtained by the second researcher by reducing the self-dual Yang Mills equation, whereas the first researcher employed the modified Lax formalism [19, 20]. Numerous researchers have taken notice of the CBS equation. Wazwaz used the tanh-coth approach to find traveling wave solutions for the CBS equation [21, 22]. Moatimid et al. [23] discovered the exact solutions and symmetry reductions for the particular case of Eq. (3). Masood et al. [24] used conservation laws for Eq. (3).

2. Characteristics of fractional derivatives

The most often used definition of the Riemann and Liouville derivative, which is defined [14, 15], is examined in this study.

$$D_t^\alpha \zeta(t) = \begin{cases} \frac{1}{\Gamma(r-\alpha)} \frac{d^r}{dt^r} \int\limits_0^t (t-\tau)^{r-\alpha-1} \zeta(\tau) d\tau & \text{if} \quad r-1 < \alpha < r,\ r \in N \\ \frac{d^r \zeta(t)}{dt^r} & \text{if } \alpha = r, \quad r \in N \end{cases}$$

where $\zeta : R \to R$, $t \to \zeta(t)$, denote a continuous function.

Properties:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},$$
$$D_t^\alpha (\zeta(t)\eta(t)) = \eta(t) D_t^\alpha \zeta(t) + \zeta(t) D_t^\alpha \eta(t),$$
$$D_t^\alpha \zeta(\eta(t)) = \frac{d\zeta(\eta(t))}{d\eta(t)} D_t^\alpha \eta(t).$$

3. Basic idea of the methods

Suppose a given The equation for nonlinear waves

$$F(u, D_t^\alpha u, u_{x_1}, u_{x_1 x_2}, u_{x_2 x_2}, ...) = 0, \tag{4}$$

where $u(x_i, t)$ is depended variable, F is PDE of $u(x_i, t)$ and its derivatives. This equation is non linear partial differential equation. So the primary steps of the main method: we seek its wave solutions

Step 1: A complex variable η is created by combining the real variables x_i and t [14]

$$u(x_i,t) = u(\eta), \;\; \eta = k_i x_i - \frac{wt^{\alpha}}{\Gamma(\alpha)}, \tag{5}$$

As a results, Eq.(4) is reduced to the ordinary differential equation (ODE):

$$U(u, u', u', u'', u'', ...) = 0. \tag{6}$$

Step 2: Assume that the following is an expression of GKM for the traveling wave solution of Eq. (6):

$$u(\eta) = \sum_{i=0}^{m} \frac{a_i}{(1+\phi(\eta))^i}, \tag{7}$$

where m is positive integer, a$_i$ is arbitrary constants. The exact solution of Eq. (5) for **GHEFM** is written as follows:

$$u(\eta) = \frac{a_{-c}[\phi(\eta)^{-c}] + ... + a_p[\phi(\eta)^p]}{r_{-d}[\phi(\eta)^{-d}] + ... + r_q[\phi(\tau)^q]}, \tag{8}$$

where c, d, p and q are arbitrary constants. In addition, $\phi(\eta)$ of Eq. (7,8) satisfies Riccati equation

$$\phi'(\eta) = A + B\phi(\eta) + C\phi^2(\eta). \tag{9}$$

Eq.(5) gives the following solutions:

Group 1: A and B are arbitrary constants, $C \neq 0$

$$\phi(\eta) = \frac{-B + \sqrt{4AC - B^2}\tan(\frac{1}{2}(\sqrt{4AC - B^2}(\eta + d_0)))}{2C}. \tag{10}$$

Group 2: $A = 0$, $B \neq 0$, and C is a free constant

$$\phi(\eta) = \frac{-B\exp(B\eta + Bd_0)}{C\exp(B\eta + Bd_0) - 1}. \tag{11}$$

Group 3: A is a free constant, $B \neq 0$, and $C = 0$

$$\phi(\eta) = \frac{-A}{B} + \frac{1}{B}\exp(B\eta). \tag{12}$$

Step 3: Substituting Eq. (7) or (8) into Eq. (6) and then collecting the power of the function $\phi(\eta)$. As a result of this substitution, equating all the coefficients of same power of $\phi(\eta)$ to zero. Yields a system of algebraic equations which can be solved to find a_i and r_i. Substituting the values of a_i and r_i into Eq. (7) to get the exact solutions of Eq. (3).

4. Travelling wave solutions

The application of this methods in previous section to officially derive various exact wave solutions for time-fractional CBS equation.

By using the suitable transform $u(\eta) = u(t, x, y, z)$, $\eta = k_1x + k_2y + k_3z - \frac{wt^{\alpha}}{\Gamma(\alpha)}$ in Eq. (3) we get:

$$-wk_1u' + k_1^2[k_2(g_1 + g_2) + k_3(g_3 + g_4)]u'u'' + k_1^3(k_2 + k_3)u'''' = 0. \tag{13}$$

We are free to select any value for k_1 , k_2, k_3 and k_4.

4.1. **Application of GKM.** Balancing between the nonlinear terms $u'u''$ and highest order linear u'''' [25] in Eq. (13). We set i=1, we have:

$$u(\eta) = a_0 + \frac{a_1}{1+\phi(\eta)}. \tag{14}$$

where the arbitrary constants a_0 and a_1.

Substituting Eq. (14) into Eq. (13), collecting all powers of $\phi(\eta)$ and equating it to zero,provides a collection of algebraic formulas for a_0 and a_1. Using Maple to solve the algebraic equation system. The following solutions are obtained:

$$w = k_1^2(4AC - B^2)(k_3 + k_2),$$

a_0 is arbitrary

$$a_1 = \frac{12k_1(k_2+k_3)(A+C-B)}{k_2(g_1+g_2)+k_3(g_3+g_4)}. \tag{15}$$

Substituting the results into Eq.(3) and obtaining the several triangular and soliton-like periodic solutions for the CBS equation that are as follows:

Case 1

$$u_1 = a_0 + \frac{2C(k_2(g_1+g_2)+k_3(g_3+g_4))}{(k_2(a+b)+k_3(c+d))(2C - B + \sqrt{4AC-B^2}\tan[\frac{1}{2}(\sqrt{4AC-B^2}(\eta+d_0))])}, \tag{16}$$

where $\eta = k_1x + k_2y + k_3z - k_1^2(4AC-B^2)(k_3+k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

Case 2

$$u_1 = a_0 + \frac{k_2(g_1+g_2)+k_3(g_3+g_4))(C\exp(B\eta+Bd_0)-1)}{(k_2(g_1+g_2)+k_3(g_3+g_4))(C\exp(B\eta+Bd_0) - B\exp(B\eta+Bd_0)-1)}, \tag{17}$$

where $\eta = k_1x + k_2y + k_3z + k_1^2B^2(k_3+k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

Case 3

$$u_2 = a_0 + \frac{k_2(g_1+g_2)+k_3(g_3+g_4)}{(k_2(g_1+g_2)+k_3(g_3+g_4))(\frac{-A}{B}+\frac{1}{B}\exp(B\eta+Bd_0)+1)}, \tag{18}$$

where $\eta = k_1x + k_2y + k_3z + k_1^2B^2(k_3+k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

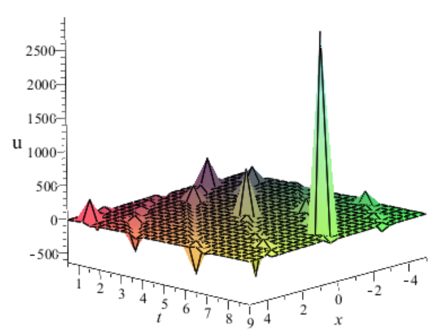

FIGURE 1. Graphical of periodic wave solution of Eq. (16) where k_1 =-1, $k_2 = k_3 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

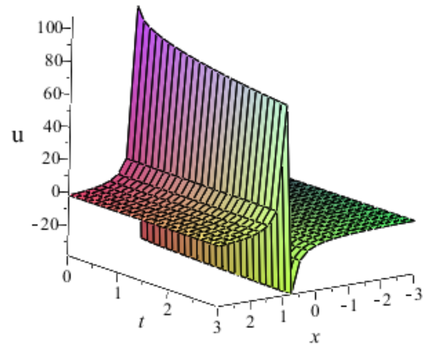

FIGURE 2. Graphical of singular wave solution of Eq. (17) where k_1 =-1, $k_2 = k_3 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

4.2. **Application of GHEFM.** Substituting Eq. (8) into Eq. (12) when p=q=1 and c=d=0. The algebraic equations for a_0, a_1, r_0 and r_1 can be solved by setting the coefficients of powers of $\phi(\eta)$ to zero $\phi(\eta)$ with the aid of Maple. We obtain the following solutions organized in the set:

$$a_1 = \frac{a_0 r_1 (k_2(g_1+g_2)+k_3(g_4+g_3)) - 12k_1(r_1^2 A(k_2+k_3) - r_0 r_1 B(k_2+k_3) + r_0^2 C(k_2+k_3))}{r_0(k_2(g_1+g_2)+k_3(g_3+g_4))},$$
$$w = k_1^2(4AC-B^2)(k_3+k_2), \quad a_0, r_0, \text{ and } r_1 \text{ are arbitrary} \tag{19}$$

Now substituting the values of a_0, a_1, r_0 and r_1 into Eq.(3) yields

Case I:

$$u_1 = \frac{2a_0 C + a_1(-B + \sqrt{4AC-B^2}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\eta+d_0))))}{2r_0 C + r_1(-B + \sqrt{4AC-B^2}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\eta+d_0))))}, \tag{20}$$

where $a_1 = \frac{a_0 r_1 (k_2(g_1+g_2)+k_3(g_4+g_3)) - 12k_1(r_1^2 A(k_2+k_3) - r_0 r_1 B(k_2+k_3) + r_0^2 C(k_2+k_3))}{r_0(k_2(g_1+g_2)+k_3(g_3+g_4))}$ and $\eta = k_1 x + k_2 y + k_3 z - k_1^2(4AC-B^2)(k_3+k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

Case 2:

$$u_2 = \frac{a_0 C \exp(B\eta + Bd_0) - a_1 B \exp(B\eta + Bd_0) - 1}{r_0 C \exp(B\eta + Bd_0) - r_1 B \exp(B\eta + Bd_0) - 1}, \tag{21}$$

where $a_1 = \frac{a_0 r_1 (k_2(g_1+g_2)+k_3(g_4+g_3)) + 12k_1(r_0 r_1 B(k_2+k_3) - r_0^2 C(k_2+k_3))}{r_0(k_2(g_1+g_2)+k_3(g_3+g_4))}$ and $\eta = k_1 x + k_2 y + k_3 z + k_1^2 B^2(k_3 + k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

Case 3:

$$u_3 = \frac{a_0 - a_1(\frac{A}{B} - \frac{1}{B}\exp(B\eta))}{r_0 - r_1(\frac{A}{B} - \frac{1}{B}\exp(B\eta))}, \tag{22}$$

where $a_1 = \frac{a_0 r_1 (k_2(g_1+g_2)+k_3(g_4+g_3)) - 12k_1(r_1^2 A(k_2+k_3) - r_0 r_1 B(k_2+k_3))}{r_0(k_2(g_1+g_2)+k_3(g_3+g_4))}$ and $\eta = k_1 x + k_2 y + k_3 z + k_1^2 B^2(k_3 + k_2)\frac{t^\alpha}{\Gamma(\alpha)}$.

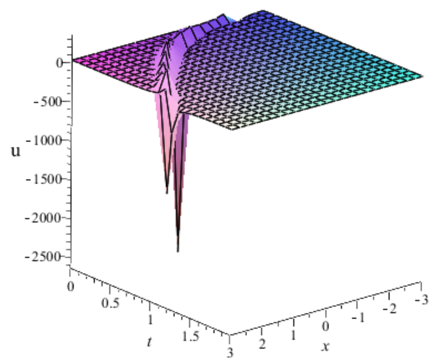

FIGURE 3. Graphical of singular kink wave solution of Eq. (18) where $k_1 = k_2 = k_3 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

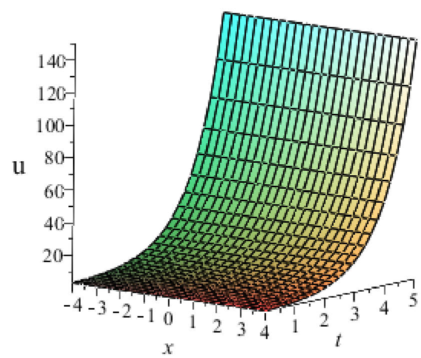

FIGURE 4. Graphical of kink wave solution of Eq. (20) where $k_3 =0.1$, $k_1 = k_2 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

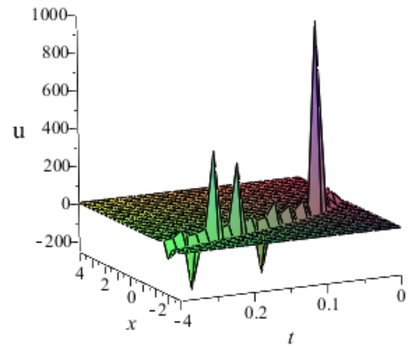

FIGURE 5. Graphical of multi-soliton solution of Eq. (21) where $k_1 =-1$, $k_3 = k_2 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

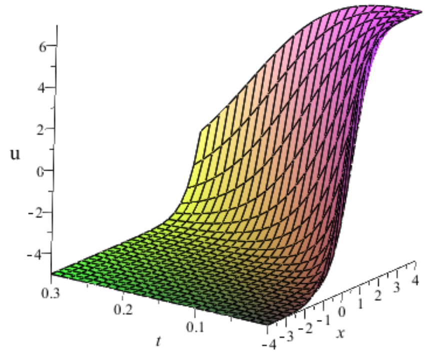

FIGURE 6. Graphical of anti-kink wave solution of Eq. (22) where $k_1 = -1$, $k_3 = k_2 = g_1 = g_2 = g_3 = g_4 = y = z = 1$.

5. Discussion

In this section, the graphical representation of the found solutions is presented together with a discussion of their physical meaning. From an application perspective, traveling waves are highly intriguing, regardless of whether their solution expressions are explicit or implicit.

According to the obtained solutions, the solutions seems to be dependent on values of the magnitude $4AC - B^2$.

In Fig. 1, the behavior of solution (16) is shown in terms of the trigonometric tangent function in the case of magnitude $4AC - B^2 > 0$, and thus results in a periodic wave solution.

Solution (17) is shown in Fig. 2 as a single wave where $4AC - B^2 < 0$. In Fig. 3, the behavior of solution (18) takes on kink waves, and this type appears when the constants A, B and c are the magnitude $4AC - B^2 < 0$. In another direction, solution (20) shows a different behavior as the wrinkle wave moves upwards as in Fig. 4. The dynamics of the solution (21) in Fig. 5 is shown in the form of a multi-soliton solution. Finally, Fig. 6 describes the properties of solution (22) in the form of anti-kink wave solution.

6. Conclusion

In this investigation, we successfully obtained a new different exact solution for the time-fractional CBS equation by using GKM and GEFM. We firstly used reduced fractional similarity transformation to reduce the time-fractional of partial differential equation to an ordinary differential equation. GKM and GEFM depend on replacing $exp(\eta)$ by arbitrary function $\phi(\eta)$ which satisfied Riccati equation to obtain many different solutions for PDE. The exact solution based on Riccati equation which has a many type of solutions.

Acknowledgment: The author would like to thank the Deanship of Scientific Research, Majmaah University Saudi Arabia for supporting this work under project No. R-2025-2121.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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