CHAPTER 8

BOUSSINESQ TYPE EQUATIONS WITH HIGH ACCURACY IN DISPERSION AND NONLINEARITY

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Abstract

Two sets of Boussinesq type equations with high accuracy in dispersion as well as in nonlinearity are presented. The first set, which is expressed in terms of the depth-averaged velocity, includes up to fifth-derivative terms in the momentum equation, while the second set, which is expressed in terms of the velocity at an arbitrary z-level, includes up to third-derivative terms in the continuity equation as well as in the momentum equation. Both sets of equations provide linear dispersion characteristics, which are accurate for wave numbers (kh) up to 6, and nonlinear characteristics which are superior to previous Boussinesq formulations. The high quality of dispersion is also achieved for the Doppler shift in connection with wave-current interaction. A numerical model based on the new equations in two horizontal dimensions is presented and verified with respect to nonlinear transformation of waves in shallow water and refraction-diffraction in deep and shallow water.

1. Introduction

The classical Boussinesq equations as formulated by e.g. Peregrine (1967) are known to incorporate only weak dispersion and weak nonlinearity. For many applications the weak dispersion is the most critical limitation and it has achieved considerable attention in the last 5 years, where a number of alternative lower order Boussinesq type equations have been presented with the purpose of improving the linear dispersion characteristics (see e.g. Madsen et al., 1991; Nwogu, 1993; Schäffer and Madsen, 1995). It has been demonstrated that the accuracy of the dispersion for larger wave numbers is sensitive to the choice of velocity equa-

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tions, and with minor modifications the lower order Boussinesq type equations can incorporate significantly improved dispersion characteristics. A similar improvement of the nonlinear properties is more difficult to obtain and will be adressed in this work.

This paper presents two sets of Boussinesq type equations with high accuracy in dispersion (μ) as well as in nonlinearity (ϵ). The first set, which is expressed in terms of the depth-averaged velocity, includes dispersive terms of order μ^4 and nonlinear terms up to order $\epsilon^5\mu^4$ (Chapter 3). The second set, which is expressed in terms of the velocity at an arbitrary z-level, includes dispersive terms of order μ^2 and nonlinear terms up to order $\epsilon^3\mu^2$ (Chapter 4). Using the technique suggested by Madsen et al. (1991) and Schäffer and Madsen (1995) we enhance the new equations and obtain excellent linear dispersion characteristics corresponding to a Pade [4,4] expansion of linear Stokes theory. A Fourier analysis also demonstrates that the accuracy of the nonlinear energy transfer is improved considerably compared to previous Boussinesq formulations. This allows for a much more accurate description of wave-wave interactions in irregular wave trains. Finally, it turns out that the high quality of dispersion is also achieved for the Doppler shift in connection with wave-current interaction and it allows for a study of wave-blocking due to opposing currents. These aspects will be studied in a companion paper at this conference by Chen et al. (1996).

A numerical model based on the new equations from Chapter 4 is presented in Chapter 5 and it is verified with respect to nonlinear transformation of waves in shallow water and refraction-diffraction in deep and shallow water.

2. Derivation of Boussinesq type equations

In the following presentation the adopted coordinate system is Cartesian with the x'-axis and y'-axis located at the still water level (SWL) and with the z'-axis pointing vertically upwards. The fluid domain is bounded by the sea bed at $z'=-h'(x',y')$ and the free surface at $z'=\eta'(x',y',t')$. Non-dimensional variables (denoted without primes) are introduced in the conventional way (see e.g. Nwogu, 1993) by the use of a characteristic water depth (h_0), wave length (l_0) and wave amplitude (a_0) and we introduce the classical measures of nonlinearity and frequency dispersion by $\epsilon=a_0/h_0$ and $\mu=h_0/l_0$.

With the usual assumptions of irrotational flow in an incompressible fluid, the nondimensional form of the governing equations and boundary conditions read:

$$\Phi_{zz} + \mu^2\nabla^2\Phi = 0 \,, \quad -h<z<\epsilon\eta \tag{1a}$$

$$\Phi_z + \mu^2\nabla h\cdot\nabla\Phi = 0 \,, \quad z = -h \tag{1b}$$

$$\Phi_t + \eta + \frac{\epsilon}{2}\left((\nabla\Phi)^2 + \frac{1}{\mu^2}(\Phi_z)^2\right) = 0\,, \quad z = \epsilon\eta \tag{1c}$$

$$\Phi_z - \mu^2\left(\eta_t + \epsilon\nabla\eta\cdot\nabla\Phi\right) = 0\,, \quad z = \epsilon\eta \tag{1d}$$

where Φ is the velocity potential and ∇ the horizontal gradient operator.

The basic idea in Boussinesq-type derivations is to reduce the three dimensional description to a two-dimensional one and this is achieved by expanding the velocity potential as a power series in the vertical coordinate:

$$\Phi(x,y,z,t) = \sum_{n=0}^{\infty} z^n \Phi^{(n)}(x,y,t) \tag{2}$$

by which $\Phi^{(0)}(x,y,t)=\Phi(x,y,0,t)$. While traditional Boussinesq theory assumes $\mu<<1$ and $\epsilon=O(\mu^2)$, the present expansion allows for arbitrary ϵ.

The individual steps in the derivation of Boussinesq-type equations are as follows: Firstly, the velocity potential is determined in terms of spatial derivatives of $\Phi^{(0)}$ by combining (2) with (1a) and (1b). By the use of the gradient operator this also defines the horizontal velocity vector in terms of the velocity, $\hat{u}$ at the still water level. Secondly, the velocity potential is inserted in the dynamic free surface condition (1c), and by using the horizontal gradient operator a momentum equation is derived in terms of $\hat{u}$. Thirdly, the horizontal velocity vector expressed in terms of $\hat{u}$ is substituted into the depth-integrated continuity equation. The resulting equations in terms of $\hat{u}$ can be found in Madsen & Schäffer (1996) and will not be given here.

3. Equations in terms of the Depth-Averaged Velocity

Traditionally, Boussinesq models are not based on equations formulated in terms of the velocity at the still water level. This is partly because of the rather complicated form of the continuity equation expressed in this variable and partly because of the relatively poor dispersion characteristics of these equations. A more common choice is the depth-averaged velocity U which is defined by

$$U \equiv \frac{1}{h+\epsilon\eta}\int_{-h}^{\epsilon\eta} u\,dz \tag{3}$$

One of the obvious advantages of using this variable is that the continuity equation becomes exact and relatively simple,

$$\eta_t + \nabla \cdot \left((h + \epsilon\eta) U \right) = 0 \tag{4}$$

3.1. Formulation of momentum equations

In order to formulate the momentum equation in terms of U we use the procedure as follows: Firstly, a relation in which U is expressed in terms of û is established by the use of (2) and (3). Secondly, this relation is inverted into a relation in which û is expressed in terms of U, by the use of successive substitutions starting at lowest order in μ^2. Now û can be eliminated from the momentum equation and replaced by functions of U. The resulting higher order momentum equations truncated at the order μ^6 take the form of

$$U_t + \nabla\eta + \frac{\epsilon}{2}\nabla(U^2) + \mu^2 \boldsymbol{\Gamma}_2^I + \mu^4 \boldsymbol{\Gamma}_4^I = O(\mu^6) \tag{5}$$

where

$$\boldsymbol{\Gamma}_2^I \equiv \left[\boldsymbol{\Lambda}_{20}^I + \epsilon \boldsymbol{\Lambda}_{21}^I + \epsilon^2 \boldsymbol{\Lambda}_{22}^I + \epsilon^3 \boldsymbol{\Lambda}_{23}^I \right] \tag{6a}$$

$$\boldsymbol{\Gamma}_4^I \equiv \left[\boldsymbol{\Lambda}_{40}^I + \epsilon \boldsymbol{\Lambda}_{41}^I + \epsilon^2 \boldsymbol{\Lambda}_{42}^I + \epsilon^3 \boldsymbol{\Lambda}_{43}^I + \epsilon^4 \boldsymbol{\Lambda}_{44}^I + \epsilon^5 \boldsymbol{\Lambda}_{45}^I \right] \tag{6b}$$

Notice that $\boldsymbol{\Lambda}_{mn}^I$ is used to express the Boussinesq terms, where subscript m accounts for the power of μ (dispersion) and subscript n for the power of ϵ (nonlinearity). The equations include full nonlinearity up to the truncated order of dispersion, i.e. retaining $\epsilon^3\mu^2$ and $\epsilon^5\mu^4$-terms, and involve higher order spatial derivatives incl. third and fifth-derivative terms. The actual expressions for the $\boldsymbol{\Lambda}_{mn}^I$ -terms can be found in Madsen & Schäffer (1996) and will not be given here. We note that if only terms up to the order $O(\epsilon,\mu^2)$ are retained we obtain the classical Boussinesq equations by Peregrine (1967) and if terms of order $O(\epsilon\mu^2,\mu^4)$ are retained as well we obtain the higher order Boussinesq equations by Dingemans (1973).

The final step in the derivation procedure is to apply the technique introduced by Madsen et al. (1991) and Schäffer & Madsen (1995) for improving the dispersion characteristics of (4) and (5). This procedure is illustrated on a horizontal bottom in the following: Firstly, (5) is truncated omitting $O(\mu^4)$, the gradient operator is applied twice and the result is multiplied by $\alpha\mu^2 h^2$, where α is a free parameter of the order O(1):

$$\alpha \mu^2 h^2 \nabla^2 \left(U_t + \nabla\eta + \frac{\epsilon}{2}\nabla(U^2) + \mu^2 \boldsymbol{\Gamma}_2^I \right) = O(\mu^6) \tag{7a}$$

Secondly, (5) is truncated omitting $O(\mu^2)$, the gradient operator is applied four times and the result is multiplied by $\beta\mu^4 h^4$, where β is a free parameter of the order O(1):

$$\beta \mu^4 h^4 \nabla^4 \left(U_t + \nabla\eta + \frac{\epsilon}{2}\nabla(U^2) \right) = O(\mu^6) \tag{7b}$$

We can now consistently modify (5) by subtracting (7a) and adding (7b), which yields an enhanced set of higher order Boussinesq equations truncated at $O(\mu^6)$. The coefficients α and β are yet to be determined.

3.2. Fourier Analysis for weakly nonlinear waves

Although the equations have been derived under the assumption of $\mu << 1$ and $\epsilon = O(1)$, we shall analyse the imbedded linear and nonlinear characteristics by assuming that $\epsilon < 1$ while μ is arbitrary. We look for analytical solutions of the form

$$\eta = a_1 \cos(\omega t - kx) + \epsilon a_2 \cos(2\omega t - 2kx) \tag{8a}$$

$$U = U_1 \cos(\omega t - kx) + \epsilon U_2 \cos(2\omega t - 2kx) \tag{8b}$$

At first order (ϵ^0) non-trivial solutions require the dispersion relation

$$\frac{\omega^2}{k^2 h} = \frac{1 + \alpha k^2 h^2 + \beta k^4 h^4}{1 + \left(\alpha + \frac{1}{3}\right) k^2 h^2 + \left(\beta + \frac{\alpha}{3} - \frac{1}{45}\right) k^4 h^4} \tag{9}$$

which should be compared with Stokes relation for linear waves on arbitrary depth, i.e. tanh(kh)/kh. The ratio between the two expressions is shown as a function of kh in Fig 1. If we omit the enhancement of the higher order equations using $(\alpha,\beta) = (0,0)$ the resulting dispersion relation (9) corresponds to a Padé [0,4] expansion in kh of the Stokes relation. For these equations the deviation is significant and in fact a singularity occurs for kh=4.2. This singularity shows up in numerical calculations as an instability even in the case of initially calm water, and actually makes (5) quite useless without the enhancement. On the other hand, by using the enhanced equations incl. (7a-b) with $\alpha = 1/9$ and $\beta = 1/945$ the resulting dispersion relation (9) corresponds to a Padé [4,4] expansion. This is an extremely good approximation to the exact linear relation even for kh as large as 6. The dispersion relation of lower order Boussinesq equations is obtained by ignoring the k^4h^4 terms in (9). With $\alpha = 0$ (Padé [0,2]) this corresponds to the classical equations of Peregrine (1967) and with $\alpha = 1/15$ it corresponds to the Padé

[2,2] formulation introduced by Madsen et al. (1991). Both cases are shown as a reference in Fig 1.

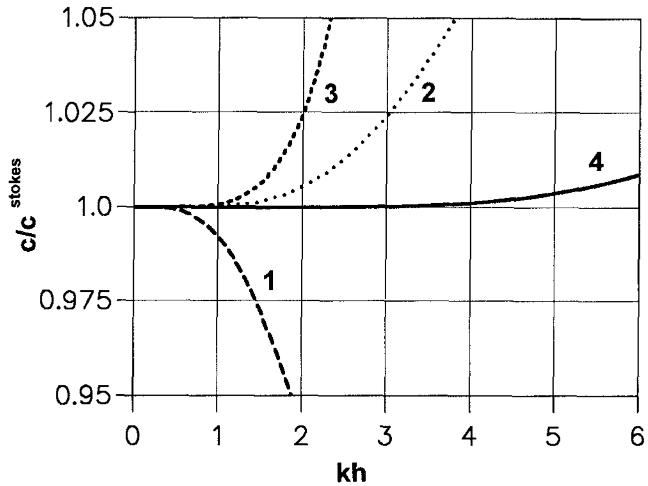

Fig 1 Relative phase celerity, c/c^{Stokes}, for various forms of the Boussinesq equations. 1: Padé [0,2]; 2: Padé [2,2]; 3: Padé [0,4]; 4: Padé [4,4].

Extending the Fourier analysis to second order we determine a_2 in terms of a_1^2/h times a transfer function. In this respect the target solution is

$$a_2^{Stokes} = \frac{1}{4}\frac{a_1^2}{h} kh \coth(kh) \left(3\coth^2(kh) - 1\right) \tag{10}$$

according to Stokes second order theory. Fig 2 shows the variation of a_2/a_2^{Stokes} as a function of kh. The curve corresponding to the new higher order equations is seen to be superior to the results obtained from the equations of Dingemans (1973) and Peregrine (1967).

It is straight-forward to extend the Fourier analysis to second order sub-harmonics and super-harmonics and the results can be found in Madsen and Schäffer (1996).

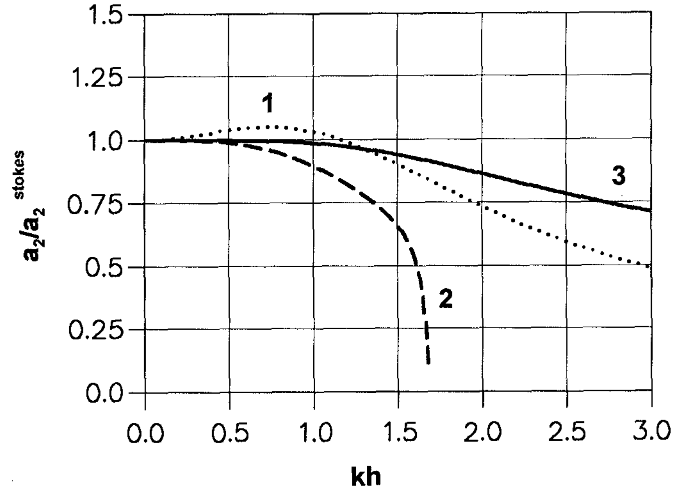

Fig 2 Ratio of second harmonic, a_2/a_2^{Stokes}. 1: Peregrine (1967); 2: Dingemans (1973) 3: Present e.g. eqs. (4) & (5) with (7a-b), $\alpha=1/9$, $\beta=1/945$.

3.3 Ambient currents and Doppler shift

Yoon and Liu (1989) introduced separate scaling of waves and currents and showed that additional terms were to be added to the classical formulation of Peregrine (1967) if a correct Doppler shift was to be obtained in connection with ambient currents. In the present work we analyse the new equations for the case of a strong but constant ambient current U_c and obtain the following dispersion relation,

$$\frac{(\omega - kU_c)^2}{k^2 h} = \frac{1 + \alpha k^2 h^2 + \beta k^4 h^4}{1 + \left(\alpha + \frac{1}{3}\right) k^2 h^2 + \left(\beta + \frac{\alpha}{3} - \frac{1}{45}\right) k^4 h^4} \tag{11}$$

This provides a correct Doppler shift including Padé [4,4] dispersion characteristics. Fig. 3 shows lines of 2 per cent wave number errors as a function of (F, h/L_0), F being the Froude number ($U_c/\sqrt{gh}$) of the current and L_0 being the deep water wave length for the case of no currents. The application range of the Padé [4,4] curve is seen to be superior to the Padé [0,2] corresponding to Yoon & Liu's formulation as shown for comparison.

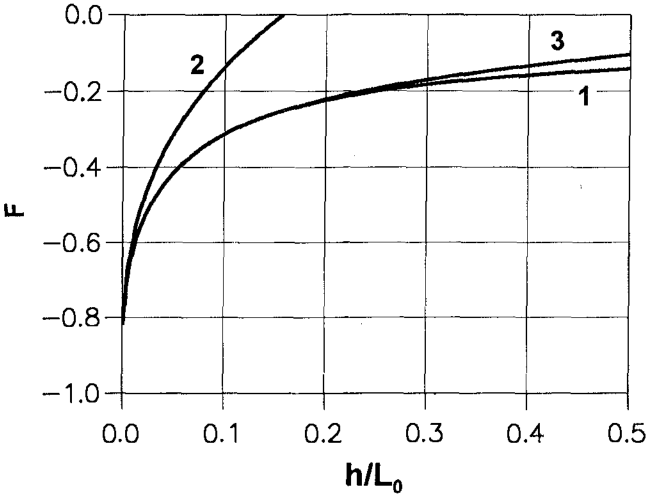

Fig 3 Waves on ambient currents. 1: Blocking curve according to Stokes theory. Tracks of 2 per cent wave number errors, $(k\text{-}k^{Stokes})/k^{Stokes}$: 2: Yoon & Liu (1989), Padé [0,2]; 3: New equations (11), Padé [4,4]

4. Equations in terms of the velocity at an arbitrary z-location

In the previous section we have demonstrated that highly accurate linear dispersion and nonlinear characteristics can be obtained by formulating higher order equations including up to fifth-derivative terms in the momentum equation. In this section we shall, however, show that almost the same accuracy can be obtained with only third-derivative terms, if the lower order equations are formulated in terms of the velocity vector at an arbitrary z-location i.e.

$$\tilde{u} \equiv u(x,y,\tilde{z},t) \tag{12}$$

This variable was introduced by Nwogu (1993).

From the expression derived for the velocity potential we can establish an expression for $\tilde{u}$ in terms of $\hat{u}$. This relation is then inverted into a relation in which $\hat{u}$ is expressed in terms of $\tilde{u}$, by the use of successive substitutions starting at lowest order in μ^2. Now $\hat{u}$ can be eliminated from the original mass and momentum equations and replaced by functions of $\tilde{u}$ and the resulting equations truncated at the order μ^4 take the form of

$$\eta_t + \nabla\cdot\left(\tilde{u}(h+\epsilon\eta) + \mu^2\Gamma_2^{II}\right) = O(\mu^4) \tag{13a}$$

$$\tilde{u}_t + \nabla\eta + \frac{\epsilon}{2}\nabla(\tilde{u}^2) + \mu^2\Gamma_2^{III} = O(\mu^4) \tag{13b}$$

where

$$\Gamma_2^{II} \equiv \left[\Lambda_{20}^{II} + \epsilon\Lambda_{21}^{II} + \epsilon^2\Lambda_{22}^{II} + \epsilon^3\Lambda_{23}^{II}\right] \tag{14a}$$

$$\Gamma_2^{III} \equiv \left[\Lambda_{20}^{III} + \epsilon\Lambda_{21}^{III} + \epsilon^2\Lambda_{22}^{III} + \epsilon^3\Lambda_{23}^{III}\right] \tag{14b}$$

These equations, which were first derived by Wei et al. (1995), include full nonlinearity up to the truncated order of dispersion i.e. retaining $\epsilon^3\mu^2$. If only terms up to the order $O(\epsilon,\mu^2)$ are retained we obtain the equations of Nwogu (1993). With a specific choice of the z-location defining the velocity variable, Nwogu and Wei et al. achieved Padé [2,2] dispersion characteristics.

Here we shall further enhance the equations (13a-b) to improve dispersion as well as nonlinearity. Again we apply the technique as described in section 3 and illustrate the procedure on a horizontal bottom: Firstly, (13a & b) are truncated omitting $O(\mu^2)$, the gradient operator is applied twice and the results are multiplied by $\mu^2 h^2$ and two free parameters (α,β) which are of the order O(1):

$$\alpha\,\mu^2 h^2 \nabla^2\left(\tilde{u}_t + \nabla\eta + \frac{\epsilon}{2}\nabla(\tilde{u}^2)\right) = O(\mu^4) \tag{15a}$$

$$\beta\,\mu^2 h^2 \nabla^2\left(\eta_t + \nabla\cdot(\tilde{u}(h+\epsilon\eta))\right) = O(\mu^4) \tag{15b}$$

We can now consistently modify (13a-b) by subtracting (15a-b), which yields an enhanced set of lower order Boussinesq equations. The detailed formulation is given in Madsen & Schäffer (1996).

The linear dispersion relation of the enhanced equations reads

$$\frac{\omega^2}{k^2 h} = \frac{1 + \left(\alpha + \beta - \gamma - \frac{1}{3}\right)k^2h^2 + \alpha\left(\beta - \gamma - \frac{1}{3}\right)k^4h^4}{1 + (\alpha + \beta - \gamma)k^2h^2 + \beta(\alpha - \gamma)k^4h^4} \tag{16}$$

where

$$\gamma \equiv \frac{\tilde{z}}{h} + \frac{1}{2}\left(\frac{\tilde{z}}{h}\right)^2 \tag{17}$$

As shown by Schäffer & Madsen (1995) Padé [4,4] characteristics can be obtained by choosing one of four different sets of parameters. It turns out that one of these sets is superior with respect to nonlinear properties, hence we recommend the parameter set

$$(\gamma,\beta,\alpha) = \left(\frac{-3-\sqrt{23/35}-2\sqrt{19/7}}{18}\,,\ \frac{28-2\sqrt{133}}{126}\,,\ \frac{105-3\sqrt{805}}{1890}\right) \tag{18}$$

Extending the Fourier analysis to second order we determine a_2 in terms of $a_1{}^2/h$ times a transfer function. Fig 4 shows the variation of $a_2/a_2{}^{Stokes}$ as a function of kh. We notice that the curve corresponding to the new enhanced equations is superior to the results obtained from Nwogu's and Wei et al's equations as shown for comparison. A full analysis of second order sub-harmonics and super-harmonics can be found in Madsen and Schäffer (1996).

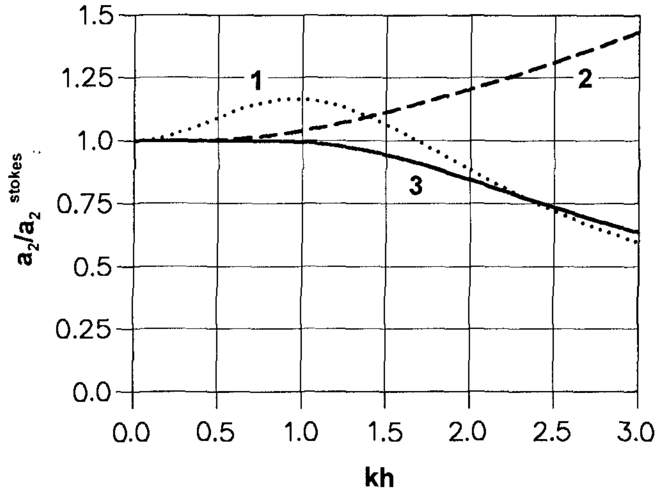

Fig 4 Ratio of second harmonic, $a_2/a_2{}^{Stokes}$. 1: Nwogu (1993); 2: Wei et al. (1995); 3: Present e.g. eqs. (13a-b) with (15a-b) & (18).

5. Numerical model and its verification

A numerical model has been developed to solve the two-dimensional equations formulated in Chapter 4. The equations are discretized in space by applying higher order central-differencing with the variables defined on a space-staggered rectangular grid while the temporal integration is performed by using a fourth order Adams-Bashforth-Moulton predictor-corrector method. More details about the numerical method can be found in Banijamali et al. (1997).

5.1 Wave transformation over a submerged bar

One of the most demanding tests for Boussinesq-type models is the study of wave transformation over a submerged bar. In this case nonlinearity increases considerably during the propagation at the upward slope and results in energy transfer to the higher harmonics. As long as the depth is decreasing the higher harmonics will be bound or phase-locked to the primary wave train, but on the downward slope the harmonics will be released and propagate as free waves. This introduces the pecularity that a linear regular shallow water wave will be converted into a linear irregular deep water wave after the passage over the bar. This situation calls for highly accurate dispersion characteristics and for this reason most Boussinesq models fail to predict the process.

Beji and Battjes (1993) and Luth et al. (1994) presented a series of accurate measurements of wave transformation over a trapezoidal bar with an upward slope of 1/20, a downward slope of 1/10, a depth of 40 cm on both sides of the bar and 10 cm on top of the bar (Fig 6a). The data have previously been used in an intercomparison study in MAST-G8M, see Dingemans (1994). As one example from this test series we have selected the case of a wave period of 2.02s and a wave height of 2.0 cm.

Fig 5 shows the measured time series of surface elevations at three locations: We notice the transformation from a sinusoidal, linear-wave profile at x=5.2m, to a profile of a strongly nonlinear wave at x=13.5m and back to a profile of a fairly linear wave at x=19.0m, where the significant frequency obviously has been doubled.

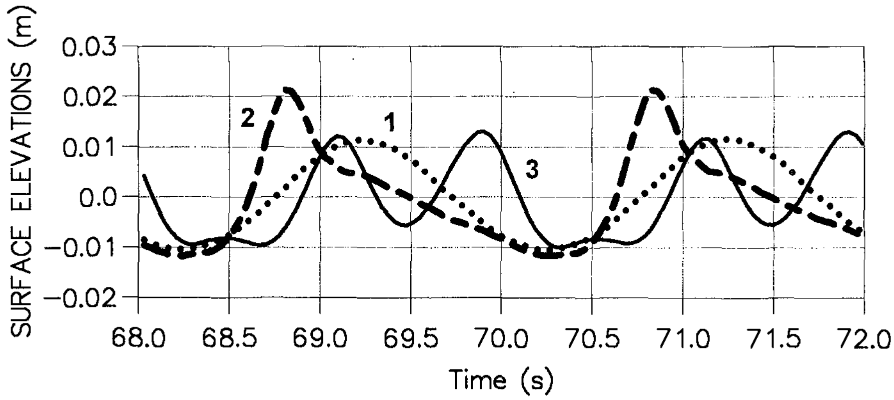

Fig 5 Harmonic generation over a submerged bar.
Measured timeseries of surface elevations at three locations
Input: wave period=2.02s, wave height=0.02m, 1: x=5.2m;
2: x=13.5m and 3: x=19.0 m.

The energy transformation to higher harmonics is in fact seen more clearly in Fig 6b-c, which is based on FFT analysis of time series from a numerical solution of the Boussinesq equations. Here we clearly notice the rapid growth of the second (and third) harmonics at the upward slope and the release of these harmonics after the bar.

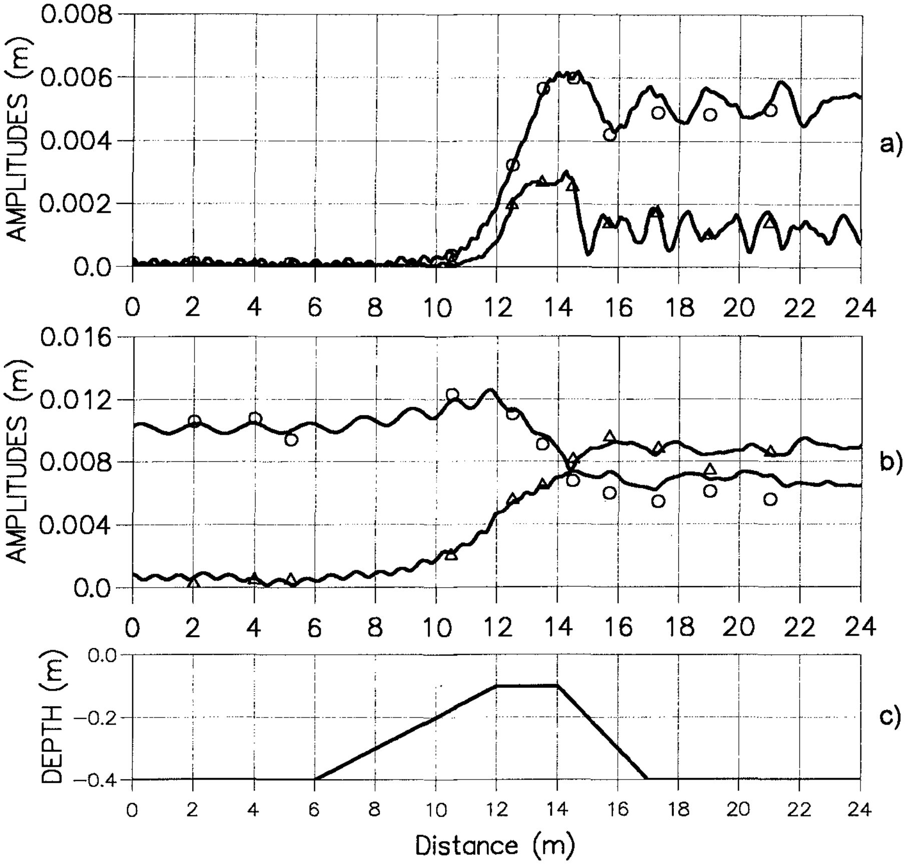

Fig 6 Harmonic generation over a submerged bar
a) Spatial evolution of harmonics (third & fourth); b) Spatial evolution of harmonics (first & second); c) Bathymetry
Markers are measurements.

In Fig 7 we compare the measured time series at x=21.0 m with the numerical results corresponding to two different versions of the Boussinesq model: One using $(\gamma,\alpha,\beta)=(-2/5,0,0)$ leading to Padé [2,2] dispersion characteristics (corresponding to the model of Wei et al.,1995) and one using the parameter set of (18) leading to Padé [4,4] characteristics.

The latter is seen to be superior and it provides a highly accurate result.

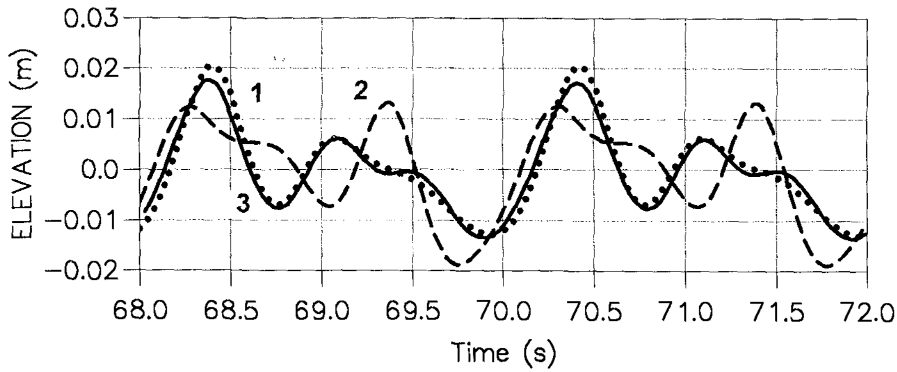

Fig 7 Computed and measured surface elevations at x=21m
1: Eqs. (13a-b) with (15a-b) & (18), Padé [4,4] 2: Wei et al. (1995), Padé [2,2] 3: Measurements

5.2 Nonlinear refraction-diffraction

As a second demanding test for the Boussinesq model we study nonlinear refraction-diffraction over a semicircular shoal with depth contours varying between 0.4572m and 0.1524m as investigated experimentally by Whalin (1971).

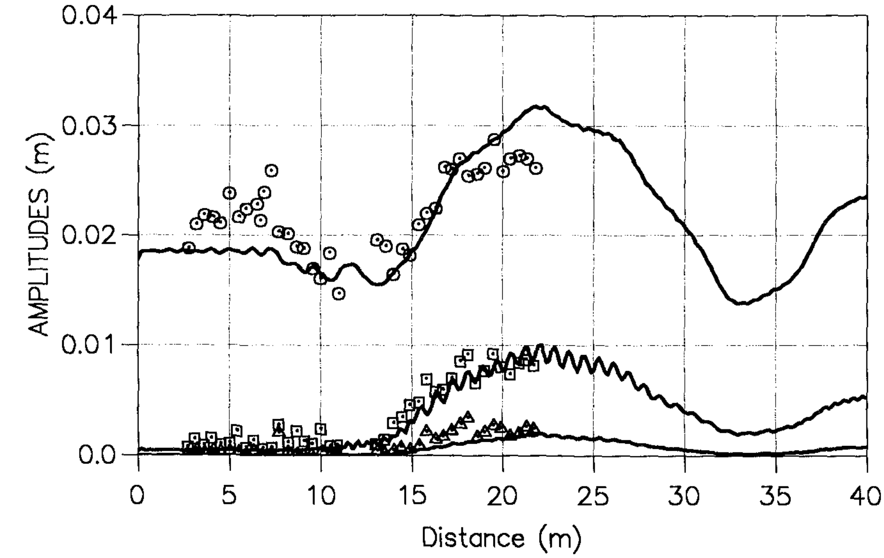

Fig 8 Whalin's nonlinear refraction-diffraction. Spatial evolution of first and second harmonics along the centreline. Input: wave period = 1.0s, wave height = 0.039m. Markers are measurements.

In the present work we focus on the case of incoming regular waves with period 1.0s and wave height 0.039m. At the boundary the waves are linear but after the focusing on the shoal higher harmonics become significant due to nonlinear effects. An FFT analysis of time series in each grid point along the centreline has been computed and the resulting spatial evolution of first and second harmonics is compared with Whalin's experimental data in Fig 8. The agreement is found to be most acceptable.

Acknowledgement

This work has been financed by the Danish National Research Foundation. Their support is greatly appreciated.

References

Beji, S. and J.A. Battjes (1993), "Experimental investigation of wave progation over a bar", Coastal Engineering, Vol. 19, pp 151-162.

Banijamali, B., O.R. Sørensen, and P.A. Madsen (1997),"A numerical model for enhanced Boussinesq type equations", In preparation.

Chen, Q., P.A. Madsen, O.R. Sørensen, and D. Basco (1996),"Boussinesq type equations with improved Doppler shift for wave/current interaction", ICCE, Orlando 1996.

Dingemans, M. (1973)," Water waves over an uneven bottom; a discussion of long-wave equations". Delft Hydraulics Report R729 part 2, 88p.

Dingemans, M. (1994),"Comparison of computations with Boussinesq-like models and laboratory measurements", MAST-G8M note, H1684, Delft Hydraulics, 32pp.

Luth, H.R., G. Klopman, and N. Kitou (1994), "Project 13G: Kinematics of waves breaking partially on an offshore bar; LDV measurements for waves with and without a net onshore current". Delft Hydraulics Report H1573, March 1994, 40pp.

Madsen, P.A., R. Murray and O.R. Sørensen (1991), "A new form of the Boussinesq equations with improved linear dispersion characteristics. Part 1", Coastal Engineering, Vol 15, pp 371-388.

Madsen, P.A. and H.A. Schäffer (1996), "Higher order Boussinesq-type equations - Derivation and Analysis". Submitted for publication, 80 pp.

Nwogu, O. (1993),"Alternative form of Boussinesq equations for nearshore wave propagation",J. WatWays, Port, Coastal Ocean Engrg., ASCE, 119(6), pp 618-638.

Peregrine, D.H. (1967),"Long waves on a beach", Journal of Fluid Mechanics, Vol 27, pp 815-827.

Schäffer, H.A. and P.A. Madsen (1995), "Further enhancements of Boussinesq-type equations". Coastal Engineering, Vol 26, pp 1-14.

Yoon, S.B. and P.L.-F. Liu (1989), "Interactions of currents and weakly nonlinear water waves in shallow water", Journal of Fluid Mechanics, Vol 205, pp 397-419.

Wei, G., J.T. Kirby, S.T. Grilli, and R. Subramanya (1995), "A fully nonlinear Boussinesq model for surface waves", Journal of Fluid Mechanics, Vol 294, pp 71-92.

Whalin, R.W. (1971), "The limit of applicability of linear wave refraction theory in a convergence zone", Res. Rep. H-71-3, U.S. Army Corps of Engineers, Waterways Expt. Station, Vicksburg, MS. 1006-01/9612.ulb/ORS