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Coefficient Problems for Bi-Univalent Functions via q–Rabotnov Kernels and q–Fibonacci Subordination

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Abstract. Motivated by the interplay between q–calculus and geometric function theory, this paper introduces and investigates a new subclass of bi-univalent functions associated with shell-like domains generated through the q–Rabotnov function and the q–analogue of Fibonacci numbers. A central contribution of this work is the definition of a novel q–derivative operator, constructed via convolution with kernels involving the q–Rabotnov function. Employing the subordination principle, we derive sharp coefficient estimates for the initial Taylor–Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, and establish Fekete–Szegö-type inequalities for the proposed class. The results obtained here unify and extend several recent contributions in the theory of bi-univalent functions, while also highlighting the role of q–special functions in generating new analytic structures. These findings enrich the structural understanding of bi-univalent functions and suggest future directions involving operator theory, convolution structures, and further applications of q–calculus in complex analysis.

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1. Introduction

Geometric function theory has long been recognized as a fertile area of complex analysis, focusing on the structural, geometric, and analytic properties of functions that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A central theme in this field is the study of subclasses of analytic and bi-univalent functions, which often

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arise through subordination, convolution operators, or fractional-calculus techniques. Classical problems such as estimating initial coefficients, growth and distortion theorems, and Fekete–Szegö type inequalities remain at the core of ongoing investigations, and their generalizations via q-calculus have opened new avenues for research.

The advent of q-calculus, sometimes referred to as the calculus of finite differences, has significantly enriched analytic function theory by providing a powerful framework for developing q-analogues of well-known operators and function classes. Through this approach, several subclasses with deep geometric and algebraic structures have been introduced and analyzed. In particular, the q-calculus has established strong links with special functions, combinatorics, and orthogonal polynomials, thus extending the applicability of classical geometric function theory to discrete and fractional domains. This versatility underscores its role in the advancement of both theoretical and applied perspectives (see, e.g., [1–21]).

The q–gamma function Γ_q, regarded as the natural q–analogue of the Euler gamma function, is a cornerstone of the modern q–calculus and plays a fundamental role in the construction of analytical operators. It is defined recursively (see [22, 23]) by

$$\Gamma_q(\kappa+1) \;=\; \frac{1-q^{\kappa}}{1-q}\,\Gamma_q(\kappa) = [\kappa]_q\,\Gamma_q(\kappa), \tag{1}$$

where the q–integer $[\kappa]_q$ is given by

$$[\kappa]_q = \begin{cases} \dfrac{1-q^{\kappa}}{1-q}, & 0<q<1,\ \kappa\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}, \\ 1, & q\mapsto 0^+,\ \kappa\in\mathbb{C}^*, \\ \kappa, & q\mapsto 1^-,\ \kappa\in\mathbb{C}^*, \\ \displaystyle\sum_{n=0}^{\gamma-1} q^n, & 0<q<1,\ \kappa=\gamma\in\mathbb{N}. \end{cases}$$

This formulation shows that Γ_q not only preserves the essential structural features of the classical gamma function, but also incorporates the discrete deformation encoded by the parameter q. Consequently, it provides a unifying framework in which fractional-order operators and kernel-type generating functions can be generalized and studied within analytic function theory.

Closely associated with Γ_q is the q–analogue of the Pochhammer symbol, or q–shifted factorial, defined by (see [23])

$$(\kappa;q)_n = \begin{cases} (1-\kappa)(1-\kappa q)\cdots(1-\kappa q^{n-1}), & n=1,2,3,\ldots, \\ 1, & n=0, \end{cases}$$

which admits the representation;

$$(\kappa;q)_n \;=\; \frac{(1-q)^n\,\Gamma_q(\kappa+n)}{\Gamma_q(\kappa)}, \qquad n>0.$$

This identity highlights the intrinsic connection between the q–shifted factorial and the q–gamma function, a relationship that underpins many of the convolution and subordination operators employed in geometric function theory. In particular, it serves as a fundamental building block in the analytic modeling of q–extensions of bi-univalent function classes.

In parallel, Rabotnov-type kernels, first introduced by Rabotnov [24] within the framework of linear viscoelasticity, have proven to be indispensable tools for modeling hereditary phenomena such as creep and relaxation. Expressed in terms of convolution operators involving Mittag–Leffler-type functions, these kernels provide a rigorous representation of fractional-order operators in constitutive equations [25]. Their remarkable flexibility has made them standard in the mathematical modeling of stress–strain relations with memory effects in mechanics and engineering. Moreover, their intrinsic connection with fractional calculus places them as a cornerstone in the analysis of viscoelastic materials and dynamical systems [26], and some applications can be found in [1, 2, 4, 27–41].

The analytic nature of Rabotnov-type kernels naturally invites their extension into geometric function theory, particularly when combined with the discrete framework of q-calculus. Such an interplay not only bridges fractional viscoelastic models with analytic operator theory but also enables the construction of novel subclasses of analytic and bi-univalent functions. These connections provide a robust mechanism for encoding hereditary behavior and nonlocal operators into analytic settings, thereby allowing classical results—such as coefficient bounds, growth and distortion estimates, and Fekete–Szegö inequalities—to be extended in new directions.

Motivated by these observations, Alsoboh et al. [42–44] recently employed subordination techniques to define a new family of q-starlike functions associated with the q-analogue of Fibonacci numbers. Their construction revealed a fundamental connection between q-Fibonacci numbers $\varkappa_q$ and the associated q-Fibonacci polynomials, expressed through the mapping

$$\Omega(z;q) = \frac{1 + q\varkappa_q^2 z^2}{1 - \varkappa_q\, z - q\varkappa_q^2 z^2}, \tag{2}$$

introduced a new family of q–starlike functions. They also established a fundamental connection between the q–analogue of Fibonacci numbers $\varkappa_q$ and their associated Fibonacci polynomials

$$\varkappa_q = \frac{1 - \sqrt{4q + 1}}{2q}. \tag{3}$$

In particular, they proved that if

$$\Omega(z;q) = 1 + \sum_{n=1}^{\infty} \widehat{p}_n\, z^n,$$

then the coefficients $\widehat{p}_n$ satisfy the recurrence relation

$$\widehat{p}_n = \begin{cases} \varkappa_q, & n = 1, \\ (2q+1)\varkappa_q^2, & n = 2, \\ (3q+1)\varkappa_q^3, & n = 3, \\ \big(\delta_{n+1}(q) + q\,\delta_{n-1}(q)\big)\varkappa_q^n, & s \geq 4, \end{cases} . \tag{4}$$

In the present work, we introduce and investigate a novel subclass of bi-univalent functions generated by the q-Rabotnov function together with the q-analogue of Fibonacci numbers. Using the principle of subordination, we derive coefficient bounds for the initial Taylor–Maclaurin coefficients and establish sharp Fekete–Szegö-type inequalities for the proposed function class. Our results extend several recent frameworks in geometric function theory and build new bridges between fractional viscoelastic modeling, q-calculus, and the analytic theory of bi-univalent functions.

2. Preliminaries

Let $\mathcal{A}$ denote the family of all analytic functions defined on the open unit disk $\mathbb{U}$, where $\mathbb{U}$ is the set of all complex numbers $z = a + ib$ (with $a, b \in \mathbb{R}$) satisfying $|z| < 1$. Geometrically, $\mathbb{U}$ represents the collection of all points in the complex plane that lie strictly inside the unit circle centered at the origin.

The functions $f \in \mathcal{A}$ are normalized to satisfy the following initial conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These normalization conditions ensure that the functions are uniquely determined and facilitate the study of their properties within the unit disk. For every function $f \in \mathcal{A}$, the Taylor-Maclaurin series expansion can be expressed in the following form:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad (z \in \mathbb{U}). \tag{5}$$

An analytic function f that satisfies $|f(z)| < 1$ and $f(0) = 0$ within the domain $\mathbb{U}$ is called a Schwartz function. When considering two functions f_1 and f_2 from $\mathcal{A}$, f_1 is referred to as subordinate to f_2, denoted by $f_1 \prec f_2$, if a Schwarz function g exists such that $f_1(z) = f_2(g(z))$ for all $z \in \mathbb{U}$. Additionally, examine the class S, which includes all functions $f \in \mathcal{A}$ that are univalent (injective) on the unit disk $\mathbb{U}$. Let P represent the collection of functions within $\mathcal{A}$ that possess positive real parts, defined as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots, \tag{6}$$

where

$$|p_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{7}$$

This is in accordance with the renowned Carathéodory's Lemma (for more details, see [45]). Essentially, $\delta \in \mathsf{P}$ if and only if $\varepsilon(z) \prec (1+z)(1-z)^{-1}$ for $z \in \mathbb{U}$.

As the foundation upon which many important subclasses of analytic functions are built, the class P is crucial to the study of analytic functions. For any function f in the subfamily S of $\mathcal{A}$, there exists an inverse function denoted f^{-1} and defined by

$$z = f^{-1}(f(z)) \text{ and } \xi = f(f^{-1}(\xi)), \quad (\mathsf{r}_0(f) \geq 0.25; \ |\xi| < \mathsf{r}_0(f); z \in \mathbb{U}). \tag{8}$$

where

$$\eta(\xi) = f^{-1}(\xi) = \xi - \alpha_2\xi^2 + \left(2\alpha_2^2 - \alpha_3\right)\xi^3 - \left(5\alpha_2^3 + \alpha_4 - 5\alpha_3\alpha_2\right)\xi^4 + \cdots. \tag{9}$$

A function $f \in \mathsf{S}$ is said to be *bi-univalent* if both f and its inverse f^{-1} belong to the class S. The family of all such functions, denoted by Σ, forms a natural and significant subclass of S within the unit disk $\mathbb{U}$. Several classical examples illustrate this concept. For example, $f_1(z) = \dfrac{z}{1+z}$ with its inverse $f_1^{-1}(z) = \dfrac{z}{1-z}$ is a typical element of Σ. Similarly, $f_2(z) = -\log(1-z)$ admits the inverse $f_2^{-1}(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$, while $f_3(z) = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$ has the inverse $f_3^{-1}(z) = \dfrac{e^z-1}{e^z}$. These examples emphasize the structural interplay between a bi-univalent function and its inverse, and they highlight the analytical richness of the class Σ in the context of geometric function theory.

Definition 1. *[46] Let $\beta, \delta, \varkappa \in \mathbb{C}$ with $\Re(\beta) > 0$, $\Re(\delta) > 0$, $\Re(\varkappa) > 0$, and $|q| < 1$. The generalized q–Mittag–Leffler function $\mathrm{E}^{\delta}_{\beta,\lambda}$ is defined by*

$$\mathrm{E}^{\delta}_{\beta,\lambda}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\delta};q)_n}{(q;q)_n} \frac{z^n}{\Gamma_q(\beta n + \lambda)}, \tag{10}$$

where Γ_q denotes the q–gamma function given in (1).

In the limiting case $q \to 1^-$, the function $\mathrm{E}^{\delta}_{\beta,\lambda}(z;q)$ reduces to the classical generalized Mittag–Leffler function. This limiting behavior elegantly bridges the discrete q–framework with its continuous analog. Motivated by this connection, we now introduce the q–analogue of the Rabotnov function as follows.

Definition 2. *Let $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, $\lambda > 0$, and $|q| < 1$. The q–Rabotnov type function $\Phi^{\delta}_{\beta,\lambda}(z;q)$ is defined by*

$$\Phi^{\delta}_{\beta,\lambda}(z;q) = z^{\beta} \sum_{n=0}^{\infty} \frac{(q^{\delta};q)_n}{(q;q)_n} \frac{[\lambda]_q^n}{\Gamma_q((n+1)(1+\beta))} z^{n(1+\beta)}. \tag{11}$$

It should be noted that in the limiting case $q \to 1^-$, the function $\Phi^{\delta}_{\beta,\lambda}(z;q)$ reduces to the classical Rabotnov function $\Phi_{\beta,\lambda}(z)$ (see [24]), thus establishing a natural link between the q framework and its classical analog. Since $\Phi^{\delta}_{\beta,\lambda}(z;q)$ is not normalized, we consider the following normalized form:

$$\begin{aligned}\mathbb{R}^{\delta}_{\beta,\lambda}(z;q) &= z^{\frac{1}{1+\beta}+1}\,\Gamma_q(1+\beta)\,\Phi^{\delta}_{\beta,\lambda}\left(z^{\frac{1}{1+\beta}};q\right)\\ &= z+\sum_{n=2}^{\infty}\frac{(q^{\delta};q)_{n-1}}{(q;q)_{n-1}}\frac{[\lambda]_q^{n-1}\,\Gamma_q(1+\beta)}{\Gamma_q((1+\beta)n)}\,z^n, \qquad z\in\mathbb{U}.\end{aligned} \tag{12}$$

Remark 1. *The function $\Phi^{\delta}_{\beta,\lambda}(z;q)$ can be interpreted as a q–analogue of kernel-type generating functions that frequently arise in the investigation of analytic and bi-univalent function classes. It encapsulates the combined effect of the generalized q –Mittag - Leffler structure together with the parameter λ, and in the limiting case $q\to 1^-$, it reduces to its classical analog expressed in terms of the Euler gamma function. Such kernel functions serve as fundamental building blocks in the development of subclasses of analytic functions defined through subordination principles, convolution structures, and operators associated with fractional q–calculus.*

We now introduce a linear operator of Hadamard–convolution type associated with the q–Rabotnov kernel.

Definition 3. *For $\beta,\delta\in\mathbb{C}$ with $\Re(\beta)>0$ and $\lambda>0$, the linear operator $\mathcal{F}^{\delta}_{\beta,\lambda}:\mathcal{A}\to\mathcal{A}$ is defined by*

$$\begin{aligned}\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q) = \mathbb{R}^{\delta}_{\beta,\lambda}(z;q)*f(z) &= z+\sum_{n=2}^{\infty}\frac{(q^{\delta};q)_{n-1}}{(q;q)_{n-1}}\frac{[\lambda]_q^{n-1}\,\Gamma_q(1+\beta)}{\Gamma_q(n(1+\beta))}\,\alpha_n z^n,\\ &= z+\frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2\,z^2+\frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\alpha_3\,z^3+O(z^4)\end{aligned} \tag{13}$$

where f is of the form (5), and $*$ denotes the Hadamard product (or coefficient-wise) of power series.

Remark 2. *The operator $\mathcal{F}^{\delta}_{\beta,\lambda}$ generalizes the classical convolution operators by incorporating q–Rabotnov kernels. Such operators play a crucial role in constructing and investigating subclasses of analytic and bi-univalent functions, particularly in deriving sharp coefficient bounds and Fekete–Szegö type inequalities.*

The advent of q-calculus has significantly advanced the study of analytic function theory by enabling the discovery of novel subclasses with intricate geometric and algebraic properties. These developments underscore the versatility of the q-calculus, demonstrating its potential to enrich the classical function theory and uncover new mathematical phenomena. The relevance of these findings extends to both theoretical and applied settings, providing a solid foundation for future research and innovation in the field [1, 2, 4, 6, 8–10, 12–14, 37–41, 47, 48].

3. Definition and example

Motivated by q-Fibonacci numbers and the q–Rabotnov operator, this section will now look at a novel subclasses of bi-univalent functions related to shell-like curves.

Definition 4. *Let $\mu \geq 0$. A function $f \in \Sigma$, defined by* (5)*, is said to belong to the class*

$$\mathfrak{R}_{\Sigma_q^{\mu}}(\beta, \delta, \lambda)$$

if the following subordinations are satisfied:

$$\mu\, \partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} \;\prec\; \Omega(z;q) := \frac{1+q\varkappa_q^2 z^2}{1-\varkappa_q z - q\varkappa_q^2 z^2}, \qquad (z \in \mathbb{U}), \tag{14}$$

and

$$\mu\, \partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} \;\prec\; \Omega(\xi;q) := \frac{1+q\varkappa_q^2 \xi^2}{1-\varkappa_q \xi - q\varkappa_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}), \tag{15}$$

where $\eta = f^{-1}$ denotes the inverse of f, ∂_q represents the q–derivative, and $\varkappa_q$ is specified in (3)*.*

By prescribing suitable specializations of the parameters q and μ, a variety of familiar subclasses of the bi-univalent function class Σ. For clarity, we present below several representative examples, illustrating how the general class $\mathfrak{R}_{\Sigma_q^{\mu}}(\beta, \delta, \lambda)$ reduces to well-known families under particular parameter choices.

Example 1. *If we take $\mu = 1$ in Definition 4, then a function $f \in \Sigma$ is said to belong to the class $\mathfrak{R}_{\Sigma_q^{1}}(\beta, \delta, \lambda)$ whenever the following subordinations hold:*

$$\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) \;\prec\; \Omega(z;q) := \frac{1+q\varkappa_q^2 z^2}{1-\varkappa_q z - q\varkappa_q^2 z^2}, \qquad (z \in \mathbb{U}), \tag{16}$$

and

$$\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) \;\prec\; \Omega(\xi;q) := \frac{1+q\varkappa_q^2 \xi^2}{1-\varkappa_q \xi - q\varkappa_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}), \tag{17}$$

where $\eta = f^{-1}$ denotes the inverse of f, ∂_q is the q–derivative, and $\varkappa_q$ is given by (3)*.*

Example 2. *If we take $\mu = 0$ in Definition 4, then a function $f \in \Sigma$ is said to belong to the class $\mathfrak{R}_{\Sigma_q^{0}}(\beta, \delta, \lambda)$ whenever the following subordinations hold:*

$$\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} \;\prec\; \Omega(z;q) := \frac{1+q\varkappa_q^2 z^2}{1-\varkappa_q z - q\varkappa_q^2 z^2}, \qquad (z \in \mathbb{U}), \tag{18}$$

and

$$\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} \;\prec\; \Omega(\xi;q) := \frac{1+q\varkappa_q^2 \xi^2}{1-\varkappa_q \xi - q\varkappa_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}), \tag{19}$$

where $\eta = f^{-1}$ denotes the inverse of f, and $\varkappa_q$ is given by (3)*.*

Example 3. *If we let $q \to 1^-$ in Definition 4, then the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta, \delta, \lambda)$ reduces to its classical analogue $\mathfrak{R}_{\Sigma^\mu}(\beta, \delta, \lambda)$. In this case, a function $f \in \Sigma$ belongs to the class if the subordinations*

$$\mu f'(z) + (1-\mu)\frac{f(z)}{z} \prec \Omega(z) := \frac{1+\varkappa^2 z^2}{1-\varkappa z-\varkappa^2 z^2}, \qquad (z \in \mathbb{U}), \tag{20}$$

and

$$\mu \eta'(\xi) + (1-\mu)\frac{\eta(\xi)}{\xi} \prec \Omega(\xi) := \frac{1+\varkappa^2 \xi^2}{1-\varkappa \xi-\varkappa^2 \xi^2}, \qquad (\xi \in \mathbb{U}), \tag{21}$$

hold, where $\eta = f^{-1}$ is the inverse of f, and $\varkappa = \frac{1-\sqrt{5}}{2} = \lim_{q\to 1^-} \varkappa_q$. Here the operator ∂_q is replaced by the classical derivative.

4. Main Results

In this section, we obtain the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_3|$ for the bi-univalent starlike and convex subclass $\mathfrak{R}_{\Sigma_q^\mu}(\beta, \delta, \lambda)$.

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$, and $p(z) \prec \Omega(z; q)$. Then there exist $\delta \in \mathsf{P}$ such that $|\varepsilon(z)| < 1$ in $\mathbb{U}$ and $p(z) = \Omega(\varepsilon(z); q)$, we have

$$\hbar(z) = (1+\varepsilon(z))(1-\varepsilon(z))^{-1} = 1 + \vartheta_1 z + \vartheta_2 z^2 + \dots \in \mathsf{P} \qquad (z \in \mathbb{U}). \tag{22}$$

It follows that

$$\varepsilon(z) = \frac{\vartheta_1 z}{2} + \left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\frac{z^2}{2} + \left(\vartheta_3 - \vartheta_1\vartheta_2 - \frac{\vartheta_1^3}{4}\right)\frac{z^3}{2} + \cdots, \tag{23}$$

and

$$\begin{aligned}
\Omega(\varepsilon(z); q) = 1 &+ \widehat{p_1}\left[\frac{\vartheta_1 z}{2} + \left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\frac{z^2}{2} + \left(\vartheta_3 - \vartheta_1\vartheta_2 - \frac{\vartheta_1^3}{4}\right)\frac{z^3}{2} + \cdots\right] \\
&+ \widehat{p_2}\left[\frac{\vartheta_1 z}{2} + \left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\frac{z^2}{2} + \left(\vartheta_3 - \vartheta_1\vartheta_2 - \frac{\vartheta_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^2 \\
&+ \widehat{p_3}\left[\frac{\vartheta_1 z}{2} + \left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\frac{z^2}{2} + \left(\vartheta_3 - \vartheta_1\vartheta_2 - \frac{\vartheta_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^3 + \cdots \\
&= 1 + \frac{\widehat{p_1}\vartheta_1}{2} z + \frac{1}{2}\left[\left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\widehat{p_1} + \frac{\vartheta_1^2}{2}\widehat{p_2}\right] z^2 \\
&+ \frac{1}{2}\left[\left(\vartheta_3 - \vartheta_1\vartheta_2 + \frac{\vartheta_1^3}{4}\right)\widehat{p_1} + \vartheta_1\left(\vartheta_2 - \frac{\vartheta_1^2}{2}\right)\widehat{p_2} + \frac{\vartheta_1^3}{4}\widehat{p_3}\right] z^3 + \cdots.
\end{aligned} \tag{24}$$

Similarly, there exists an analytic function ν such that $|\nu(\xi)| < 1$ in $\mathbb{U}$ and $p(\xi) = \Omega(\nu(\xi); q)$. Therefore, the function

$$\kappa(\xi) = (1+\nu(\xi))(1-\nu(\xi))^{-1} = 1 + \upsilon_1 \xi + \upsilon_2 \xi^2 + \dots \in \mathsf{P}. \tag{25}$$

It follows that

$$\nu(\xi)=\frac{\upsilon_1\xi}{2}+\left(\upsilon_2-\frac{\upsilon_1^2}{2}\right)\frac{\xi^2}{2}+\left(\upsilon_3-\upsilon_1\upsilon_2-\frac{\upsilon_1^3}{4}\right)\frac{\xi^3}{2}+\cdots, \tag{26}$$

and

$$\begin{aligned}\Omega(\nu(\xi);q)=1&+\frac{\widehat{p_1}\upsilon_1}{2}\xi+\frac{1}{2}\left[\left(\upsilon_2-\frac{\upsilon_1^2}{2}\right)\widehat{p_1}+\frac{\upsilon_1^2}{2}\widehat{p_2}\right]\xi^2\\ &+\frac{1}{2}\left[\left(\upsilon_3-\upsilon_1\upsilon_2+\frac{\upsilon_1^3}{4}\right)\widehat{p_1}+\upsilon_1\left(\upsilon_2-\frac{\upsilon_1^2}{2}\right)\widehat{p_2}+\frac{\upsilon_1^3}{4}\widehat{p_3}\right]\xi^3+\cdots.\end{aligned} \tag{27}$$

In the following theorem we determine the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_2|$ for the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$. Later we will reduce these bounds to other classes for special cases.

Theorem 1. *Let f given by* (5) *be in the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$. Then*

$$|\alpha_2| \le \min\left\{\begin{array}{l}\frac{|\varkappa_q|}{[\lambda]_q}\sqrt{\frac{[2]_q\,\Gamma_q\big(3(1+\beta)\big)\,\Gamma_q^2\big(2(1+\beta)\big)}{\left\{\begin{array}{l}\varkappa_q\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\big(2(1+\beta)\big)\\ \quad-\left(1+\mu\,q\right)^2[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\left((2q+1)\,\varkappa_q-1\right)\end{array}\right\}}},\\ \frac{\varkappa_q^2\,\Gamma_q^2\left(2(1+\beta)\right)}{\left(1+\mu\,q\right)^2[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)}\end{array}\right\},$$

and

$$\left|\alpha_3\right|\le\frac{\varkappa_q^2\,\Gamma_q^2\left(2(1+\beta)\right)}{\left(1+\mu\,q\right)^2[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)}+\frac{[2]_q\left|\varkappa_q\right|\Gamma_q\big(3(1+\beta)\big)}{\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}.$$

Proof. Let $f\in\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$ and $\eta=f^{-1}$. Considering (14) and (15) we have

$$\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big)+(1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)}{z}=\Omega(\varepsilon(z);q),\qquad(z\in\mathbb{U}), \tag{28}$$

and

$$\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)\Big)+(1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)}{\xi}=\Omega(\nu(\xi);q),\qquad(\xi\in\mathbb{U}). \tag{29}$$

Using (12), we have

$$\begin{aligned}\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big)+(1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)}{z}&=1+\frac{\left(1+\mu\,q\right)[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2\,z\\ &+\frac{\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\alpha_3\,z^2\;+\;O(z^3).\end{aligned} \tag{30}$$

and

$$\mu\,\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big)+(1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi}=1-\frac{\big(1+\mu\,q\big)\,[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2\,\xi$$
$$+\frac{\big(1+q\,\mu\,[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\big(2\,\alpha_2^2-\alpha_3\big)\,\xi^2\;+\;O(\xi^3). \tag{31}$$

By comparing (28) and (30), along (24), yields

$$\frac{\big(1+\mu\,q\big)\,[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2\,z+\frac{\big(1+q\,\mu\,[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\alpha_3\,z^2+\cdots$$
$$=\frac{\widehat{p_1}\vartheta_1}{2}z+\frac{1}{2}\left[\left(\vartheta_2-\frac{\vartheta_1^2}{2}\right)\widehat{p_1}+\frac{\vartheta_1^2}{2}\widehat{p_2}\right]z^2+\cdots. \tag{32}$$

Besied that, by comparing (24) and (31), along (27), yields

$$-\frac{\big(1+\mu\,q\big)\,[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2\,\xi$$
$$+\frac{\big(1+q\,\mu\,[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\big(2\alpha_2^2-\alpha_3\big)\,\xi^2+\cdots \tag{33}$$
$$=\frac{\widehat{p_1}\,\upsilon_1}{2}\,\xi+\frac{1}{2}\left[\left(\upsilon_2-\tfrac{\upsilon_1^2}{2}\right)\widehat{p_1}+\tfrac{\upsilon_1^2}{2}\,\widehat{p_2}\right]\xi^2+\cdots.$$

Equating the pertinent coefficient in (32) and (33), we obtain

$$\frac{\big(1+\mu\,q\big)\,[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2=\frac{\widehat{p_1}\vartheta_1}{2} \tag{34}$$
$$-\frac{\big(1+\mu\,q\big)\,[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\big(2(1+\beta)\big)}\,\alpha_2=\frac{\widehat{p_1}\upsilon_1}{2} \tag{35}$$
$$\frac{\big(1+q\,\mu\,[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\alpha_3=\frac{1}{2}\left[\left(\vartheta_2-\frac{\vartheta_1^2}{2}\right)\widehat{p_1}+\frac{\vartheta_1^2}{2}\widehat{p_2}\right] \tag{36}$$
$$\frac{\big(1+q\,\mu\,[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\,\big(2\,\alpha_2^2-\alpha_3\big)=\frac{1}{2}\left[\left(\upsilon_2-\frac{\upsilon_1^2}{2}\right)\widehat{p_1}+\frac{\upsilon_1^2}{2}\widehat{p_2}\right] \tag{37}$$

From (34) and (35), we have

$$\vartheta_1=-\upsilon_1\quad\Longleftrightarrow\quad\vartheta_1^2=\upsilon_1^2, \tag{38}$$

and using (4), we have

$$\alpha_2^2 = \frac{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}{8\left(1+\mu\, q\right)^2 [\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)}\left(\vartheta_1^2+\upsilon_1^2\right), \tag{39}$$

or equivalent to

$$\left(\vartheta_1^2+\upsilon_1^2\right) = \frac{8\left(1+\mu\, q\right)^2 [\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)}{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}\alpha_2^2, \tag{40}$$

Now, by summing (36) and (37), we obtain

$$\frac{2\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\big(3(1+\beta)\big)}\alpha_2^2 = \frac{(\vartheta_2+\upsilon_2)\varkappa_q}{2} + \left[\frac{(2q+1)\varkappa_q^2}{4} - \frac{\varkappa_q}{4}\right]\left(\vartheta_1^2+\upsilon_1^2\right). \tag{41}$$

By putting (39) in (41), with doing some calculations, yields to

$$\alpha_2^2 = \frac{(\vartheta_2+\upsilon_2)\,\varkappa_q^2\,[2]_q\,\Gamma_q\big(3(1+\beta)\big)\,\Gamma_q^2\big(2(1+\beta)\big)}{4\,[\lambda]_q^2\left\{\begin{array}{l}\varkappa_q\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\big(2(1+\beta)\big)\\ \quad -\left(1+\mu\, q\right)^2[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\big((2q+1)\,\varkappa_q-1\big)\end{array}\right\}}. \tag{42}$$

Using (7) for (42), we have

$$|\alpha_2| \le \frac{|\varkappa_q|}{[\lambda]_q}\sqrt{\frac{[2]_q\,\Gamma_q\big(3(1+\beta)\big)\,\Gamma_q^2\big(2(1+\beta)\big)}{\left\{\begin{array}{l}\varkappa_q\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\big(2(1+\beta)\big)\\ \quad -\left(1+\mu\, q\right)^2[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\big((2q+1)\,\varkappa_q-1\big)\end{array}\right\}}}. \tag{43}$$

Besided that, from (39)

$$|\alpha_2| \le \frac{\varkappa_q^2\,\Gamma_q^2\left(2(1+\beta)\right)}{\left(1+\mu\, q\right)^2[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)}.$$

Now, so as to find the bound on $|\alpha_3|$, let's subtract from (36) and (37) along (39), we obtain

$$\alpha_3 = \alpha_2^2 + \frac{[2]_q\,\varkappa_q\,\Gamma_q\big(3(1+\beta)\big)}{4\big(1+q\,\mu\,[2]_q\big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}\left(\vartheta_2-\upsilon_2\right). \tag{44}$$

Hence, we get

$$|\alpha_3| = |\alpha_2|^2 + \frac{[2]_q\,|\varkappa_q|\,\Gamma_q\big(3(1+\beta)\big)}{\big(1+q\,\mu\,[2]_q\big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}. \tag{45}$$

Then, in view of (39), we obtain

$$|\alpha_3| \le \frac{\varkappa_q^2\,\Gamma_q^2\left(2(1+\beta)\right)}{\left(1+\mu\, q\right)^2[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)} + \frac{[2]_q\,|\varkappa_q|\,\Gamma_q\big(3(1+\beta)\big)}{\big(1+q\,\mu\,[2]_q\big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}. \tag{46}$$

In the following theorem, we find the Fekete-Szegö functional for $f \in \Re_{\Sigma_q^\mu}(\beta,\delta,\lambda)$.

Theorem 2. *Let f given by* (5) *be in the class* $\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$ *and* $\rho\in\mathbb{R}$. *Then we have*

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\le\begin{cases}\frac{\varkappa_q\,[2]_q\,\Gamma_q\left(3(1+\beta)\right)}{[\lambda]_q^2\,\Gamma_q(1+\beta)\,[\delta]_q\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}, & 0\le\left|\mathscr{D}(\rho)\right|\le\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\\ 4\left|\mathscr{D}(\rho)\right|, & \left|\mathscr{D}(\rho)\right|\ge\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\end{cases}.$$

where

$$\mathscr{D}(\rho)=\frac{(1-\rho)\,\varkappa_q\,\Gamma_q^2\left(2(1+\beta)\right)}{\left\{\begin{array}{l}\varkappa_q\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q\,\Gamma_q^2\left(2(1+\beta)\right)\\ \quad-\left(1+\mu\,q\right)^2[\delta]_q\,[2]_q\,\Gamma_q(1+\beta)\,\Gamma_q\left(3(1+\beta)\right)\left((2q+1)\,\varkappa_q-1\right)\end{array}\right\}}\tag{47}$$

Proof. Let $f\in\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$, from (42) and (44) we have

$$\begin{aligned}\alpha_3-\rho\,\alpha_2^2&=\frac{[2]_q\,\varkappa_q\,\Gamma_q\left(3(1+\beta)\right)}{4\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}\left(\vartheta_2-\upsilon_2\right)\\&\quad+\frac{(1-\rho)(\vartheta_2+\upsilon_2)\,\varkappa_q^2\,[2]_q\,\Gamma_q\left(3(1+\beta)\right)\Gamma_q^2\left(2(1+\beta)\right)}{4\,[\lambda]_q^2\left\{\begin{array}{l}\varkappa_q\left(1+q\,\mu\,[2]_q\right)[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\left(2(1+\beta)\right)\\ \quad-\left(1+\mu\,q\right)^2[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\left(3(1+\beta)\right)\left((2q+1)\,\varkappa_q-1\right)\end{array}\right\}}\\&=\frac{\varkappa_q\,[2]_q\,\Gamma_q\left(3(1+\beta)\right)}{4\,[\lambda]_q^2\,\Gamma_q(1+\beta)\,[\delta]_q}\left[\left(\mathscr{D}(\rho)+\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\right)\vartheta_2\right.\\&\qquad\left.+\left(\mathscr{D}(\rho)-\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\right)\upsilon_2\right]\end{aligned}\tag{48}$$

where $\mathscr{D}(\rho)$ is given by (47).

Then, by taking modulus of (48), we conclude that

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\le\begin{cases}\frac{\varkappa_q\,[2]_q\,\Gamma_q\left(3(1+\beta)\right)}{[\lambda]_q^2\,\Gamma_q(1+\beta)\,[\delta]_q\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}, & 0\le\left|\mathscr{D}(\rho)\right|\le\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\\ 4\left|\mathscr{D}(\rho)\right|, & \left|\mathscr{D}(\rho)\right|\ge\frac{1}{\left(1+q\,\mu\,[2]_q\right)[\delta+1]_q}\end{cases}.$$

5. Corollaries

The general coefficient estimates established in Theorems 1 and 2 give rise to several noteworthy special cases under suitable choices of the parameters μ and q. In particular, when one considers the purely q–differential subclass ($\mu=1$), the ratio-type subclass ($\mu=0$), and the classical limiting case ($q\to1^-$), the results simplify to the following corollaries.

Corollary 1 ($\mu = 1$). *Let f given by (5) belong to $\mathfrak{R}_{\Sigma_q^1}(\beta,\delta,\lambda)$. Then*

$$|\alpha_2| \leq \min\left\{ \begin{array}{l} \dfrac{|\varkappa_q|}{[\lambda]_q}\sqrt{\dfrac{[2]_q\,\Gamma_q\big(3(1+\beta)\big)\,\Gamma_q^2\big(2(1+\beta)\big)}{\begin{array}{c}\varkappa_q\,\big(1+q[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\big(2(1+\beta)\big)\\ -\,(1+q)^2\,[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\,\big((2q+1)\,\varkappa_q-1\big)\end{array}}}, \\ \dfrac{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}{(1+q)^2\,[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)} \end{array} \right\},$$

and

$$|\alpha_3| \leq \frac{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}{(1+q)^2\,[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)} + \frac{[2]_q\,|\varkappa_q|\,\Gamma_q\big(3(1+\beta)\big)}{\big(1+q[2]_q\big)\,[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}.$$

Moreover, for any $\rho \in \mathbb{R}$,

$$\left|\alpha_3-\rho\,\alpha_2^2\right| \leq \begin{cases} \dfrac{\varkappa_q\,[2]_q\,\Gamma_q\big(3(1+\beta)\big)}{[\lambda]_q^2\,\Gamma_q(1+\beta)\,[\delta]_q\,\big(1+q[2]_q\big)\,[\delta+1]_q}, & 0 \leq \left|\mathscr{D}_1(\rho)\right| \leq \dfrac{1}{\big(1+q[2]_q\big)\,[\delta+1]_q}, \\ 4\left|\mathscr{D}_1(\rho)\right|, & \left|\mathscr{D}_1(\rho)\right| \geq \dfrac{1}{\big(1+q[2]_q\big)\,[\delta+1]_q}, \end{cases}$$

where

$$\mathscr{D}_1(\rho) = \frac{(1-\rho)\,\varkappa_q\,\Gamma_q^2\big(2(1+\beta)\big)}{\begin{array}{c}\varkappa_q\,\big(1+q[2]_q\big)\,[\delta+1]_q\,\Gamma_q^2\big(2(1+\beta)\big)\\ -\,(1+q)^2\,[\delta]_q\,[2]_q\,\Gamma_q(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\,\big((2q+1)\,\varkappa_q-1\big)\end{array}}.$$

Corollary 2 ($\mu = 0$). *Let f given by (5) belong to $\mathfrak{R}_{\Sigma_q^0}(\beta,\delta,\lambda)$. Then*

$$|\alpha_2| \leq \min\left\{ \begin{array}{l} \dfrac{|\varkappa_q|}{[\lambda]_q}\sqrt{\dfrac{[2]_q\,\Gamma_q\big(3(1+\beta)\big)\,\Gamma_q^2\big(2(1+\beta)\big)}{\begin{array}{c}\varkappa_q\,[\delta]_q\,[\delta+1]_q\,\Gamma_q(1+\beta)\,\Gamma_q^2\big(2(1+\beta)\big)\\ -\,[\delta]_q^2\,[2]_q\,\Gamma_q^2(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\,\big((2q+1)\,\varkappa_q-1\big)\end{array}}}, \\ \dfrac{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}{[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)} \end{array} \right\},$$

and

$$|\alpha_3| \leq \frac{\varkappa_q^2\,\Gamma_q^2\big(2(1+\beta)\big)}{[\delta]_q^2\,[\lambda]_q^2\,\Gamma_q^2(1+\beta)} + \frac{[2]_q\,|\varkappa_q|\,\Gamma_q\big(3(1+\beta)\big)}{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}.$$

Moreover, for any $\rho \in \mathbb{R}$,

$$\left|\alpha_3-\rho\,\alpha_2^2\right| \leq \begin{cases} \dfrac{\varkappa_q\,[2]_q\,\Gamma_q\big(3(1+\beta)\big)}{[\lambda]_q^2\,\Gamma_q(1+\beta)\,[\delta]_q\,[\delta+1]_q}, & 0 \leq \left|\mathscr{D}_0(\rho)\right| \leq \dfrac{1}{[\delta+1]_q}, \\ 4\left|\mathscr{D}_0(\rho)\right|, & \left|\mathscr{D}_0(\rho)\right| \geq \dfrac{1}{[\delta+1]_q}, \end{cases}$$

where

$$\mathscr{D}_0(\rho) = \frac{(1-\rho)\,\varkappa_q\,\Gamma_q^2\big(2(1+\beta)\big)}{\varkappa_q\,[\delta+1]_q\,\Gamma_q^2\big(2(1+\beta)\big) - [\delta]_q\,[2]_q\,\Gamma_q(1+\beta)\,\Gamma_q\big(3(1+\beta)\big)\,\big((2q+1)\,\varkappa_q-1\big)}.$$

Corollary 3 (Classical limit $q \to 1^-$)**.** *Let f given by (5) belong to $\mathfrak{R}_{\Sigma^\mu}(\beta,\delta,\lambda)$, the classical limit of $\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$ as $q \to 1^-$. With $[n]_q \to n$, $\Gamma_q \to \Gamma$, $[\delta]_q \to \delta$, $[\delta+1]_q \to \delta+1$, $[\lambda]_q \to \lambda$, and $\varkappa_q \to \varkappa$, we obtain*

$$|\alpha_2| \le \min\left\{ \begin{array}{l} \dfrac{|\varkappa|}{\lambda}\sqrt{\dfrac{2\,\Gamma\big(3(1+\beta)\big)\,\Gamma^2\big(2(1+\beta)\big)}{\varkappa\,\big(1+2\mu\big)\,\delta(\delta+1)\,\Gamma(1+\beta)\,\Gamma^2\big(2(1+\beta)\big) - (1+\mu)^2\,\delta^2\,2\,\Gamma^2(1+\beta)\,\Gamma\big(3(1+\beta)\big)\,\big(3\,\varkappa-1\big)}}, \\ \dfrac{\varkappa^2\,\Gamma^2\big(2(1+\beta)\big)}{(1+\mu)^2\,\delta^2\,\lambda^2\,\Gamma^2(1+\beta)} \end{array} \right\},$$

and

$$|\alpha_3| \le \frac{\varkappa^2\,\Gamma^2\big(2(1+\beta)\big)}{(1+\mu)^2\,\delta^2\,\lambda^2\,\Gamma^2(1+\beta)} + \frac{2\,|\varkappa|\,\Gamma\big(3(1+\beta)\big)}{\big(1+2\mu\big)\,\delta(\delta+1)\,\lambda^2\,\Gamma(1+\beta)}.$$

Moreover, for any $\rho \in \mathbb{R}$,

$$\big|\alpha_3 - \rho\,\alpha_2^2\big| \le \begin{cases} \dfrac{\varkappa\,2\,\Gamma\big(3(1+\beta)\big)}{\lambda^2\,\Gamma(1+\beta)\,\delta\,\big(1+2\mu\big)\,(\delta+1)}, & 0 \le \big|\mathscr{D}_{\rm cl}(\rho)\big| \le \dfrac{1}{\big(1+2\mu\big)\,(\delta+1)}, \\ 4\,\big|\mathscr{D}_{\rm cl}(\rho)\big|, & \big|\mathscr{D}_{\rm cl}(\rho)\big| \ge \dfrac{1}{\big(1+2\mu\big)\,(\delta+1)}, \end{cases}$$

where

$$\mathscr{D}_{\rm cl}(\rho) = \frac{(1-\rho)\,\varkappa\,\Gamma^2\big(2(1+\beta)\big)}{\varkappa\,\big(1+2\mu\big)\,(\delta+1)\,\Gamma^2\big(2(1+\beta)\big) - (1+\mu)^2\,\delta\,2\,\Gamma(1+\beta)\,\Gamma\big(3(1+\beta)\big)\,(3\,\varkappa-1)}.$$

6. Conclusion

In this work, we have introduced and studied the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta,\delta,\lambda)$, constructed through convolution operators involving the q–Rabotnov function and subordinated to the q–Fibonacci structure. A key feature of our investigation is the definition of a new q–derivative operator based on q–Rabotnov kernels, which provides a flexible framework for analyzing subclasses of bi-univalent functions. Within this setting, we have derived sharp coefficient estimates for the initial Taylor–Maclaurin coefficients and established corresponding Fekete–Szegö type inequalities.

The general results obtained in Theorems 1 and 2 unify and extend several recent contributions to the theory of bi-univalent functions, while naturally reducing to important special cases under suitable parameter choices. In particular, the framework recovers the purely q–differential subclass ($\mu = 1$), the ratio-type subclass ($\mu = 0$), and the classical limit ($q \to 1^-$), thereby illustrating both the flexibility and the unifying character of the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta, \delta, \lambda)$ in geometric function theory.

For future research, it would be of significant interest to develop analogous subclasses generated by other q–special functions or higher-order convolution operators, and to examine possible applications of the proposed q–derivative operator in operator theory, multivariable geometric mappings, and related analytic inequalities.

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