EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6225
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Roughness of Bipolar Soft Sets via Ideals and Its Applications

Dali Shi[1], H. M. Khiamy[2], S. E. Abbas[2], Ismail Ibedou[3,*]

[1] *College of Accounting, Guangzhou College of Technology and Business, Guangzhou 528138, China*
[2] *Mathematics Department, Faculty of Science, Sohag University, Sohag 82524, Egypt*
[3] *Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt*

Abstract. The essential objectives of this study are to propose enhancements and modifications to the bipolar soft rough sets methodology by incorporating ideals. The paper introduces two distinct types of ideal bipolar soft approximation operators, which serve as extensions to the existing bipolar soft rough approximation operator. Furthermore, two approaches are employed to establish and investigate a novel type of bipolar approximation space, referred to as the bi-ideal bipolar soft approximation space. This work also explores the relationships between these proposed techniques and previous methods, detailing their respective characteristics and advantages. By enlarging the ideal bipolar lower approximations and reducing the ideal bipolar upper approximations, these strategies significantly reduce the ambiguity and uncertainty within the decision-making process. The paper additionally outlines several key metrics related to ideal bipolar soft spaces, enriching the theoretical understanding of these structures. A practical application of the proposed spaces is presented in the context of multi-attribute group decision-making (MAGDM) problems. To support this, an algorithm is developed to facilitate the selection of the most optimal alternative from a range of options, accompanied by a practical example to demonstrate its effectiveness. The analysis highlights the reliability, adaptability, and superiority of the proposed MAGDM framework. Furthermore, a concise comparison with existing methodologies is provided, showcasing the advantages and robustness of the proposed approach in addressing complex decision-making challenges.

2020 Mathematics Subject Classifications: 03E72, 03B52

Key Words and Phrases: Bipolar soft set, bipolar ideal rough soft sets, ideal bipolar soft approximation space

Abbreviations

PA $\Rightarrow$ positive approximation , NA $\Rightarrow$ negative approximation, UA $\Rightarrow$ upper approximation
LA $\Rightarrow$ lower approximation, SA $\Rightarrow$ soft approximation , BR $\Rightarrow$ boundary region

1. Introduction

In numerous fields such as social sciences, economics, engineering, environmental sciences, artificial intelligence, and medical sciences, uncertainty and imprecision in data pose significant challenges. These issues often stem from limitations in representing knowledge and the inherent

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6225

Email addresses: shidali@gzgs.edu.cn (D. Shi),
hossam.khiamy@gmail.com (H. M. Khiamy), salaheldin_ahmed@science.sohag.edu.eg (S. E. Abbas),
ismail.abdelaziz@fsc.bu.edu.eg (I. Ibedou)

roughness of available information. To address these complexities, researchers have proposed a variety of mathematical modeling approaches, including interval mathematics, vague set theory, fuzzy set theory, theory of rough sets [1], and theory of probability. However, despite their utility, each of these approaches has inherent limitations [2], prompting the need for alternative frameworks.

Soft set theory introduced by Molodtsov [3], offers a flexible and effective mathematical tool to manage uncertainty. Its applicability spans various domains, including game theory, functional analysis, probability theory, operational research, and medical diagnostics. This framework has undergone rapid development, with significant contributions such as the introduction of algebraic operations on soft sets by Maji et al. [4] and presented by Ali et al. [5] as well as some fundamental operations on soft sets. Soft topological spaces are introduced by Cagman et al. [6]. N-soft sets [7, 8], multipolar neutrosophic soft set [9], sum of soft topological spaces [10] were introduced. A number of scholars have examined these properties and the have used soft sets in decision-making [11, 12]. The joint of rough sets and soft sets has given rise to soft rough sets, which address these challenges of incomplete and uncertain information in intelligent systems. Feng [13] pioneered this integration to refine approximation techniques, while Alkhazaleh and Marei [14] put forward new soft rough sets approximations. Furthermore, the concept of ideals, defined as non-empty collections of sets closed under hereditary properties and finite additivity [15], has been instrumental in improving these approximations. The introduction of bi-ideals [16] has provided a new framework for reducing BRs and enhancing the precision of approximations, enabling the resolution of complex real-life problems [16–19]. Alharbi et al. [20] improved the SAs given by Feng et al. [13], Alkhazaleh and Marei [14] utilizing ideals. Bipolar soft sets, first explored in [21], extend the notion of fuzzy sets by incorporating a bipolar membership scale ranging from -1 to 1. The notion of a bipolar valued fuzzy set [22] was presented by upgrading the fuzzy set's membership grade from $[0, 1]$ to $[-1, 1]$. This concept has been generalized further to include bipolar fuzzy relations [23], bipolar complex fuzzy sets [24], bipolar fuzzy soft sets [25], fuzzy bipolar soft sets [26], bipolar neutrosophic soft sets [27], hesitant bipolar-valued fuzzy soft sets [28], multi-fuzzy bipolar soft sets [29], bipolar fuzzy soft graphs [30], bipolar fuzzy soft expert sets [31] and rough fuzzy bipolar soft sets [32] are some ways that scholars have generalized. These advancements have opened new avenues for addressing decision-making challenges.

The motivations of this paper introduces an innovative approach for refining and extending bipolar soft rough sets by leveraging the concept of ideals. This novel framework builds on and enhances existing methodologies [33], [34] and [35] by presenting two types of ideal bipolar soft rough approximation operators. These operators generalize traditional bipolar soft rough approximations, providing greater accuracy and effectiveness in addressing uncertainty and vagueness in data. The main properties of the presented technique are shown and many comparisons between our techniques and the previous ones are proposed. The bipolar soft rough approximations [34] are special cases of the presented ideal bipolar SAs. Also, we discusses the ideal bipolar SA-related measures. The proposed approximations utilizing ideals are more accurate than [33], [34] and [35]. Therefore, the proposed techniques are very useful in real life applications representing and discussing the vagueness of data. Additionally, bipolar soft bi-ideal approximation spaces, which are new bipolar SA spaces created using two ideals, are introduced. Two distinct techniques are described for these approximations. The key contributions of this study include the introduction of bipolar soft bi-ideal approximation spaces, which employ two ideals to construct advanced approximation spaces. These spaces represent a significant step forward in capturing the dual characteristics of datasets. Two distinct methods for constructing these spaces are detailed, offering new tools for data analysis and decision-making. To demonstrate the practical utility of that proposed framework, the study applies these techniques to multi-attribute group decision-making (MAGDM) problems. A novel algorithm is presented to facilitate the selection of optimal alternatives, supported by a comprehensive example illustrating its effectiveness. The reliability, adaptability, and superiority of the proposed methods are further validated through a comparative analysis with existing decision-making techniques, emphasizing their potential to address complex real-world problems.

2. Preliminaries

The main ideas employed in this study are reviewed in this section. In the paper, the universal set, the parameter set, and the power set are denoted by $\mathscr{Q}, \wp$ and $2^{\mathscr{Q}}$, respectively.

Definition 2.1. [1] If $\mathscr{Q}$ is a universal set of objects, R is an equivalence relation on $\mathscr{Q}$, and $[\daleth]_R$ is the equivalence class containing $\daleth$. Then, for any $\Im \subseteq \mathscr{Q}$, the lower, UAs and the BR of $\Im$ are defined respectively by:

$$\underline{Lower}(\Im) = \{\daleth \in \mathscr{Q} : [\daleth]_R \subseteq \Im\},$$
$$\overline{Upper}(\Im) = \{\daleth \in \mathscr{Q} : [\daleth]_R \cap \Im \neq \emptyset\},$$
$$BND(\Im) = \overline{Upper}(\Im) - \underline{Lower}(\Im).$$

Definition 2.2. [3] A soft set over $\mathscr{Q}$ is a pair $(f, \wp)$ where $f : \wp \longrightarrow 2^{\mathscr{Q}}$. Consequently, a parameterised collection of subsets of $\mathscr{Q}$ is provided by a soft set over $\mathscr{Q}$.

Definition 2.3. [13] Let $(f, \wp)$ be a soft set over $\mathscr{Q}$. If for any $\varsigma_1, \varsigma_2 \in \wp$, there is $\varsigma_3 \in \wp$ such that $f(\varsigma_3) = f(\varsigma_1) \cap f(\varsigma_2)$ whenever $f(\varsigma_1) \cap f(\varsigma_2) \neq \emptyset$, then $(f, \wp)$ is called an intersection complete soft set.

Definition 2.4. [21] Let $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, ..., \varsigma_n\}$ be the set of parameters, and the not set of $\wp$ is defined by $\aleph = \{\neg\varsigma_1, \neg\varsigma_2, \neg\varsigma_3, ..., \neg\varsigma_n\}$ where, for all i, $\neg\varsigma_i =$ not ς_i.

The family of all bipolar soft sets over $\mathscr{Q}$ will be denoted by $\mathfrak{BSS}^{\mathscr{Q}}$.

Definition 2.5. [21] A bipolar soft set on $\mathscr{Q}$ is an object of form $\mathfrak{B} = (f, g : \wp)$ where $f : \wp \longrightarrow 2^{\mathscr{Q}}$ and $g : \wp \longrightarrow 2^{\mathscr{Q}}$ with the property that for each $\varsigma \in \wp$, we have $f(\varsigma) \cap g(\neg\varsigma) = \emptyset$.

Definition 2.6. [15] A non-empty family $\mathscr{L}$ of subsets of $\mathscr{Q}$ is called an ideal on $\mathscr{Q}$ if it fulfills these conditions

(1) If $\Im \in \mathscr{L}$ and $\vartheta \subseteq \Im$, then $\vartheta \in \mathscr{L}$,

(2) If $\Im, \vartheta \in \mathscr{L}$, then $\Im \cup \vartheta \in \mathscr{L}$.

Definition 2.7. [16] Let $\mathscr{L}_1, \mathscr{L}_2$ be two ideals on $\Im$. Then, the set of subsets from $\mathscr{L}_1, \mathscr{L}_2$ is denoted by $< \mathscr{L}_1, \mathscr{L}_2 >$ and is given by:

$$< \mathscr{L}_1, \mathscr{L}_2 >= \{\Im_1 \cup \Im_2 : \ \Im_1 \in \mathscr{L}_1, \ \Im_2 \in \mathscr{L}_2\}.$$

Proposition 2.1. *[16] Let $\mathscr{L}_1, \mathscr{L}_2$ be two ideals on $\Im$ and $\Im, \vartheta \subseteq \mathscr{Q}$. Then, the family $< \mathscr{L}_1, \mathscr{L}_2 >$ has the following proprieties.*

(1) $< \mathscr{L}_1, \mathscr{L}_2 >\neq \emptyset$;

(2) $\Im \in < \mathscr{L}_1, \mathscr{L}_2 >, \ \vartheta \subseteq \Im \ \Rightarrow \ \vartheta \in < \mathscr{L}_1, \mathscr{L}_2 >$;

(3) $\Im, \vartheta \in < \mathscr{L}_1, \mathscr{L}_2 > \ \Rightarrow \ \Im \cup \vartheta \in < \mathscr{L}_1, \mathscr{L}_2 >$.

Definition 2.8. [20] Let $(\mathscr{Q}, \mathbf{S}, \mathscr{L})$ be a soft ideal approximation space with $\mathbf{S} = (f, \wp)$ be a soft set over a universe $\mathscr{Q}$, $\mathscr{L}$ be an ideal on $\mathscr{Q}$, $\Im \subseteq \mathscr{Q}$. Then, the lower and UAs, $\underline{\mathfrak{SR}}^{\mathscr{L}}(\Im)$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}(\Im)$, are defined respectively by:

$$\underline{\mathfrak{SR}}^{\mathscr{L}}(\Im) = \cup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathscr{L}\}$$
$$\overline{\mathfrak{SR}}^{\mathscr{L}}(\Im) = [\underline{\mathfrak{SR}}^{\mathscr{L}}(\Im^c)]^c, \text{ where } \Im^c \text{ be the complement of } \Im.$$

Definition 2.9. [34] Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a bipolar soft set on $\mathscr{Q}$ and the pair $\beta = (\mathscr{Q}, (f, g : \wp))$ is called a bipolar soft approximation space (BSA-space for short). Depending on β, the following operators are defined for any $\Im \subseteq \mathscr{Q}$ by:

$$\left.\begin{array}{l}\underline{\mathfrak{SR}}_{\beta^+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \subseteq \Im\},\\ \overline{\mathfrak{SR}}_{\beta^+}(\Im) = \left(\underline{\mathfrak{SR}}_{\beta^+}(\Im^c)\right)^c,\\ \overline{\mathfrak{SR}}_{\beta^-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \subseteq \Im^c\},\\ \underline{\mathfrak{SR}}_{\beta^-}(\Im) = \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im^c)\right)^c,\end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be soft β-lower positive, soft β upper positive, soft β-upper negative, and soft β-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l}\underline{\mathfrak{PSR}}_{\beta}(\Im) = \left(\underline{\mathfrak{SR}}_{\beta^+}(\Im), \underline{\mathfrak{SR}}_{\beta^-}(\Im)\right),\\ \overline{\mathfrak{PSR}}_{\beta}(\Im) = \left(\overline{\mathfrak{SR}}_{\beta^+}(\Im), \overline{\mathfrak{SR}}_{\beta^-}(\Im)\right),\end{array}\right\}$$

are called the ideal bipolar SAs of $\Im$ with respect to the BSA-space.

Definition 2.10. [33] Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a bipolar soft set on $\mathscr{Q}$. Then, $\beta = (\mathscr{Q}, (f, g : \wp))$ is the corresponding bipolar soft approximation space. Depending on $\beta_{\mathscr{L}}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$ by:

$$\left.\begin{array}{l}\underline{\mathfrak{SF}}_{\beta^+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \subseteq \Im\},\\ \overline{\mathfrak{SF}}_{\beta^+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im \neq \emptyset\},\\ \overline{\mathfrak{SF}}_{\beta^-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \subseteq \Im^c\},\\ \underline{\mathfrak{SF}}_{\beta^-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im \neq \emptyset\}\end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be soft β-lower positive, ideal soft β upper positive,ideal soft β-upper negative, and soft β-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l}\underline{\mathfrak{PSF}}_{\beta}(\Im) = \left(\underline{\mathfrak{SF}}_{\beta^+}(\Im), \underline{\mathfrak{SF}}_{\beta^-}(\Im)\right)\\ \overline{\mathfrak{PSF}}_{\beta}(\Im) = \left(\overline{\mathfrak{SF}}_{\beta^+}(\Im), \overline{\mathfrak{SF}}_{\beta^-}(\Im)\right)\end{array}\right\}$$

are called the ideal bipolar SAs of $\Im$ with respect to the IBSA-space

Definition 2.11. [33] Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a bipolar soft set on $\mathscr{Q}$ and $\beta = (\mathscr{Q}, (f, g : \wp))$ is the corresponding bipolar soft approximation space. Then, $\mathfrak{B} = (f, g : \wp)$ is called a semi-intersection bipolar soft set, if $f(\varsigma_i) \cap g(\neg\varsigma_i) = \emptyset$ for all $\varsigma \in \wp$ and $\neg\varsigma \in \neg\wp$.

Definition 2.12. [35] Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a bipolar soft set on $\mathscr{Q}$ and $\mathscr{L}$ is an ideal on $\mathscr{Q}$. The triple $\beta = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ is called ideal bipolar soft approximation space

(IBSA-space for short). Depending on $\beta_{\mathscr{L}}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$ by:

$$\left.\begin{array}{l} \underline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathscr{L}\}, \\ \overline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im \notin \mathscr{L}\}, \\ \overline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im \in \mathscr{L}\}, \\ \underline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im^c \notin \mathscr{L}\} \end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be ideal soft $\beta_{\mathscr{L}}$-lower positive, ideal soft $\beta_{\mathscr{L}}$ upper positive,ideal soft $\beta_{\mathscr{L}}$-upper negative, and ideal soft $\beta_{\mathscr{L}}$-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l} \underline{\mathfrak{P}\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta}(\Im) = \left(\underline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta+}(\Im), \underline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta-}(\Im)\right) \\ \overline{\mathfrak{P}\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta}(\Im) = \left(\overline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta+}(\Im), \overline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta-}(\Im)\right) \end{array}\right\}$$

are called the ideal bipolar SAs of $\Im$ with respect to the IBSA-space.

3. Novel style of bipolar soft rough sets approximation based on ideals

Definition 3.1. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\mathscr{L}$ be an ideal on $\mathscr{Q}$. The triple $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ is called an ideal bipolar soft approximation space (IBSA-space for short). Based on $\beta_{\mathscr{L}}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$ by:

$$\left.\begin{array}{l} \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathscr{L}\}, \\ \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im^c)\right)^c, \\ \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im \in \mathscr{L}\}, \\ \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c \end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be ideal soft $\beta_{\mathscr{L}}$-lower positive, ideal soft $\beta_{\mathscr{L}}$ upper positive,ideal soft $\beta_{\mathscr{L}}$-upper negative, and ideal soft $\beta_{\mathscr{L}}$-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l} \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) \\ \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) \end{array}\right\}$$

are called the ideal bipolar SAs of $\Im$ with respect to the IBSA-space. Moreover, when $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im) \neq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im)$ then, $\Im$ is termed as an ideal bipolar soft rough set and $\Im$ is said to be ideal bipolar soft $\beta_{\mathscr{L}}$-rough; otherwise $\Im$ is called ideal bipolar soft $\beta_{\mathscr{L}}$-definable. The related positive, boundary, and negative regions with respect to the ideal bipolar SAs are given by

$$\begin{aligned} \mathcal{POS}^{\mathscr{L}}_{\beta}(\Im) &= \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right), \\ \mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) - \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im),\ \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) - \overline{SR}_{\beta-}(\Im)\right) \\ \mathcal{NEG}^{\mathscr{L}}_{\beta}(\Im) &= (\mathscr{Q}, \mathscr{Q}) - \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)\right)^c, \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c\right). \end{aligned}$$

Remark 3.1. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. From Definition 3.1, we deduce that:

(1) $\Im \subseteq \mathscr{Q}$ is ideal bipolar soft rough definable when $\mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) = (\emptyset, \emptyset)$.

(2) The ideal soft lower PAs in Definition 2.12 in [35] and ideal soft $\beta_{\mathscr{L}}$-lower PAs of $\Im$ are identical. That is, $\underline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta+}(\Im) = \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)$.

(3) The ideal soft upper NAs in Definition 2.12 in [35] and ideal soft $\beta_{\mathscr{L}}$-upper NAs of $\Im$ are identical. That is, $\overline{\mathfrak{S}\mathfrak{F}}^{\mathscr{L}}_{\beta-}(\Im) = \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$.

(4) If $\mathscr{L} = \emptyset$ in Definition 3.1, then these ideal bipolar SAs coincide with these bipolar soft rough approximations in Definition 2.9 in [34]. So, the bipolar soft rough approximations in [34] are special cases of these ideal bipolar SAs.

(5) For $\Im \subseteq \mathscr{Q}$, we have $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \Im$, $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \Im^c$, $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \supseteq \Im$ and $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \supseteq \Im^c$ did not hold in general. Moreover, it is not necessary to be $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \emptyset$.

(6) We see that $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)$ coincides with the Definition 2.8 given in [20]. That is, $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}(\Im)$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}(\Im)$.

Here, we offer the following example to make the idea of ideal bipolar SAs clear.

Example 3.1. *Let* $(f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ *with* $Q = \{\daleth_1,\ \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$. *The maps* f *and* g *are given as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$
$$\varsigma \mapsto \begin{cases} \{\daleth_1, \daleth_6\}, & \mathit{if}\ \varsigma = \varsigma_1, \\ \{\daleth_3, \daleth_4\}, & \mathit{if}\ \varsigma = \varsigma_2, \\ \{\}, & \mathit{if}\ \varsigma = \varsigma_3, \\ \{\daleth_2, \daleth_5\}, & \mathit{if}\ \varsigma = \varsigma_4, \end{cases} \quad \mathit{and} \quad \neg\varsigma \mapsto \begin{cases} \{\daleth_3, \daleth_5\}, & \mathit{if}\ \neg\varsigma = \neg\varsigma_1, \\ \{\daleth_5\}, & \mathit{if}\ \neg\varsigma = \neg\varsigma_2, \\ \{\daleth_2, \daleth_6\}, & \mathit{if}\ \neg\varsigma = \neg\varsigma_3, \\ \{\daleth_4\}, & \mathit{if}\ \neg\varsigma = \neg\varsigma_4. \end{cases}$$

Define an ideal $\mathscr{L}$ *on* $\mathscr{Q}$ *as* $\mathscr{L} = \{\emptyset, \{\daleth_2\}, \{\daleth_3\}, \{\daleth_2, \daleth_3\}\}$. *According to Definition 3.1, we can evaluate the ideal bipolar SAs of* $\Im = \{\daleth_3, \daleth_4, \daleth_5\} \subseteq \mathscr{Q}$ *as follows.*

$$\begin{aligned} \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) &= \{\daleth_3, \daleth_4\} \\ \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) &= \{\daleth_2, \daleth_5\}, \\ \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) &= \{\daleth_2, \daleth_6\}, \\ \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) &= \{\daleth_1, \daleth_6\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) &= \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) = (\{\daleth_3, \daleth_4\}, \{\daleth_1, \daleth_6\}) \\ \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) = (\{\daleth_2, \daleth_5\}, \{\daleth_2, \daleth_6\}) \end{aligned}$$

Consequently, $\Im$ *is an ideal bipolar soft* $\beta_{\mathscr{L}}$*-rough because* $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \neq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im)$. *Moreover, by direct calculations, we obtain*

$$\begin{aligned} &\mathcal{POS}^{\mathscr{L}}_{\beta}(\Im) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) = (\{\daleth_3, \daleth_4\}, \{\daleth_1, \daleth_6\}), \\ &\mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) - \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im),\ \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) - \overline{SR}_{\beta-}(\Im)\right) = (\{\daleth_2, \daleth_5\}, \{\daleth_1\}), \\ &\mathcal{NEG}^{\mathscr{L}}_{\beta}(\Im) == \left(\left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)\right)^c, \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c\right) = (\{\daleth_1, \daleth_3, \daleth_4, \daleth_6\}, \{\daleth_2, \daleth_3, \daleth_4, \daleth_5\}). \end{aligned}$$

Remark 3.2. From Example 3.1, the relationship between the containment of $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im)$ and $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im)$ is in general:

(1) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \Im \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsupseteq \Im \nsupseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$,

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \nsubseteq \Im \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \nsupseteq \Im \nsupseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$.

Theorem 3.1. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Let $\Im \subseteq \mathscr{Q}$, $f(\varsigma) \notin \mathscr{L}$ and $g(\neg\varsigma) \notin \mathscr{L}$ for all $\varsigma \in \wp$, then $\Im$ is ideal bipolar soft $\beta_{\mathscr{L}}$-definable if and only if $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \Im \in \mathscr{L}$.*

Proof. Let $\Im$ be ideal bipolar soft $\beta_{\mathscr{L}}$-definable, then $\underline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \neq \overline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)$, So, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Thus, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$. In fact, if $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset$, then $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$. Assume that $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \neq \emptyset$ and $\daleth \in \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. We have two cases.
The first case, if $\daleth \notin \Im$, then $\daleth \in \Im^c$ and there exists $\varsigma \in \wp$ such that $\daleth \in f(\varsigma)$ and $f(\varsigma) \cap \Im^c \in \mathscr{L}$. Therefore, $\daleth \in f(\varsigma) \cap \Im^c \in \mathscr{L}$ and hence $\{\daleth\} \in \mathscr{L}$. Hence, $\bigcup_{\daleth \in \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)}^{\daleth \in \Im^c} \{\daleth\} \in \mathscr{L}$.

The second case, if $\daleth \in \Im$, then $\daleth \notin \Im^c$ and so $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c = \emptyset \in \mathscr{L}$. The two cases lead to $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$. Since, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Then, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c = \emptyset \in \mathscr{L}$. Similarly, the other part could be given.

Conversely, assume that $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \Im \in \mathscr{L}$. Since, $f(\varsigma) \notin \mathscr{L}$ and $g(\neg\varsigma) \notin \mathscr{L}$ for all $\varsigma \in \wp$. Then, it is obvious that $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \supseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. So, it sufficient to show that $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \supseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Let $\daleth \in \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Then, there exists $\varsigma \in \wp$ such that $\daleth \in f(\varsigma) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Since, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c \in \mathscr{L}$ and $f(\varsigma) \cap \Im^c \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \Im^c$, then $f(\varsigma) \cap \Im^c \in \mathscr{L}$ and hence $\daleth \in \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Also, let $\daleth \in \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Then, there exists $\neg\varsigma \in \neg\wp$ such that $\daleth \in g(\neg\varsigma) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Since, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \Im \in \mathscr{L}$ and $g(\neg\varsigma) \cap \Im \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \Im$, then $g(\neg\varsigma) \cap \Im \in \mathscr{L}$ and hence $\daleth \in \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Consequently, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$.

Proposition 3.1. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Let $\Im \subseteq \mathscr{Q}$, $f(\varsigma) \notin \mathscr{L}$ and $g(\neg\varsigma) \notin \mathscr{L}$ for all $\varsigma \in \wp$. Then, $\Im$ is ideal bipolar soft $\beta_{\mathscr{L}}$-definable if $\Im \cap \left[\bigcup_{\varsigma \in \wp} f(\varsigma) \ \cup \bigcup_{\neg\varsigma \in \neg\wp} g(\neg\varsigma) \right] \in \mathscr{L}$.*

Proof. Since, $f(\varsigma) \notin \mathscr{L}$ and $g(\neg\varsigma) \notin \mathscr{L}$ for all $\varsigma \in \wp$. Then,
$\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \supseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Let $\Im \subseteq \mathscr{Q}$ such that
$\Im \cap \left[\bigcup_{\varsigma \in \wp} f(\varsigma) \ \cup \bigcup_{\neg\varsigma \in \neg\wp} g(\neg\varsigma) \right] \in \mathscr{L}$. Then, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset$. In fact, assume that $\daleth \in \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Then, there exists $\varsigma \in \wp$ such that $\daleth \in f(\varsigma)$ and $f(\varsigma) \cap \Im \notin \mathscr{L}$, Since
$f(\varsigma) \cap \Im \subseteq \Im \cap \left[\bigcup_{\varsigma \in \wp} f(\varsigma) \ \cup \bigcup_{\neg\varsigma \in \neg\wp} g(\neg\varsigma) \right]$, then $\Im \cap \left[\bigcup_{\varsigma \in \wp} f(\varsigma) \ \cup \bigcup_{\neg\varsigma \in \neg\wp} g(\neg\varsigma) \right] \notin \mathscr{L}$ a contradiction. So, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset$. Hence,
$\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Let $\daleth \in \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Then, there exists $\neg\varsigma \in \neg\wp$ such that $\daleth \in g(\neg\varsigma)$ and $g(\neg\varsigma) \cap \Im^c \notin \mathscr{L}$, Since, $g(\neg\varsigma) \cap \Im \subseteq \Im \cap \left[\bigcup_{\varsigma \in \wp} f(\varsigma) \ \cup \bigcup_{\neg\varsigma \in \neg\wp} g(\neg\varsigma) \right] \in \mathscr{L}$, then $g(\neg\varsigma) \cap \Im \in \mathscr{L}$. Hence, $\daleth \in \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$, therefore $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$.

Theorem 3.2. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space. For* $\Im, \vartheta \subseteq \mathscr{Q}$*, the following properties hold:*

(1) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im^c) = [\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]^c$;

(2) $\Im \subseteq \vartheta \Rightarrow \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$;

(3) $\Im \subseteq \vartheta \Rightarrow \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$;

(4) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)] = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$;

(5) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)] = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$;

(6) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cap \vartheta) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$;

(7) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cup \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta)$;

(8) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cap \vartheta) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$;

(9) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta)$.

Proof. The proof is analogous to the proof of a proposition given in [20] where the soft ideal approximations $\underline{\mathfrak{SR}}^{\mathscr{L}}(\Im)$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}(\Im)$ are used instead of $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$, respectively.

Remark 3.3. $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. For $\Im, \vartheta \subseteq \mathscr{Q}$, we deduce by the next examples that in general:

(1) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q}) \neq \mathscr{Q}$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) \neq \emptyset$.

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) \neq \emptyset$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q}) \neq \mathscr{Q}$.

(3) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \Im$,$\Im \nsubseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \Im$, $\Im \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$.

(4) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) \nRightarrow \Im \subseteq \vartheta$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) \nRightarrow \Im \subseteq \vartheta$.

(5) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsupseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$.

(6) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsupseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$.

(7) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cap \vartheta) \neq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta) \neq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$.

Example 3.2. *Let* $(f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$*, where* $Q = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}$*. The maps* f *and* g *are given as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$

$$\varsigma \mapsto \begin{cases} \{\daleth_1, \daleth_6\}, & \text{if } \varsigma = \varsigma_1, \\ \{\}, & \text{if } \varsigma = \varsigma_2, \\ \{\daleth_3\}, & \text{if } \varsigma = \varsigma_3, \\ \{\daleth_1, \daleth_2, \daleth_3\}, & \text{if } \varsigma = \varsigma_4, \\ \{\daleth_1, \daleth_2, \daleth_5\}, & \text{if } \varsigma = \varsigma_5 \end{cases} \quad \text{and} \quad \neg\varsigma \mapsto \begin{cases} \{\daleth_4, \daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\daleth_2, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\daleth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\daleth_3, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_5. \end{cases}$$

Consider $\mathscr{L} = \{\emptyset, \{\daleth_1\}, \{\daleth_6\}, \{\daleth_1, \daleth_6\}\}$*. Then,* *(1)* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \cup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap X^c \in \mathscr{L})\} = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6\} \neq \Im$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) = [\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]^c = \{\gimel_4\} \neq \emptyset$.

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) = \cup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \emptyset^c \in \mathscr{L})\} = \{\gimel_1, \gimel_6\} \neq \emptyset$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = [\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset)]^c = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\} \neq \Im$.

(3) Let $\Im = \{\gimel_4\}$. *Then,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_1, \gimel_6\}$. *Hence,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \Im$ *and* $\Im \nsubseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$.

(4) From part (3), if $\Im = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6\}$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$. *Hence,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \Im$ *and* $\Im \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$.

(5) Let $\Im = \{\gimel_1, \gimel_2, \gimel_3\}, B = \{\gimel_1, \gimel_2, \gimel_4\}$. *Then,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_1, \gimel_2, \gimel_3, \gimel_6\}$ *and* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) = \{\gimel_1, \gimel_6\}$. *So,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$, *but* $\vartheta \nsubseteq \Im$. *Also, if* $\Im = \{\gimel_4, \gimel_5, \gimel_6\}, B = \{\gimel_3, \gimel_5, \gimel_6\}$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_4, \gimel_5\}$ *and* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$. *So,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta)$ *but* $\Im \nsubseteq \vartheta$.

(6) From part (4), if $\Im = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6\}$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$. *So,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)] = X$. *Hence,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsupseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$.

(7) From part (6), if $\Im = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5, \gimel_6\}$. *Then,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \Im$ *but* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)] = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$. *Hence,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$ *and* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsupseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$.

(8) Consider $\mathscr{L} = \{\emptyset, \{\gimel_4\}, \{\gimel_6\}, \{\gimel_4, \gimel_6\}\}$ *and* $\Im = \{\gimel_1, \gimel_2, \gimel_5\}$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\gimel_1, \gimel_2, \gimel_4, \gimel_5, \gimel_6\}$. *So,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)] = \{\gimel_1, \gimel_6\}$. *Hence,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \nsubseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}[\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)]$.

(9) Consider $\mathscr{L} = \{\emptyset, \{\gimel_4\}, \{\gimel_5\}, \{\gimel_4, \gimel_5\}\}$. *If* $\Im = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5\}$ *and* $B = \{\gimel_1, \gimel_6\}$. *Then,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) = \{\gimel_1, \gimel_2, \gimel_3, \gimel_5\} \cap \{\gimel_1, \gimel_6\} = \{\gimel_1\}$ *but* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cap \vartheta) = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\{\gimel_1\}) = \emptyset$. *Also, if* $\Im = \{\gimel_2, \gimel_3\}$ *and* $B = \{\gimel_4, \gimel_5, \gimel_6\}$. *Then,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\vartheta) = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5\} \cup \{\gimel_4, \gimel_6\} = \{\gimel_2, \gimel_3, \gimel_4, \gimel_5, \gimel_6\}$ *but* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta) = X$.

Theorem 3.3. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space. For* $\Im, \vartheta \subseteq \mathscr{Q}$, *the following properties hold:*

(1) $\Im \subseteq \vartheta \Longrightarrow \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$;

(2) $\Im \subseteq \vartheta \Longrightarrow \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$;

(3) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) \supseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta)$;

(4) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \subseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta)$;

(5) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) \supseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cup \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta)$;

(6) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \subseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\vartheta)$;

(7) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im^c) = \left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$.

Proof.

(1) Let $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \vartheta \in \mathscr{L}\}$. Therefore, there exist some $g(\neg\varsigma)$ such that $\daleth \in g(\neg\varsigma)$ such that $g(\neg\varsigma) \cap \vartheta \in \mathscr{L}$. Because $\Im \subseteq \vartheta$ and $\mathscr{L}$ is an ideal, thus in particular, $\daleth \in g(\neg\varsigma)$ and $g(\neg\varsigma) \cap \Im \in \mathscr{L}$. Therefore, $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$. Consequently, $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta) \subseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$

(2) As $\Im \subseteq \vartheta$, so $\Im^c \supseteq \vartheta^c$. By part (1), we can infer that $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c) \subseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)$. Thus, it implies that, $\left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c \supseteq \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)\right)^c$. Hence, $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$.

(3) Assume that $\daleth \notin \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap (\Im \cap \vartheta)\mathscr{L}\}$. Therefore, for all $\neg\varsigma \in \aleph, \daleth \in g(\neg\varsigma)$, we have $g(\neg\varsigma) \cap [\Im \cap \vartheta] \notin \mathscr{L}$. Then, for all $\neg\varsigma \in \aleph$, $\daleth \in g(\neg\varsigma)$, we have $g(\neg\varsigma) \cap \Im \notin \mathscr{L}$ or $g(\neg\varsigma) \cap \vartheta \notin \mathscr{L}$. Consequently, $\daleth \notin \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$ or $\daleth \notin \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$. This implies that, $\daleth \notin \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$. Hence, $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) \supseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$

(4) Assume that $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap (\Im \cup \vartheta) \in \mathscr{L}\}$. Thus, there exists some $g(\neg\varsigma)$ such that $\daleth \in g(\neg\varsigma)$ and $g(\neg\varsigma) \cap [\Im \cup \vartheta] \in \mathscr{L}$. This implies that $\daleth \in g(\neg\varsigma)$ such that $g(\neg\varsigma) \cap \Im \in \mathscr{L}$ and $g(\neg\varsigma) \cap \vartheta \in \mathscr{L}$. Consequently, $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$ and $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$. Therefore, $\daleth \in \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$. Hence, we obtain $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \subseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

(5) By Definition 3.1, it follows that

$$\begin{aligned}\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta)^c\right)^c = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c \cup \vartheta^c)\right)^c \supseteq \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)\right)^c \\ &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c \cup \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)\right)^c = \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\end{aligned}$$

(6) By Definition 3.1 it follows that

$$\begin{aligned}\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta)^c\right)^c = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c \cap \vartheta^c)\right)^c \subseteq \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)\right)^c \\ &= \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c \cap \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta^c)\right)^c = \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta).\end{aligned}$$

Hence, $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

(7) By definition of ideal soft $\beta_{\mathscr{L}}$-lower NA of $\Im$, we have $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c$. This indicates that $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im^c) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$. This completes this proof.

Remark 3.4. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. For $\Im, \vartheta \subseteq \mathscr{Q}$, the following example indicates that in general:

(1) $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) \neq \mathscr{Q}$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\emptyset) \neq \emptyset$.

(2) $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\emptyset) \neq \emptyset$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) \neq \mathscr{Q}$.

(3) $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) \nsubseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

(4) $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \nsupseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

(5) $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cap \vartheta) \nsubseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$,

(6) $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \nsupseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

Example 3.3. *Let* $(f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *with* $\mathscr{Q} = \{\beth_1, \beth_2, \beth_3, \beth_4, \beth_5, \beth_6\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$. *The maps* f *and* g *are as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$

$$\varsigma \mapsto \begin{cases} \{\beth_1, \beth_2, \beth_3\}, & if\ \varsigma = \varsigma_1, \\ \{\beth_1, \beth_4\}, & if\ \varsigma = \varsigma_2, \\ \{\beth_1, \beth_3\}, & if\ \varsigma = \varsigma_3, \\ \{\beth_3, \beth_5, \beth_6\}, & if\ \varsigma = \varsigma_4, \\ \{\beth_2, \beth_4\}, & if\ \varsigma = \varsigma_5, \\ \{\beth_1, \beth_2, \beth_5\}, & if\ \varsigma = \varsigma_6, \end{cases} \quad and \quad \neg\varsigma \mapsto \begin{cases} \{\beth_4, \beth_5\}, & if\ \neg\varsigma = \neg\varsigma_1, \\ \{\beth_5\}, & if\ \neg\varsigma = \neg\varsigma_2, \\ \{\beth_2, \beth_6\}, & if\ \neg\varsigma = \neg\varsigma_3, \\ \{\beth_4\}, & if\ \neg\varsigma = \neg\varsigma_4, \\ \{\beth_3, \beth_6\}, & if\ \neg\varsigma = \neg\varsigma_5, \\ \{\beth_3, \beth_4, \beth_6\}, & if\ \neg\varsigma = \neg\varsigma_6. \end{cases}$$

Consider an ideal $\mathscr{L}$ *on* $\mathscr{Q}$ *as* $\mathscr{L} = \{\emptyset, \{\beth_1\}, \{\beth_3\}, \{\beth_1, \beth_3\}\}$.

(1) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\emptyset) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \emptyset \in \mathscr{L}\} = \{\beth_2, \beth_3, \beth_4, \beth_5, \beth_6\} \neq \emptyset$, *Also,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\mathscr{Q}) = [\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\emptyset)]^c = \{\beth_1\} \neq \mathscr{Q}$.

(2) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\mathscr{Q}) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \mathscr{Q} \in \mathscr{L}\} = \emptyset \neq \mathscr{Q}.$, *Also,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\emptyset) = [\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im)]^c = \mathscr{Q} \neq \emptyset$.

(3) If we take, $\Im, \vartheta \subseteq \mathscr{Q}$ *such that* $\Im = \{\beth_2\}$ *and* $\vartheta = \{\beth_6\}$. *Then,* $\Im \cap \vartheta = \emptyset$. *By direct computation, we obtain*
$\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) = \{\beth_3, \beth_4, \beth_5, \beth_6\}$ *and* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) = \mathscr{Q}, \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta) = \{\beth_4, \beth_5\}$ *and* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta) = \mathscr{Q}$, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) = \{\beth_2, \beth_3, \beth_4, \beth_5, \beth_6\}$. *Clearly,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) = \{\beth_2, \beth_3, \beth_4, \beth_5, \beth_6\} \supset \{\beth_3, \beth_4, \beth_5, \beth_6\} = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$. *So,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) \supset \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$. *Hence,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) \nsubseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$.

(4) In part (1), $\Im \cup \vartheta = \{\beth_2, \beth_6\}$. *Thus,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) = \{\beth_1, \beth_4, \beth_5\} \subset \mathscr{Q} = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$, *That is,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) \subset \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$. *That is* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) \nsupseteq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$.

(5) Now, if we consider that $\Im, \vartheta \subseteq \mathscr{Q}$ *such that* $\Im = \{\beth_3, \beth_4, \beth_5\}$ *and* $\vartheta = \{\beth_1, \beth_2, \beth_5\}$. *Then,* $\Im \cup \vartheta = \{\beth_1, \beth_2, \beth_3, \beth_4, \beth_5\}$. *Therefore,*
$\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) = \{\beth_2, \beth_3, \beth_6\}, \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta) = \{\beth_3, \beth_4, \beth_6\}$ *and* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) = \emptyset$
Clearly, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) = \emptyset \subset \{\beth_3, \beth_6\} = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$. *That is,* $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) \subset \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im)$, *which implies that* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) \nsubseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$.

(6) Now, if we assume that $\Im, \vartheta \subseteq \mathscr{Q}$ *is such that* $\Im = \{\beth_3, \beth_4, \beth_6\}$ *and* $\vartheta = \{\beth_2, \beth_6\}$. *Then,* $\Im \cap \vartheta = \{\beth_6\}$. *Then,*
$\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) = \{\beth_1, \beth_2, \beth_5\}, \underline{S}_{\mathscr{L}^-}(\vartheta) = \{\beth_1, \beth_4, \beth_5\}$, *and* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) = \mathscr{Q}$.
Clearly, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) = \mathscr{Q} \supset \{\beth_1, \beth_2, \beth_4, \beth_5\} = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$, *That is,* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cap \vartheta) \supset \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cup \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$, *which shows that* $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im \cup \vartheta) \nsupseteq \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\Im) \cap \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\vartheta)$.

Remark 3.5. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Then, the above example shows that in general:

(1) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^+}(\emptyset) \neq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^+}(\emptyset)$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\emptyset) \neq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta^-}(\emptyset)$.

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q}) \neq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q})$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) \neq \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q})$.

The condition in which the ideal soft $\beta_{\mathscr{L}}$-upper positive and deal soft $\beta_{\mathscr{L}}$-lower PAs of $\mathscr{Q}$ and $\emptyset$ coincide is shown in the following results.

Proposition 3.2. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a full bipolar soft set, that is $\bigcup_{e\in\wp} f(\varsigma) = \mathscr{Q}$ and $\bigcup_{\neg\varsigma\in\wp} g(\neg\varsigma) = \mathscr{Q}$. Then,*

(1) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q}) = \mathscr{Q}$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) = \emptyset$.

(2) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\emptyset) = \emptyset$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) = \mathscr{Q}$.

(3) $\Im \in \mathscr{L} \Rightarrow \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) = \mathscr{Q}$ and $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) = \emptyset$.

(4) $\Im^c \in \mathscr{L} \Rightarrow \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \mathscr{Q}$ and $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset$.

(5) If $\mathscr{L} = 2^{\mathscr{Q}}$, then $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) = \emptyset$.

Proof.

(1) Based on Definition 3.1 we get $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q}) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \emptyset \in \mathscr{L}\}$. Since $\bigcup_{e\in\wp} f(\varsigma) = \mathscr{Q}$, therefore $\underline{\mathcal{S}}_{\beta+}(\mathscr{Q}) = \bigcup_{e\in\wp} f(\varsigma) = \mathscr{Q}$. Now, by Definition 3.1, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) = \left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset^c)\right)^c = \left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\mathscr{Q})\right)^c = \mathscr{Q}^c = \emptyset$. That is, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\emptyset) = \emptyset$.

(2) From Definition 3.1, we get that $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\emptyset) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap (\emptyset) \in \mathscr{L}\}$. Since, $\bigcup_{\neg\varsigma\in\aleph} g(\neg\varsigma) = \mathscr{Q}$. Therefore, it implies that $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\emptyset) = \bigcup_{\neg\varsigma\in\aleph} g(\neg\varsigma) = \mathscr{Q}$. That is, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\emptyset) = \mathscr{Q}$. By Definition 3.1, we get $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) = \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}^c)\right)^c = \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\emptyset)\right)^c = \mathscr{Q}^c = \emptyset$. That is, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\mathscr{Q}) = \emptyset$.

(3), (4) and (5) are Similar to (1),(2).

Corollary 3.1. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and then,*

(1) $\mathfrak{B} = (f, g : \wp)$ is a full bipolar soft set;

(2) $\underline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\mathscr{Q}) = (\mathscr{Q}, \mathscr{Q})$;

(3) $\overline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\emptyset) = (\emptyset, \emptyset)$.

Proof. Direct consequence of Proposition 3.2.

The coming result explains the relationship between the ideal soft upper positive, ideal soft lower NAs in [35] and our ideal soft $\beta_{\mathscr{L}}$-upper positive, and ideal soft $\beta_{\mathscr{L}}$-lower NAs of $\Im$.

Proposition 3.3. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a full bipolar soft set. Then, for any $\Im \subseteq \mathscr{Q}$, the following properties hold.*

(1) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im)$;

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta-}(\Im)$.

Proof.

(1) Let $\daleth \notin \overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im) = \{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im \notin \mathscr{L}\}$. Then, for all $e \in \wp$, we have $f(\varsigma) \cap \Im \in \mathscr{L}$. So, $\daleth \in (\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im^c))$. Therefore, $\daleth \notin \left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im^c)\right)^c = \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)$. Consequently, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im)$.

(2) Assume that $\daleth \notin \underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta-}(\Im) = \{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im^c \notin \mathscr{L}\}$. Then, for all $\neg\varsigma \in \aleph$, we have $g(\neg\varsigma) \cap \Im^c \in \mathscr{L}$. Therefore, $\daleth \in \overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im^c)$. Thus, $\daleth \notin \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im^c)\right)^c = \underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)$. Hence, $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) \subseteq \underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta-}(\Im)$.

Remark 3.6. The previous proposition indicates that the ideal soft upper PA of $\Im \subseteq \mathscr{Q}$ in [35] is finer than our ideal soft $\beta_{\mathscr{L}}$-upper PA of X . Similarly, the ideal soft lower NA of $\Im \subseteq \mathscr{Q}$ in [35] is finer than the ideal soft $\beta_{\mathscr{L}}$-lower NA of $\Im \subseteq \mathscr{Q}$. The coming example shows that the inclusions in parts (1) and (2) of the above proposition may strictly hold.

Example 3.4. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space, where $\mathscr{Q} = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5\}$ and $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$. The maps f and g are as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$

$$\varsigma \mapsto \begin{cases} \{\daleth_1, \daleth_3\}, & \text{if } \varsigma = \varsigma_1 \\ \{\daleth_1, \daleth_4, \daleth_5\}, & \text{if } \varsigma = \varsigma_2 \\ \{\daleth_2\}, & \text{if } \varsigma = \varsigma_3 \\ \{\daleth_2, \daleth_4, \daleth_5\}, & \text{if } \varsigma = \varsigma_4 \\ \{\daleth_1, \daleth_2\}, & \text{if } \varsigma = \varsigma_5 \\ \{\daleth_3, \daleth_5\}, & \text{if } \varsigma = \varsigma_6 \end{cases} \quad \text{and} \quad \neg\varsigma \mapsto \begin{cases} \{\daleth_2, \daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\daleth_3\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\daleth_3, \daleth_4, \daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\daleth_1, \daleth_3\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_5, \\ \{\daleth_2, \daleth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_6. \end{cases}$$

Consider, $\mathscr{L} = \{\emptyset, \{\daleth_1\}, \{\daleth_2\}, \{\daleth_1, \daleth_2\}\}$. If we take $\Im = \{\daleth_1, \daleth_3, \daleth_5\}$. Then,

$$\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) = \{\daleth_3, \daleth_4, \daleth_5\}$$
$$\overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im) = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5\}$$

Clearly, $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im) \subset \overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im)$, which indicates that the inclusions in part (1) of Proposition 3.3 may strictly hold. Here, if $\Im = \{\daleth_3, \daleth_4\}$. Then,

$$\underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta-}(\Im) = \{\daleth_2, \daleth_3, \daleth_4, \daleth_5\}$$
$$\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im) = \{\daleth_5\}$$

Clearly, we can see the following: $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-} \subset \underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta-}(\Im)$, indicating that the inclusion in part (2) of Proposition 3.3 might be strict.

Theorem 3.4. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\Im \subseteq \mathscr{Q}$. Then, the following properties hold.*

(1) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}\left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right) = \left(\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$;

(2) $\underline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}\left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right) = \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$;

(3) $\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}\left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right) \subseteq \left(\overline{\mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$.

(4) $\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) \supseteq \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}$.

Proof.

(1) *Let* $\vartheta = \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}$ *and* $\daleth \in \vartheta = \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c} = \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im^{c}) = \bigcup\{g(\neg\varsigma),\ \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im^{c} \in \mathscr{L}\}$. *Then, there exists some* $\neg\varsigma \in \aleph$ *such that* $\daleth \in g(\neg\varsigma)$ *and* $g(\neg\varsigma) \cap \vartheta^{c} \in \mathscr{L}$. *So,* $\daleth \in \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\vartheta^{c}) = \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left[\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}\right]^{c} = \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *Therefore,* $\daleth \in \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$.

Hence, $\vartheta \subseteq \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *This implies that* $\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c} \subseteq \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$.

Conversely, let $\daleth \notin \vartheta = \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c} = \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im^{c}) = \bigcup\{g(\neg\varsigma),\ \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im^{c} \in \mathscr{L}\}$. *Then, for all* $\neg\varsigma \in \aleph$ *with* $\daleth \in g(\neg\varsigma)$, *we have* $g(\neg\varsigma) \cap \vartheta^{c} \notin \mathscr{L}$. *So,* $\daleth \notin \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\vartheta^{c}) = \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *Therefore,* $\daleth \notin \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *Hence,* $\vartheta \supseteq \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *This implies that,* $\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c} \supseteq \overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)$. *Hence,* $\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) \supseteq \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}$. *Consequently,* $\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) \supseteq \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}$.

(2) *By Definition 3.1, it follows that*

$$\begin{aligned}
\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) &= \left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}\right)^{c} \\
&= \left[\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im^{c})\right)\right]^{c} \quad \text{by part (7) of Theorem 3.3} \\
&= \left[\left[\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im^{c})\right]^{c}\right]^{c} \quad \text{by part (1)} \\
&= \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im^{c}) = \left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c} \quad \text{by part (7) of Theorem 3.3.}
\end{aligned}$$

Hence, $\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) = \left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right)^{c}$.

The proofs of parts (3) and (4) are similar to that of part (1).

The following example explains that the inclusions in part (1) and (4) of Theorem 3.4 might strictly hold.

Example 3.5. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space with* $\mathscr{Q} = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6\}$. *The maps* f *and* g *are as follows:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad \varsigma \mapsto \begin{cases} \{\daleth_1, \daleth_2, \daleth_3\}, & \text{if } \varsigma = \varsigma_1, \\ \{\daleth_1, \daleth_4\}, & \text{if } \varsigma = \varsigma_2, \\ \{\daleth_1, \daleth_3\}, & \text{if } \varsigma = \varsigma_3, \\ \{\daleth_3, \daleth_5, \daleth_6\}, & \text{if } \varsigma = \varsigma_4, \\ \{\daleth_2, \daleth_4\}, & \text{if } \varsigma = \varsigma_5, \\ \{\daleth_1, \daleth_2, \daleth_5\}, & \text{if } \varsigma = \varsigma_6, \end{cases} \quad \text{and} \quad g : \aleph \longrightarrow 2^{\mathscr{Q}}, \qquad \neg\varsigma \mapsto \begin{cases} \{\daleth_4, \daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\daleth_2, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\daleth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\daleth_3, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_5, \\ \{\daleth_3, \daleth_4, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_6. \end{cases}$$

If we consider $\mathscr{L} = \{\emptyset, \{\daleth_1\}\}$ *and* $\Im = \{\daleth_1, \daleth_3, \daleth_5, \daleth_6\}$, *then* $\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) = \{\daleth_1, \daleth_2, \daleth_4\}$ *and* $\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) = \{\daleth_4\}$. *Therefore,* $\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) = \{\daleth_2, \daleth_3, \daleth_5, \daleth_6\}$. *Also,*
$\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}\left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im)\right) = \{\daleth_1, \daleth_2, \daleth_3, \daleth_5, \daleth_6\}$.

Clearly, $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}\left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) = \{\daleth_2, \daleth_3, \daleth_5, \daleth_6\} \subset \{\daleth_1, \daleth_2, \daleth_3, \daleth_5, \daleth_6\} = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$*, indicating that the inclusion in part (3) of Theorem 3.4 might be strict. Also,* $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}\left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) = \{\daleth_1, \daleth_2, \daleth_3, \daleth_5, \daleth_6\} \supset \{\daleth_3, \daleth_5, \daleth_6\} = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^c$*, which shows that the inclusions in part (4) of Theorem 3.4 may strictly hold.*

Definition 3.2. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Let $\Im, \vartheta \subseteq \mathscr{Q}$. Then,

(1) $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta) \Longleftrightarrow \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta)$ and $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \supseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

(2) $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta) \Longleftrightarrow \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta)$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \supseteq \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$.

Definition 3.3. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Let $\Im, \vartheta \subseteq \mathscr{Q}$. Then,

(1) $\left(\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)\right) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\right)$.

(2) $\left(\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)\right) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\right)$.

Definition 3.4. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space. Let $\Im, \vartheta \subseteq \mathscr{Q}$. Then,

(1) $\left(\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcap \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)\right) = \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\right)$.

(2) $\left(\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcap \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)\right) = \left(\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta), \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\right)$.

Theorem 3.5. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space and* $\Im \subseteq \mathscr{Q}$*. Then, the following properties hold.*

(1) If $\Im \subseteq \vartheta$*, then* $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$ *and* $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$;

(2) $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im \cup \vartheta) \sqsupseteq \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$;

(3) $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im \cap \vartheta) \sqsubseteq \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcap \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$;

(4) $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im \cup \vartheta) \sqsupseteq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$;

(5) $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im \cap \vartheta) \sqsubseteq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcap \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta)$.

Proof.

(1) Assume that $\Im \subseteq \vartheta$. Since $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta)$ and $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$ by Theorems 3.2 and 3.3. Then, $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta) \sqsubseteq \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im)$ by Definition 3.2. The other part is similar.

(2) Since $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta) \supseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta)$ and $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta) \subseteq \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)$ by Theorems 3.2 and 3.3. Then,

$$\begin{aligned}
\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im \cup \vartheta) &= \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im \cup \vartheta), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im \cup \vartheta)\right) \\
&\sqsupseteq \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\vartheta), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\vartheta)\right) \\
&= \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)\right) \sqcup \left(\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im), \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right) \\
&= \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\vartheta).
\end{aligned}$$

The other parts can be proved similarly. Any one can add examples to induce that the inclusions in parts (3) and (5) in Theorem 3.5 might be precise.

Proposition 3.4. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$, $\mathscr{L}, \mathcal{J}$ be two ideals on U and let $\Im \subseteq \mathscr{Q}$. Then,*

(1) $\mathscr{L} \subseteq \mathcal{J} \Longrightarrow \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqsubseteq \underline{\mathfrak{PSR}}_{\beta}^{\mathcal{J}}(\Im)$.

(2) $\mathscr{L} \subseteq \mathcal{J} \Longrightarrow \overline{\mathfrak{PSR}}_{\beta}^{\mathcal{J}}(\Im) \sqsubseteq \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im)$.

Proof. *Straightforward by Theorems 3.2 and 3.3, and Definition 3.2.*

The coming example shows that the inclusions in the above proposition might be precise.

Example 3.6. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\mathscr{Q} = \{\gimel_1.\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$ and $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$. The maps f and g are as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$

$$\varsigma \mapsto \begin{cases} \{\gimel_1, \gimel_4\}, & \text{if } \varsigma = \varsigma_1, \\ \{\gimel_5\}, & \text{if } \varsigma = \varsigma_2, \\ \{\gimel_2, \gimel_3\}, & \text{if } \varsigma = \varsigma_3, \\ \{\gimel_1, \gimel_3\}, & \text{if } \varsigma = \varsigma_4 \end{cases} \quad \text{and} \quad \neg\varsigma \mapsto \begin{cases} \{\gimel_2, \gimel_3\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\gimel_1, \gimel_2, \gimel_4\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\gimel_4\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\gimel_2\}, & \text{if } \neg\varsigma = \neg\varsigma_4. \end{cases}$$

Consider tow ideals defined on $\mathscr{Q}$ as $\mathscr{L} = \{\emptyset, \{\gimel_1\}\}$ and $\mathcal{J} = \{\emptyset, \{\gimel_4\}\}$. Let $\Im = \{\gimel_1, \gimel_5\}$, then $\overline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) = \{\gimel_5\}$ and $\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}} = \{\gimel_1, \gimel_2, \gimel_3, \gimel_4\}$. Therefore,
$\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) = (\{\gimel_5\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4\})$. Also, $\overline{\mathfrak{SR}}_{\beta+}^{\mathcal{J}}(\Im) = \{\gimel_1, \gimel_4, \gimel_5\}$ and $\overline{\mathfrak{SR}}_{\beta-}^{\mathcal{J}}(\Im) = \{\gimel_2, \gimel_3, \gimel_4\}$. Therefore, $\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) = (\{\gimel_1, \gimel_3, \gimel_4, \gimel_5\}, \{\gimel_1, \gimel_2, \gimel_3, \gimel_4\})$. This means that
$\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqsubseteq \overline{\mathfrak{PSR}}_{\beta}^{\mathcal{J}}(\Im)$ but $\mathscr{L} \nsubseteq \mathcal{J}$.

Proposition 3.5. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$, $\mathscr{L}, \mathcal{J}$ be two ideals on U. Let $\Im \subseteq \mathscr{Q}$. Then, the following properties hold*

(1) $\underline{\mathfrak{SR}}_{\beta+}^{(\mathscr{L} \cap \mathcal{J})}(\Im) = \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cap \underline{\mathfrak{SR}}_{\beta+}^{\mathcal{J}}(\Im)$;

(2) $\underline{\mathfrak{SR}}_{\beta-}^{(\mathscr{L} \cap \mathcal{J})}(\Im) = \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) \cup \underline{\mathfrak{SR}}_{\beta-}^{\mathcal{J}}(\Im)$;

(3) $\underline{\mathfrak{SR}}_{\beta+}^{(\mathscr{L} \cup \mathcal{J})}(\Im) = \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cup \underline{\mathfrak{SR}}_{\beta+}^{\mathcal{J}}(\Im)$;

(4) $\underline{\mathfrak{SR}}_{\beta-}^{(\mathscr{L} \cup \mathcal{J})}(\Im) = \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) \cap \underline{\mathfrak{SR}}_{\beta-}^{\mathcal{J}}(\Im)$;

Proof.

(1)

$$\begin{aligned} \underline{\mathfrak{SR}}_{\beta+}^{(\mathscr{L} \cap \mathcal{J})}(\Im) &= \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in (\mathscr{L} \cap \mathcal{J})\} \\ &= \left[\bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathscr{L}\}\right] \text{ and } \left[\bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathcal{J}\}\right] \\ &= \left[\bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathscr{L}\}\right] \cap \left[\bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in \mathcal{J}\}\right] \\ &= \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cap \underline{\mathfrak{SR}}_{\beta+}^{\mathcal{J}}(\Im). \end{aligned}$$

The other parts can be proved similarly.

Theorem 3.6. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$, $\mathscr{L}, \mathcal{J}$ be two ideals on U. Let $\Im \subseteq \mathscr{Q}$. Then, the following properties hold*

(1) $\underline{\mathfrak{PSR}}_{\beta}^{(\mathscr{L}\cap\mathcal{J})}(\Im) = \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcap \underline{\mathfrak{PSR}}_{\beta}^{\mathcal{J}}(\Im)$;

(2) $\underline{\mathfrak{PSR}}_{\beta}^{(\mathscr{L}\cup\mathcal{J})}(\Im) = \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcup \underline{\mathfrak{PSR}}_{\beta}^{\mathcal{J}}(\Im)$.

Proof. The proofs are obvious from Proposition 3.5.

Proposition 3.6. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space. If* f *is the intersection complete soft set, then* $\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im \cap \vartheta) \subseteq \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcap \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta)$ *for all* $\Im, \vartheta \subseteq U$.

Proof. From [20] $\underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im \cap \vartheta) = \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cap \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\vartheta)$. *By Theorem 3.3,* $\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im \cap \vartheta) \supseteq \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) \cup \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\vartheta)$. *Therefore,* $\underline{\vartheta}_{\mathscr{L}P}(\Im \cap \vartheta) \subseteq \underline{\vartheta}_{\mathscr{L}P}(\Im) \sqcap \underline{\vartheta}_{\mathscr{L}P}(\vartheta)$ *by Definitions 3.2 and 3.4.*

Definition 3.5. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and let $\Im \subseteq U$. Then, the complement of the ideal bipolar soft LA and ideal bipolar soft UA $\Im$ with respect to the IBSA-space are defined respectively by

$$(\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} = \left(\underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im), \underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im)\right)$$
$$(\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} = \left(\overline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im), \overline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im)\right)$$

Proposition 3.7. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space. For* $\Im, \vartheta \subseteq \mathscr{Q}$, *the following properties hold:*

(1) $\left[(\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c}\right]^{c} = \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im)$;

(2) $\left[(\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c}\right]^{c} = \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im)$;

(3) $\left(\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcup \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta)\right)^{c} = (\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} \sqcap (\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c}$;

(4) $\left(\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcup \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta)\right)^{c} = (\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} \sqcap (\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c}$;

(5) $\left(\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcap \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta)\right)^{c} = (\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} \sqcup (\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c}$;

(6) $\left(\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqcap \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta)\right)^{c} = (\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c} \sqcup (\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c}$;

(7) $\underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqsubseteq \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta) \Longleftrightarrow \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c} \sqsubseteq \underline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c}$;

(8) $\overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im) \sqsubseteq \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta) \Longleftrightarrow \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\vartheta))^{c} \sqsubseteq \overline{\mathfrak{PSR}}_{\beta}^{\mathscr{L}}(\Im))^{c}$.

Proof. Direct by using Theorem 3.5 and Definition 3.5.

Proposition 3.8. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *be a semi-intersection bipolar soft set and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be the corresponding IBSA-space and* $\Im \subseteq U$. *Then,*

(1) $\underline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cap \underline{\mathfrak{SR}}_{\beta-}^{\mathscr{L}}(\Im) = \emptyset$

(2) $\overline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) \cap \overline{\mathfrak{SR}}_{\beta+}^{\mathscr{L}}(\Im) = \emptyset$.

Proof. Since $\mathfrak{B} = (f, g : \wp)$ is a semi-intersection bipolar soft set over U, then $\mathfrak{B} = (f, g : \wp)$ is said to be a semi-intersection bipolar soft set, if $f(\varsigma_i) \cap g(\neg\varsigma_i) = \emptyset$ for all $\varsigma \in \wp$ and $\neg\varsigma \in \neg\wp$.. So it is clear that $\underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \emptyset$ and $\overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset$.

The coming theorem establishes the relationship between the current ideal bipolar SAs in Definition 3.1 and the previous bipolar SAs in definition 2.9 in [34].

Theorem 3.7. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ be a semi-intersection bipolar soft set and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be the corresponding IBSA-space and $\Im \subseteq U$. Then,*

(1) $\overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im)$ *and* $\underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \underline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im)$.

(2) $\mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) \subseteq \mathcal{BND}_{\beta}(\Im)$.

Proof. Immediately from Definitions 2.9 and 3.1.

Remark 3.7. It is noted from Theorem 3.7 that the ideal bipolar SAs in Definition 3.1 decreases the ideal bipolar soft UAs and increases the ideal bipolar soft BR. Consequently, a decision taken based on the computations of the approach in our Definition 3.1 is more applicable since we get a higher accuracy value than the accuracy values given by the previous approach discussed in [34].

4. Another style of bipolar soft ideal rough sets approximation

Definition 4.1. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{B}\mathfrak{S}\mathfrak{S}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ is the corresponding IBSA-space. Depending on $\beta_{\mathscr{L}}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$ by:

$$\left.\begin{array}{l}
\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \Im \cap \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im),\\
\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \Im \cup \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im),\\
\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \Im^{c} \cap \overline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im),\\
\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \Im^{c} \cup \underline{\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)
\end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be $*-$ideal soft $\beta_{\mathscr{L}}$-lower positive, $*-$ideal soft $\beta_{\mathscr{L}}$ upper positive, $*-$ideal soft $\beta_{\mathscr{L}}$-upper negative, and $*-$ideal soft $\beta_{\mathscr{L}}$-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l}
\underline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)\\
\overline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)
\end{array}\right\}$$

are called the $*-$ideal bipolar SAs of $\Im$ with respect to the IBSA-space. Moreover, when $\underline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im) \neq \overline{\mathfrak{P}\mathfrak{S}\mathfrak{R}}_{\beta}(\Im)$. Then, $\Im$ is termed as an $*-$ideal bipolar soft rough set and $\Im$ is said to be $*-$ideal bipolar soft $\beta_{\mathscr{L}}$-rough; otherwise $\Im$ is called $*-$ideal bipolar soft $\beta_{\mathscr{L}}$-definable. The corresponding $*-$positive, $*-$boundary, and $*-$negative regions with respect to the $*-$ideal bipolar SAs are given as

$$\begin{aligned}
* - \mathcal{POS}^{\mathscr{L}}_{\beta}(\Im) &= \left(\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right),\\
* - \mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) &= \left(\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) - \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im),\ \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) - \overline{* - \mathfrak{S}\mathfrak{R}}_{\beta-}(\Im)\right)\\
* - \mathcal{NEG}^{\mathscr{L}}_{\beta}(\Im) &= (\mathscr{Q}, \mathscr{Q}) - \overline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta}(\Im) = \left(\left(\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)\right)^{c}, \left(\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)\right)^{c}\right).
\end{aligned}$$

The links between the existing approximations in Definitions 3.1 and 4.1 are shown in the coming theorem.

Theorem 4.1. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *is the corresponding IBSA-space. Let* $\Im \subseteq \mathscr{Q}$*. Then,*

(1) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im)$;

(2) $\overline{\mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im)$;

(3) $\mathcal{BND}^{\mathscr{L}}_{\beta}(\Im) \subseteq * - \mathcal{BND}^{\mathscr{L}}_{\beta}(\Im)$.

Proof. Straightforward.

Proposition 4.1. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *is the corresponding IBSA-space. Let* $\Im \subseteq \mathscr{Q}$*. Then,*

(1) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\emptyset) = (\emptyset, \mathscr{Q})$ *and* $\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\mathscr{Q}) = (\mathscr{Q}, \emptyset)$;

(2) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq (\Im, \Im^c) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)$;

(3) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im^c) = [\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)]^c$ *and* $\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im^c) = [\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)]^c$;

(4) $\Im \sqsubseteq \vartheta \Rightarrow \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta)$ *and* $\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta)$;

(5) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}[\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)] = \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)$ *and*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}[\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)] = \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)$;

(6) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}[\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)] \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)$ *and*
$\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}[\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im)]$;

(7) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im \sqcup \vartheta) \sqsubseteq \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta)$ *and*
$\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta) \sqsubseteq \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im \sqcup \vartheta)$;

(8) $\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im \sqcup \vartheta) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta)$ *and*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im) \sqcup \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\vartheta) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta}(\Im \sqcup \vartheta)$.

Proof. Straightforward.

The connections between the prior definitions in [33], [34], [35] and the current approximations in Definition 4.1 are shown in the following theorem.

Theorem 4.2. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *be a full bipolar soft set and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *is the corresponding IBSA-space. Let* $\Im \subseteq \mathscr{Q}$*. Then,*

(1) $\underline{\mathfrak{SF}}_{\beta+}(\Im) \subseteq \underline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im) = \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq (\Im, \Im^c) \subseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SF}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SF}}_{\beta+}(\Im)$;

(2) $\underline{\mathfrak{SF}}_{\beta+}(\Im) = \underline{\mathfrak{PSR}}_{\beta+}(\Im) \subseteq \underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq (\Im, \Im^c) \subseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{PSR}}_{\beta+}(\Im) \subseteq \overline{\mathfrak{SF}}_{\beta+}(\Im)$;

Proof. Immediately.

Remark 4.1. Theorem 4.2 states that, when the methods in Definition 2.10 in [33], Definition 2.12 in [35], Definition 2.9 in [34] and our proposed method in Definition 4.1 are compared, it is observed that Definition 4.1 enhances the bipolar BR and enlarges the bipolar AM of a set $\Im$ by enlarging the bipolar LA and shrinking the bipolar UA. So, the suggested method is more accurate than [33], [34] and [35] in decision making. As a special case: If $\mathscr{L} = \emptyset$, then Definition 4.1 coincide with the previous Definition in [34].

5. Bi-ideal bipolar SA spaces

Definition 5.1. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\mathscr{L}_1$, $\mathscr{L}_2$ be two ideals on $\mathscr{Q}$. Then, $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ is called bi-ideal bipolar soft approximation space (Bi-IBSA-space for short). Based on $\beta_{\mathscr{L}_1,\mathscr{L}_2}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$:

$$\left.\begin{array}{l}
\underline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \bigcup\{f(\varsigma), \varsigma \in \wp : f(\varsigma) \cap \Im^c \in <\mathscr{L}_1, \mathscr{L}_2>\}, \\
\overline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\underline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im^c)\right)^c, \\
\overline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \bigcup\{g(\neg\varsigma), \neg\varsigma \in \aleph : g(\neg\varsigma) \cap \Im \in <\mathscr{L}_1, \mathscr{L}_2>\}, \\
\underline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\overline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im^c)\right)^c
\end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-lower positive, bi-ideal soft
$\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$ upper positive, bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-upper negative, and bi-ideal soft $\beta_{\mathscr{L}}$-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l}
\underline{\mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\underline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \underline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)\right) \\
\overline{\mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\overline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \overline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)\right)
\end{array}\right\}$$

are called the bi-ideal bipolar SAs of $\Im$ with respect to the Bi-IBSA-space.

Remark 5.1. The bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-lower positive and NAs and the bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-upper positive and NAs 5.1 will coincide with the corresponding approximations given in Definition 3.1 if $\mathscr{L}_1 = \mathscr{L}_2$. Also, the fulfilled properties of the recent bi-ideal bipolar SAs in Definition 5.1 are those given in Theorems 3.2 and 3.3.

Definition 5.2. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\mathscr{L}_1$, $\mathscr{L}_2$ be two ideals on $\mathscr{Q}$ and $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ is the corresponding Bi-IBSA-space. Based on $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$, the following operators are defined for any $\Im \subseteq \mathscr{Q}$:

$$\left.\begin{array}{l}
\underline{* - \mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \Im \cap \underline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \\
\overline{* - \mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \Im \cup \overline{\mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \\
\overline{* - \mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \Im^c \cap \overline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \\
\underline{* - \mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \Im^c \cup \underline{\mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)
\end{array}\right\}$$

are called the approximations of $\Im$ and are considered to be $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-lower positive, $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$ upper positive, $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-upper negative, and $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-lower negative, respectively. Moreover, the ordered pairs are given as

$$\left.\begin{array}{l}
\underline{* - \mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\underline{* - \mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \underline{* - \mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)\right) \\
\overline{* - \mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) = \left(\overline{* - \mathfrak{SR}}_{\beta+}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im), \overline{* - \mathfrak{SR}}_{\beta-}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)\right)
\end{array}\right\}$$

are called the $*-$bi-ideal bipolar SAs of $\Im$ with respect to the Bi-IBSA-space.

Remark 5.2. The $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-lower positive and NAs and the $*-$bi-ideal soft $\beta_{<\mathscr{L}_1,\mathscr{L}_2>}$-upper positive and NAs 5.2 will coincide with the corresponding approximations given in Definition 4.1 if $\mathscr{L}_1 = \mathscr{L}_2$. Also, the fulfilled properties of the recent $*-$bi-ideal bipolar SAs in Definition 5.2 are the same of Definition 4.1.

Definition 5.3. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\mathscr{L}_1,\ \mathscr{L}_2$ be two ideals on $\mathscr{Q}$ and $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ is the corresponding Bi-IBSA-space. We the follwing operators:

(1) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) = \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1}(\Im) \sqcup \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_2}(\Im)$;

(2) $\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) = \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1}(\Im) \sqcap \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_2}(\Im)$;

where $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_i}(\Im)$ and $\overline{\mathbf{S}R_{\mathscr{L}_i}}(\Im)$ are the ideal bipolar soft lower and the UAs of $\Im$ related to $\mathscr{L}_i$, $i \in \{1, 2\}$ as in Definition 4.1.

Remark 5.3. The ideal bipolar soft lower and the UAs given in Definition 5.3 will coincide with the approximations given in Definition 4.1 if $\mathscr{L}_1 = \mathscr{L}_2$.

Proposition 5.1. *Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\mathscr{L}_1,\ \mathscr{L}_2$ be two ideals on $\mathscr{Q}$ and $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ is the corresponding Bi-IBSA-space. Then, these properties are fulfilled.*

(1) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\emptyset) = (\emptyset, \mathscr{Q})$ *and* $\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\mathscr{Q}) = (\mathscr{Q}, \emptyset)$;

(2) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq (\Im, \Im^c) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)$;

(3) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}[\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)] = \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)$;

(4) $\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}[\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)] = \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)$;

(5) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}[\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)] \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)$;

(6) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}[\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)]$;

(7) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im^c) = [\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)]^c$ *and*
$\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im^c) = [\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im)]^c$;

(8) $\Im \subseteq \vartheta \Rightarrow \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta)$ *and*
$\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta)$;

(9) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im \cap \vartheta) \sqsubseteq \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqcap \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta)$ *and*
$\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \cup \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta) \sqsubseteq \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im \sqcup \vartheta)$;

(10) $\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im \cap \vartheta) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqcap \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta)$ *and*
$\overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqcup \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\vartheta) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im \cup \vartheta)$.

Proof. Straightforward.

As shown by the following example, equality relations cannot take the place of the inclusions in part (9).

Example 5.1. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\mathscr{Q} = \{\beth_1.\beth_2, \beth_3, \beth_4, \beth_5\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$. *The maps* f *and* g *are as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad\qquad g : \aleph \longrightarrow 2^{\mathscr{Q}},$$
$$\varsigma \mapsto \begin{cases} \{\beth_1, \beth_2\}, & \text{if } \varsigma = \varsigma_1, \\ \{\beth_3, \beth_5\}, & \text{if } \varsigma = \varsigma_2, \\ \{\beth_2, \beth_4\}, & \text{if } \varsigma = \varsigma_3, \\ \{\beth_1, \beth_3\}, & \text{if } \varsigma = \varsigma_4 \end{cases} \quad \text{and} \quad \neg\varsigma \mapsto \begin{cases} \{\beth_3, \beth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\beth_1, \beth_2\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\beth_3, \beth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\beth_2, \beth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_4. \end{cases}$$

Consider tow ideals defined on $\mathscr{Q}$ *as* $\mathscr{L}_1 = \{\emptyset, \{\beth_1\}\}$ *and* $\mathscr{L}_2 = \{\emptyset, \{\beth_3\}, \{\beth_5\}, \{\beth_3, \beth_5\}\}\}$. *Let* $\Im = \{\beth_1, \beth_2, \beth_3\}$ *and* $\vartheta = \{\beth_4, \beth_5\}$. *Then,* $\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\Im) = \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im), \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^-}(\Im)\right) = (\mathscr{Q}, \emptyset)$ *and*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\Im) = \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im), \overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^-}(\Im)\right) = (\{\beth_1, \beth_2, \beth_3, \beth_4\}, \{\beth_4, \beth_5\})$. *Hence,*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im) = \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\Im) \sqcap \overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\Im) = (\{\beth_1, \beth_2, \beth_3, \beth_4\}, \{\beth_4, \beth_5\})$. *Also,*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\vartheta) = (\overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\vartheta), \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^-}(\vartheta)) = (\{\beth_4, \beth_5\}, \{\beth_1, \beth_2\})$ *and*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\vartheta) = \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\vartheta), \overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^-}(\vartheta)\right) = (\{\beth_2, \beth_4, \beth_5\}, \{\beth_1, \beth_2, \beth_3\})$. *Hence,*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\vartheta) = \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\vartheta) \sqcap \overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\vartheta) = (\{\beth_4, \beth_5\}, \{\beth_1, \beth_2\})$.
Also, $\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\Im \cup \vartheta) = \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im \cup \vartheta), \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^-}(\Im \cup \vartheta)\right) = (\mathscr{Q}, \emptyset)$ *and*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\Im \cup \vartheta) = \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im \cup \vartheta), \overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^-}(\Im \cup \vartheta)\right) = (\mathscr{Q}, \emptyset\})$. *Hence,*
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im \cup \vartheta) = \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta}(\vartheta) \sqcap \overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta}(\vartheta) = (\mathscr{Q}, \emptyset)$.
Clearly, $\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im) \sqcup \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\vartheta) \neq \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im \cup \vartheta)$.

Theorem 5.1. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ *is the corresponding Bi-IBSA-space. Then, these properties are fulfilled for* $\Im \subseteq \mathscr{Q}$

(1) $\overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im)$.

(2) $\underline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im) \sqsubseteq \underline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta}(\Im)$.

(3) $* - \mathcal{BND}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta}(\Im) \sqsubseteq * - \mathcal{BND}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im)$.

Proof.

(1) Let $\beth \in \overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta^+}(\Im)$. Then, $\beth \in \Im$ or $\beth \in \overline{* - \mathfrak{SR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta^+}(\Im)$. So, $\beth \in \Im$ or $\forall \varsigma \in \wp : \beth \in f(\varsigma)$, and we get $f(\varsigma) \cap \Im \notin <\mathscr{L}_1, \mathscr{L}_2>$. First choice, if $\beth \in \Im$, then $\beth \in \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im)$ and $\beth \in \overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im)$. Thus, $\beth \in \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta^+}(\Im)$. Second choice, if $\forall \varsigma \in \wp : \beth \in f(\varsigma)$, we have $f(\varsigma) \cap \Im \notin <\mathscr{L}_1, \mathscr{L}_2>$, then $f(\varsigma) \cap \Im \notin \mathscr{L}_1$ and $f(\varsigma) \cap \Im \notin \mathscr{L}_2$ $\forall \varsigma \in \wp : \beth \in f(\varsigma)$, and thus $\beth \notin \underline{\mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im^c)$ and $\beth \notin \underline{\mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im^c)$. That means $\beth \in \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im)$ and $\beth \in (\overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im)$ by Definitions 3.1, 4.1. Thus, $\beth \in \overline{* - \mathfrak{SR}}^{\mathscr{L}_1}_{\beta^+}(\Im) \cap \overline{* - \mathfrak{SR}}^{\mathscr{L}_2}_{\beta^+}(\Im)$. This means that $\beth \in \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta^+}(\Im)$. Hence, $\overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta^+}(\Im) \subseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta^+}(\Im)$.

Let $\beth \in \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta^-}(\Im) = \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta^-}(\Im) \cup \overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta^-}(\Im)$, then
$\beth \in \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1}_{\beta^-}(\Im)$ or $\beth \in \overline{* - \mathfrak{PSR}}^{\mathscr{L}_2}_{\beta^-}(\Im)$. Then, $\beth \in \Im^c$ and $\exists \neg\varsigma_1 \in \wp : \beth \in g(\neg\varsigma_1)$, and $g(\neg\varsigma_1) \cap \Im \in \mathscr{L}_1$ or $\exists \neg\varsigma_2 \in \wp : \beth \in g(\neg\varsigma_2)$, and $g(\neg\varsigma_2) \cap \Im \in \mathscr{L}_2]$.
Since $\mathscr{L}_1, \mathscr{L}_2 \subseteq <\mathscr{L}_1, \mathscr{L}_2>$. Then, $\beth \in \overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta^-}(\Im)$. Hence,
$\overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta^-}(\Im) \subseteq \overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta^-}(\Im)$. Thus,
$\overline{* - \mathfrak{PSR}}^{<\mathscr{L}_1,\mathscr{L}_2>}_{\beta}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}^{\mathscr{L}_1,\mathscr{L}_2}_{\beta}(\Im)$.

(2) Similarly as part (1).

(3) It is immediately by parts (1), (2).

Proposition 5.2. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{<\mathscr{L}_1,\mathscr{L}_2>} = (\mathscr{Q}, (f, g : \wp), \mathscr{L}_1, \mathscr{L}_2)$ *is the corresponding Bi-IBSA-space. and* $\Im \subseteq \mathscr{Q}$*. Then,*

(1) $\overline{* - \mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq \overline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_i}(\Im)(\Im)\ \forall i \in \{1, 2\}.$

(2) $\underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_i}(\Im)(\Im) \sqsubseteq \underline{* - \mathfrak{PSR}}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq \underline{* - \mathfrak{PSR}}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)\ \forall i \in \{1, 2\}.$

(3) $* - \mathcal{BND}_{\beta}^{<\mathscr{L}_1,\mathscr{L}_2>}(\Im)(\Im) \sqsubseteq * - \mathcal{BND}_{\beta}^{\mathscr{L}_1,\mathscr{L}_2}(\Im) \sqsubseteq * - \mathcal{BND}_{\beta}^{\mathscr{L}_i}(\Im)\ \forall i \in \{1, 2\}.$

Proof. Straightforward follows from Definitions 5.2, 5.3 and Theorem 5.1.

Remark 5.4. From Theorem 5.1, we deduce that the bi-ideal bipolar soft BR defined by Definition 5.2 is lower than that bi-ideal bipolar soft BR computed by Definition 5.3. Consequently, a decision made according to the computations of the approach in Definition 5.2 is more applicable than the approach in Definition 5.3.

6. Some important measures related to IBSA-Space

In [1], Pawlak established two quantitative metrics to assess the imprecision of rough set approximations. These metrics might help determine the exact degree to which the data is associated with a certain equivalence relation for a given classification. In general, a set becomes unclear when a BR exists.A set's accuracy decreases with increasing boundary area size.

Regarding to Pawlak [1], the accuracy value and measures of roughness of $\Im \subseteq q$ are defined as

$$Acc(\Im) = \frac{|\underline{Lower}(\Im)|}{|\overline{Upper}(\Im)|},$$
$$Rough(\Im) = 1 - Acc(\Im).$$

where $|\bullet|$ denotes the order of the set. However, the degree of completeness of the knowledge of a set $\Im$ is captured by $Acc(\Im)$, whereas the degree of incompleteness of the knowledge of the set $\Im$ is perceived by $Rough(\Im)$.

As a generalization of these measures, here we propose some measures in the framework of the IBSA-space and investigate some of its essential properties.

Definition 6.1. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be the corresponding IBSA-space. Then, the AM of $\emptyset \neq \Im \subseteq \mathscr{Q}$ in the IBSA environment is given as follows:

$$Acc_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\beta+}, \Im^{\mathscr{L}}_{\beta-}\right),$$

where

$$\Im^{\mathscr{L}}_{\beta+} = \frac{|\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)|}{|\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta+}(\Im)|}, \text{ and } \Im^{\mathscr{L}}_{\beta-} = \frac{|\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)|}{|\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta-}(\Im)|}.$$

The measure of roughness for $\emptyset \neq \Im \subseteq \mathscr{Q}$ in the IBSA-space is characterized as follows:

$$Rough_{\mathscr{L}B}(\Im) = (1, 1) - \left(\Im^{\mathscr{L}}_{\beta+}, \Im^{\mathscr{L}}_{\beta-}\right) = \left(1 - \Im^{\mathscr{L}}_{\beta+}, 1 - \Im^{\mathscr{L}}_{\beta-}\right).$$

Clearly, $0 \leq \Im^{\mathscr{L}}_{\beta+} \leq 1$ and $0 \leq \Im^{\mathscr{L}}_{\beta-} \leq 1$.

Remark 6.1. If $Acc_{\mathscr{L}B} = (\Im) = \left(\Im^{\mathscr{L}}_{\beta+}, \Im^{\mathscr{L}}_{\beta-}\right)$ is the accuracy of $\Im$. Then,

(1) $\Im \subseteq \mathscr{Q}$ is $*-$ideal bipolar soft $\beta_{\mathscr{L}}$-definable if and only if $Rough_{\mathscr{L}B} = (\Im) = (\emptyset, \emptyset)$.

(2) If $\Im^{\mathscr{L}}_{\beta+} < 1$ and $\Im^{\mathscr{L}}_{\beta-} < 1$, the set $\Im$ has some nonempty BR and consequently is $*-$ideal bipolar soft $\beta_{\mathscr{L}}$-rough.

Proposition 6.1. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\Im, \vartheta \subseteq \mathscr{Q}$. Then,*

(1) $Acc_{\mathscr{L}B}(\Im) = (0,0) \Longleftrightarrow \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset = \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$;

(2) $Acc_{\mathscr{L}B}(\Im) = (1,1) \Longleftrightarrow \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)$ *and* $\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im) = \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$;

(3) $\Im \subseteq \vartheta \Longrightarrow Acc_{\mathscr{L}B}(\Im) \leq Acc_{\mathscr{L}B}(\vartheta)$.

Proof. Straightforward from Definition 6.1.

In 2001, Gediga and Düntsch [36] proposed a measure of precision of the approximation of $\emptyset \neq \Im \subseteq \mathscr{Q}$, which is given by:

$$\mathbf{P}(\Im) = \frac{|\underline{Lower}(\Im)|}{|\Im|}.$$

This is a relative number of elements of $\Im$ which can be approximated by $\underline{Lower}$. It should be noted that $\mathbf{P}(\Im)$ needs a complete knowledge of $\Im$, while $Acc(\Im)$ does not exist. It could be generalized in the framework of the IBSA-space as follow:

Definition 6.2. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be the corresponding IBSA-space. Then, the precision measure of $\emptyset \neq \Im \subseteq \mathscr{Q}$ in the IBSA environment is given as follows:

$$\mathbf{P}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{*\beta+}, \Im^{\mathscr{L}}_{*\beta-}\right),$$

where

$$\Im^{\mathscr{L}}_{*\beta+} = \frac{|\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im)|}{|\Im|}, \text{ and } \Im^{\mathscr{L}}_{*\beta-} = \frac{|\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)|}{|\Im|}.$$

Clearly, $0 \leq \Im^{\mathscr{L}}_{*\beta+} \leq 1$ and $0 \leq \Im^{\mathscr{L}}_{*\beta-} \leq 1$.

From the definition given above, we can infer the following properties of $\mathbf{P}_{\mathscr{L}B} = (\Im)$:

Proposition 6.2. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\Im, \vartheta \subseteq \mathscr{Q}$. Then,*

(1) $\mathbf{P}_{\mathscr{L}B}(\Im) = (0,0) \Longleftrightarrow \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \emptyset = \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$;

(2) $\mathbf{P}_{\mathscr{L}B}(\Im) = (1,1) \Longleftrightarrow \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta+}(\Im) = \Im = \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta-}(\Im)$;

(3) $\mathbf{P}_{\mathscr{L}B}(\Im) \geq Acc_{\mathscr{L}B}(\Im)$, *that is,* $\Im^{\mathscr{L}}_{*\beta+} \geq \Im^{\mathscr{L}}_{\beta+}$ *and* $\Im^{\mathscr{L}}_{*\beta-} \geq \Im^{\mathscr{L}}_{\beta-}$;

(4) $\Im \subseteq \vartheta \Longrightarrow Acc_{\mathscr{L}B}(\Im) \leq Acc_{\mathscr{L}B}(\vartheta)$.

Proof. Straightforward.

Yao [37] studied a few of the AM's properties provided by Pawlak [1] and introduced another measure known as the measure of the completeness of knowledge, given by

$$\mathbf{C}(\Im) = \frac{\left||\underline{Lower}(\Im)\right| + \left|\underline{|Lower}\,(\Im^c)\right|}{|Q|}.$$

Definition 6.3. Let $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ and $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be the corresponding IBSA-space. Then, the measure of the completeness of knowledge of $\emptyset \neq \Im \subseteq \mathscr{Q}$ in the IBSA environment is given as follows:

$$\mathbf{C}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\sharp\beta+}, \Im^{\mathscr{L}}_{\sharp\beta-}\right),$$

where

$$\Im^{\mathscr{L}}_{\sharp\beta+} = \frac{|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im)| + |\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im^c)|}{|Q|}, \text{ and } \Im^{\mathscr{L}}_{\sharp\beta-} = \frac{|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im)| + |\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im^c)|}{|Q|}.$$

Clearly, $0 < \Im^{\mathscr{L}}_{\sharp\beta+} < 1$ and $0 \leq \Im^{\mathscr{L}}_{\sharp\beta-} \leq 1$. In other words, $\mathbf{C}_{\mathscr{L}B}(\Im)$ cannot be zero for any $\Im \subseteq \mathscr{Q}$.

The condition under which $\mathbf{C}_{\mathscr{L}B}(\Im)$ reaches its greatest value is given by the following proposition.

Proposition 6.3. *Let $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ be IBSA-space and $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a full bipolar soft set. Then, $\mathbf{C}_{\mathscr{L}B}(\Im) = (1, 1)$ whenever $\Im = \emptyset$ or $\Im = Q$,*

Proof. Because $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ be a full bipolar soft set, $\bigcup_{e\in\wp} f(\varsigma) = \mathscr{Q}$ and $\bigcup_{\neg\varsigma\in\wp} g(\neg\varsigma) = \mathscr{Q}$. We now demonstrate the necessary result for the two cases.

Case 1: When $\Im = \emptyset$. Then, $\Im^{\mathscr{L}}_{\sharp\beta+} =$

$$\frac{\left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\emptyset)\right| + \left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\emptyset^c)\right|}{|\mathscr{Q}|} = \frac{\left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\emptyset)\right| + \left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\mathscr{Q})\right|}{|\mathscr{Q}|} = \frac{|\emptyset| + |\mathscr{Q}|}{|\mathscr{Q}|} = 1.$$

Similarly, $\Im^{\mathscr{L}}_{\sharp\beta-} =$

$$\frac{\left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\emptyset)\right| + \left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\emptyset^c)\right|}{|\mathscr{Q}|} = \frac{\left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\emptyset)\right| + \left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\mathscr{Q})\right|}{|\mathscr{Q}|} = \frac{|\mathscr{Q}| + |\emptyset|}{|\mathscr{Q}|} = 1.$$

Therefore, $\mathbf{C}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\sharp\beta+}, \Im^{\mathscr{L}}_{\sharp\beta-}\right) = (1, 1)$.

Case 2: When $\Im = \mathscr{Q}$, then $\Im^{\mathscr{L}}_{\sharp\beta+} =$

$$\frac{\left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\mathscr{Q})\right| + \underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\mathscr{Q}^c)\,|}{|\mathscr{Q}|} = \frac{\left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\mathscr{Q})\right| + \left|\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\emptyset)\right|}{|\mathscr{Q}|} = \frac{|\mathscr{Q}| + |\emptyset|}{|\mathscr{Q}|} = 1.$$

Also, $\Im^{\mathscr{L}}_{\sharp\beta-} =$

$$\frac{\left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\mathscr{Q})\right| + \left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\mathscr{Q}^c)\right|}{|\mathscr{Q}|} = \frac{\left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\mathscr{Q})\right| + \left|\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\emptyset)\right|}{|\mathscr{Q}|} = \frac{|\emptyset| + |\mathscr{Q}|}{|\mathscr{Q}|} = 1.$$

Consequently, $\mathbf{C}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\sharp\beta+}, \Im^{\mathscr{L}}_{\sharp\beta-}\right) = (1, 1)$. Hence, in both of cases we have $\mathbf{C}_{\mathscr{L}B}(\Im) = (1, 1)$.

To clarify the concepts of $Acc_{\mathscr{L}B}(\Im), \mathbf{P}_{\mathscr{L}B}(\Im)$ and $\mathbf{C}_{\mathscr{L}B}(\Im)$, we expand on the following examples below.

Example 6.1. *(Continued from Example 3.1) The $*-$ideal bipolar SAs of* $\Im = \{\daleth_3, \daleth_4, \daleth_5\} \subseteq \mathscr{Q} = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$ *are as follows:*

$$\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im) = \Im \cap \underline{\mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im) = \{\daleth_3, \daleth_4\},$$
$$\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im) = \Im \cup \overline{\mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im) = \{\daleth_2, \daleth_3, \daleth_4.\daleth_5\},$$
$$\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im) = \Im^c \cap \overline{\mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im) = \{\daleth_2, \daleth_6\},$$
$$\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im) = \Im^c \cup \underline{\mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im) = \{\daleth_1, \daleth_2, \daleth_6\}.$$

Also,

$$\underline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im^c) = \Im^c \cap \underline{\mathfrak{SR}^{\mathscr{L}}_{\beta+}}(\Im^c) = \{\daleth_1, \daleth_6\},$$
$$\overline{* - \mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im^c) = \Im \cap \underline{\mathfrak{SR}^{\mathscr{L}}_{\beta-}}(\Im^c) = \{\daleth_3, \daleth_4, \daleth_5\}.$$

Therefore,

$$Acc_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\beta+}, \Im^{\mathscr{L}}_{\beta-}\right) = \left(\frac{2}{5}, \frac{2}{3}\right) = (0.400, 0.666)$$
$$\mathbf{P}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{*\beta+}, \Im^{\mathscr{L}}_{*\beta-}\right) = \left(\frac{2}{3}, \frac{2}{3}\right) = (0.666, 0.666).$$
$$\mathbf{C}_{\mathscr{L}B}(\Im) = \left(\Im^{\mathscr{L}}_{\sharp\beta+}, \Im^{\mathscr{L}}_{\sharp\beta-}\right) = \left(\frac{2+2}{6}, \frac{2+3}{6}\right) = \left(\frac{4}{6}, \frac{5}{6}\right) = (0.666, 0.833).$$

Remark 6.2. In the next example, we calculate the AMs of the bipolar soft rough approximations techniques given in [33], [34], [35] and our suggested technique Definition 4.1 according to the AM in Definition 6.1, which illustrates that our suggested technique produce the higher accuracy values. Thus, the investigated technique is more accurate in the decision making.

Example 6.2. *Let* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *and* $\beta_{\mathscr{L}} = (\mathscr{Q}, (f, g : \wp), \mathscr{L})$ *be IBSA-space with* $\mathscr{Q} = \{\daleth_1, \daleth_2, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}$ *and* $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6, \varsigma_7\}$. *The mappings f and g are as follow:*

$$f : \wp \longrightarrow 2^{\mathscr{Q}}, \quad \varsigma \mapsto \begin{cases} \{\daleth_1, \daleth_3, \daleth_4, \daleth_5, \daleth_6\}, & \text{if } \varsigma = \varsigma_1, \\ \{\daleth_1, \daleth_2\}, & \text{if } \varsigma = \varsigma_2, \\ \{\daleth_1, \daleth_2, \daleth_4\}, & \text{if } \varsigma = \varsigma_3, \\ \{\daleth_1\}, & \text{if } \varsigma = \varsigma_4, \\ \{\daleth_3, \daleth_4\}, & \text{if } \varsigma = \varsigma_5, \\ \{\daleth_2, \daleth_4\}, & \text{if } \varsigma = \varsigma_6, \\ \{\daleth_1, \daleth_3, \daleth_5, \daleth_6\}, & \text{if } \varsigma = \varsigma_7, \end{cases} \quad \text{and} \quad g : \aleph \longrightarrow 2^{\mathscr{Q}}, \quad \neg\varsigma \mapsto \begin{cases} \{\daleth_2\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\daleth_5\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\daleth_3, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\daleth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\daleth_5, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_5, \\ \{\daleth_3, \daleth_5, \daleth_6\}, & \text{if } \neg\varsigma = \neg\varsigma_6. \\ \{\daleth_2, \daleth_4\}, & \text{if } \neg\varsigma = \neg\varsigma_7. \end{cases}$$

Consider, $\mathscr{L} = \{\emptyset, \{\daleth_1\}, \{\daleth_2\}, \{\daleth_3\}, \{\daleth_1, \daleth_2\}, \{\daleth_1, \daleth_3\}, \{\daleth_2, \daleth_3\}, \{\daleth_1, \daleth_2, \daleth_3\}\}$. *The comparison between the proposed technique and the previous techniques is shown in Table 1. According to Table 1, if* $\mathfrak{B} = (f, g : \wp) \in \mathfrak{BSS}^{\mathscr{Q}}$ *is a full bipolar soft set, then Definition 4.1 gives us a lower an upper accuracy value than the calculated ones given in [33], [34], [35]. Moreover, it is clear that the proposed approach in [35] and its counterpart introduced in [34] are different in general. Hence, a decision made according to the calculations of our current technique in Definition 4.1 is more accurate.*

Table 1: Comparison The AM in Definition 6.1 of a set $\Im \subseteq \mathscr{Q}$ by using the proposed approach in Definitions 4.1 and other previous approaches

$\Im \subseteq \mathscr{Q}$	technique in [33]	technique in [34]	technique in [35]	Our suggested technique
$\{\gimel_1\}$	(1/6,1)	(1/3,5/6)	(1,1)	(1,1)
$\{\gimel_1, \gimel_2\}$	(1/3,4/5)	(1/2,4/5)	(1,4/5)	(1,4/5)
$\{\gimel_1, \gimel_3\}$	(1/6,3/5)	(1/4,2/3)	(1/2,4/5)	(1/2,2/3)
$\{\gimel_1, \gimel_5\}$	(1/6,4/5)	((1/3,4/5)	(1/5,4/5)	(1/3,4/5)
$\{\gimel_2, \gimel_4\}$	(1/3,3/4)	(1,3/4)	(1/3,3/4)	(1,3/4)
$\{\gimel_3, \gimel_4\}$	(1/3,3/5)	(1/2,3/5)	(1/3,3/4)	(1/2,3/5)
$\{\gimel_1, \gimel_2, \gimel_5\}$	(1/3,3/5)	(1/2,3/5)	(1/3,3/5)	(1/2,3/4)
$\{\gimel_1, \gimel_3, \gimel_5\}$	(1/6,2/5)	(1/4,2/5)	(1/5,3/5)	(1/4,3/5)
$\{\gimel_1, \gimel_5, \gimel_6\}$	(1/6, 2/5)	(1/3,1/2)	(1/5,2/3)	(1/3,1/2)
$\{\gimel_2, \gimel_3, \gimel_4\}$	(1/2, 2/3)	(3/5, 1/4)	(1/2, 1/2)	(3/5,1/2)
$\{\gimel_2, \gimel_4, \gimel_5\}$	(1/3, 1/2)	(2/5, 1/2)	(1/3,1/2)	(2/5,2/3)
$\{\gimel_3, \gimel_4, \gimel_5\}$	(1/3, 1/5)	(1/2,1/5)	(1/3, 2/5)	(1/2,1/2)
$\{\gimel_1, \gimel_3, \gimel_5, \gimel_6\}$	(2/3, 1)	(1,1)	(4/5, 1)	(1,2/3)
$\{\gimel_2, \gimel_3, \gimel_4, \gimel_5\}$	(1/2, 0)	(3/5, 0)	(1/2,0)	(3/5,0)
$\{\gimel_2, \gimel_4, \gimel_5, \gimel_6\}$	(1/3, 0)	(2/5, 0)	(5/6,0)	(1,0)
$\{\gimel_3, \gimel_4, \gimel_5, \gimel_6\}$	(1/3, 1/4)	(1/2, 1/4)	(2/3,1/4)	(1,1/2)
$\{\gimel_2, \gimel_3, \gimel_4, \gimel_5, \gimel_6\}$	(1/2, 0)	(3/5, 0)	(5/6,0)	(1,0)

7. MAGDM using the IBSA-spaces

Group decision-making (GDM) is a useful technique for handling complex decision-making (DM) situations when a number of experts select a set of options. The objective is to include the viewpoints of experts in order to identify a solution that the group of experts finds most agreeable. In a complex society, GDM approaches must take into account multiple attributes. Due to the rapid expansion in many sectors, studies on GDM that explicitly incorporate several qualities have made significant progress and are the major emphasis.

MAGDM is, in general, a strategy where a group of experts (DMs) works together to identify the best choice over a range of options which are classified depending on their attributes in a particular situation. This section explains how to create a reliable MAGDM method that makes use of IBSA-spaces. In the framework of the IBSA-spaces, we give an overview of a MAGDM problem. Next, we give a general mathematical formulation of the MAGDM problem based on the $*-$ideal bipolar SAs of $\Im \subseteq \mathscr{Q}$ in the IBSA-spaces.

7.1. Problem description

Consider $\mathscr{Q} = \{\gimel_1, \gimel_2, \ldots, \gimel_n\}$ is a set of n objects, and $\wp = \{\varsigma_1, \varsigma_2, \ldots, \varsigma_m\}$ is a set of all possible object attributes. Assume we have a board of professional experts. $\mathscr{D} = \{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k\}$ consisting of k invited DMs. All of the items in $\mathscr{Q}$ must be examined by each expert, who will then be asked to evaluate each one and identify just "the optimal alternatives" based on their knowledge and abilities. Consequently, the fundamental evaluation result for each expert is a subset of $\mathscr{Q}$: Let $\Im_1, \Im_2, \ldots, \Im_k \subseteq \mathscr{Q}$ stands for the primary evaluations of DMs $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_k$, respectively, and $\beta_1, \beta_2, \ldots, \beta_r \in \mathcal{BSSQ}$ are the real data that were previously obtained for problems at various locations or times. To keep things simple, we'll suppose that every expert's assessment in $\mathscr{D}$ is equally significant. The DM for this MAGDM issue is therefore: "how to compromise differences in these evaluations expressed by individual experts to find the alternatives that are most acceptable by the group of experts as a whole" .

7.2. Mathematical modelling

Now, we provide a mathematical model and the progress of the MAGDM method using IBSA theory.

Definition 7.1. Let us consider $\underline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q}(\Im_j) = \left(\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j)\right)$; and $\overline{* - \mathfrak{P}\mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q}(\Im_j) = \left(\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j)\right)$ be the $*-$ideal bipolar soft lower and UAs of $\Im$,
$\Im_j (j = 1, 2, \ldots, k)$ is related to $\beta_q = (f_q, g_q : \wp) \in \mathcal{BSSQ}; (\mathscr{Q} = 1, 2, \ldots, r)$. Then, $[\underline{\mathcal{M}}] =$

$$\begin{pmatrix} \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_1), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_1)\right] & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_2), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_2)\right] & \cdots & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_k), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_k)\right] \\ \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_1), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_1)\right] & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_2), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_2)\right] & \cdots & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_k), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_k)\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_1), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_1)\right] & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_2), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_2)\right] & \cdots & \left[\underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_k), \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_k)\right] \end{pmatrix}$$

and $[\overline{\mathcal{M}}] =$

$$\begin{pmatrix} \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_1), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_1)\right] & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_2), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_2)\right] & \cdots & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^+}(\Im_k), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_1^-}(\Im_k)\right] \\ \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_1), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_1)\right] & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_2), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_2)\right] & \cdots & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^+}(\Im_k), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_2^-}(\Im_k)\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_1), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_1)\right] & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_2), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_2)\right] & \cdots & \left[\overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^+}(\Im_k), \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_r^-}(\Im_k)\right] \end{pmatrix}$$

are called the $*-$ideal bipolar soft lower and UA matrices, respectively. Here

$$\begin{aligned} \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j) &= \left(\underline{\gimel_{1j}}_{f_q}, \underline{\gimel_{2j}}_{f_q}, \ldots, \underline{\gimel_{nj}}_{f_q}\right), \\ \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j) &= \left(\underline{\gimel_{1j}}_{g_q}, \underline{\gimel_{2j}}_{g_q}, \ldots, \underline{\gimel_{nj}}_{g_q}\right), \\ \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j) &= \left(\overline{\gimel_{1j}}_{f_q}, \overline{\gimel_{2j}}_{f_q}, \ldots, \overline{\gimel_{nj}}_{f_q}\right), \\ \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j) &= \left(\overline{\gimel_{1j}}_{g_q}, \overline{\gimel_{2j}}_{g_q}, \ldots, \overline{\gimel_{nj}}_{g_q}\right), \end{aligned}$$

where

$$\underline{\gimel_{ij}}_{f_q} = \begin{cases} 1, & \text{if } \gimel_i \in \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j), \\ 0, & \text{otherwise} \end{cases}$$

$$\underline{\gimel_{ij}}_{g_q} = \begin{cases} \frac{1}{2}, & \text{if } \gimel_i \in \underline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j), \\ 0, & \text{otherwise} \end{cases}$$

$$\overline{\gimel_{ij}}_{f_q} = \begin{cases} \frac{1}{2}, & \text{if } \gimel_i \in \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^+}(\Im_j), \\ 0, & \text{otherwise} \end{cases}$$

$$\overline{\gimel_{ij}}_{g_q} = \begin{cases} 1, & \text{if } \gimel_i \in \overline{* - \mathfrak{S}\mathfrak{R}}^{\mathscr{L}}_{\beta_q^-}(\Im_j), \\ 0, & \text{otherwise} \end{cases}$$

Definition 7.2. Let $\underline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ be the $*-$ideal bipolar soft lower and UA matrices with respect to

$\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_q}(\Im_j)$ and $\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_q}(\Im_j)$, where: $j = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, r$. Then,

$$\mathbb{V}^f = \bigoplus_{j=1}^{k}\bigoplus_{q=1}^{r}\left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^+}(\Im_j) \oplus \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^+}(\Im_j)\right),$$
$$\mathbb{V}^g = \bigoplus_{j=1}^{k}\bigoplus_{q=1}^{r}\left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^-}(\Im_j) \oplus \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^-}(\Im_j)\right),$$

are called the positive and negative $*-$ideal bipolar SA vectors, respectively. Here, the operations $\bigoplus$ represent the vector summation.

Definition 7.3. Let $\mathbb{V}^f$ and $\mathbb{V}^g$ be positive and negative $*-$ideal bipolar SA vectors, respectively. Then,

$$\mathbb{V}_d = \mathbb{V}^f - \mathbb{V}^g = (\delta_1, \delta_2, \cdots, \delta_n)$$

is said to be a decision vector where each δ_i is called the score value ($\mathcal{SVL}$) of $\beth_i \in \mathscr{Q}$.

- $\beth_i \in \mathscr{Q}$ is viewed as an optimal alternative if its $\mathcal{SVL}$ is a maximum of $\delta_i; \forall i = 1, 2, \ldots, n$.
- $\beth_i \in \mathscr{Q}$ is viewed as the worst alternative if its $\mathcal{SVL}$ is a minimum of $\delta_i; \forall i = 1, 2, \ldots, n$.

In cases when $\mathscr{Q}$ has several optimum alternatives, we select any one of them.

7.3. The proposed algorithm

We propose a DM algorithm for the MAGDM established problem considered in subsection 7.2. The associated flowchart depicting the above subsection is illustrated in Figure 1.

7.4. Appointment of a faculty member problem

Here, we present a case study to explain the essential methodology of the proposed mathematical modeling technique and its related concepts.

Example 7.1. *In universities, the selection of academic members to high posts entails DM and extremely complex reviews. A candidate may be evaluated based on a number of factors, including research output, management skills, and stress tolerance. In order to accurately assess the applicants according to these attributes, It makes sense to conduct professional experts with their opinions.*

Let $\mathscr{Q} = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_5\}$ be the group of five candidates who, in the opinion of the institution, would be a good match for senior professor positions. An expert panel is assembled to select the best qualified candidate for this role. Based on this collection of qualities, the panel assesses applicants. $\wp = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_4, \varsigma_6\}$, where ς_1 = productivity of research, ς_2 = managerial abilities, ς_3 = influence on the research community, ς_4 = ability for working under pressure, ς_5 = attributes of academic leadership, and ς_6 = contribution to X University. Define, an ideal on $\mathscr{Q}$ as $\mathscr{L} = \{\emptyset, \{\beth_1\}\}$.

Step 1: *Let $\mathfrak{D} = \{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3\}$ is the board on the round table of experts who decided the evaluations for those candidates as:*

$$\Im_1 = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}, \Im_2 = \{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\} \text{ and } \Im_3 = \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}$$

Step 2: *The results of the board in two different meetings and times for the candidates are presented in the form of bipolar soft sets as $\beta_1 = (f_1, g_1 : \wp)$ and $\beta_2 = (f_2, g_2 : \wp)$, where positive*

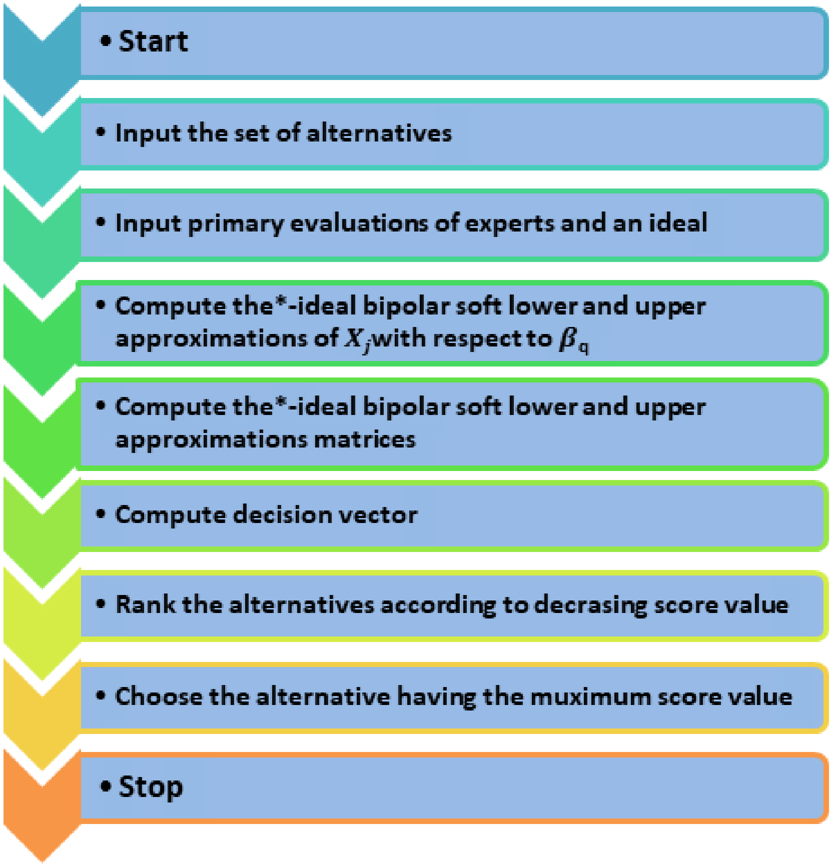

Figure 1: Summary of the proposed mathematical modelling in subsection 7.2 for MAGDM.

membership map of bipolar soft sets denotes the ability of candidates and negative membership denotes non-ability of candidates in a certain attribute:

$$f_1 : \wp \longrightarrow 2^{\mathscr{Q}}, \qquad \varsigma \mapsto \begin{cases} \{\mathscr{C}_1, \mathscr{C}_3\}, & \text{if } \varsigma = \varsigma_1, \\ \{\mathscr{C}_1, \mathscr{C}_4, \mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_2, \\ \{\mathscr{C}_2\}, & \text{if } \varsigma = \varsigma_3, \\ \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_4, \\ \{\mathscr{C}_1, \mathscr{C}_2\}, & \text{if } \varsigma = \varsigma_5, \\ \{\mathscr{C}_3, \mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_6, \end{cases} \quad \text{and} \quad g : \aleph \longrightarrow 2^{\mathscr{Q}}, \qquad \neg\varsigma \mapsto \begin{cases} \{\mathscr{C}_2, \mathscr{C}_5\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\mathscr{C}_3\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\mathscr{C}_1, \mathscr{C}_3\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\mathscr{C}_5\}, & \text{if } \neg\varsigma = \neg\varsigma_5, \\ \{\mathscr{C}_2, \mathscr{C}_4\}, & \text{if } \neg\varsigma = \neg\varsigma_6, \end{cases}$$

Also,

$$f_2 : \wp \longrightarrow 2^{\mathscr{Q}} \qquad \varsigma \mapsto \begin{cases} \{\mathscr{C}_2, \mathscr{C}_3\}, & \text{if } \varsigma = \varsigma_1, \\ \{\mathscr{C}_1, \mathscr{C}_3\}, & \text{if } \varsigma = \varsigma_2, \\ \{\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}, & \text{if } \varsigma = \varsigma_3, \\ \{\mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_4, \\ \{\mathscr{C}_1, \mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_5, \\ \{\mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}, & \text{if } \varsigma = \varsigma_6, \end{cases} \quad \text{and} \quad g_2 : \aleph \longrightarrow 2^{\mathscr{Q}}, \qquad \neg\varsigma \mapsto \begin{cases} \{\mathscr{C}_4\}, & \text{if } \neg\varsigma = \neg\varsigma_1, \\ \{\mathscr{C}_4, \mathscr{C}_5\}, & \text{if } \neg\varsigma = \neg\varsigma_2, \\ \{\mathscr{C}_1\}, & \text{if } \neg\varsigma = \neg\varsigma_3, \\ \{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_4\}, & \text{if } \neg\varsigma = \neg\varsigma_4, \\ \{\mathscr{C}_2, \mathscr{C}_3\}, & \text{if } \neg\varsigma = \neg\varsigma_5, \\ \{\mathscr{C}_1, \mathscr{C}_2\}, & \text{if } \neg\varsigma = \neg\varsigma_6. \end{cases}$$

Step 3: *The $*-$ideal bipolar soft lower and UAs of $\Im_j; (j = 1, 2, 3)$ with respect to $\beta_q =$*

$(f_q, g_q : \wp) \in \mathcal{BSS2}; (\mathcal{Q} = 1, 2)$ *could be computed as follow:*

$$\begin{aligned}
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_1) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_1), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_1)\right) = (\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_1) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_1), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_1)\right) = (\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}, \{\mathscr{C}_5\});\\
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_2) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_2), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_2)\right) = (\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_2) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_2), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_2)\right) = (\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\});\\
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_3) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_3), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_3)\right) = (\{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_3\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_1}(\Im_3) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^+}(\Im_3), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_1^-}(\Im_3)\right) = (\{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_3\}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_1) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_1), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_1)\right) = (\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_4, \mathscr{C}_5\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_1) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_1), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_1)\right) = (\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_4, \mathscr{C}_5\});\\
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_2) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_2), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_2)\right) = (\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_2) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_2), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_2)\right) = (\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\});\\
\underline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_3) &= \left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_3), \underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_3)\right) = (\{\mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_3\});\\
\overline{* - \mathfrak{PSR}}^{\mathscr{L}}_{\beta_2}(\Im_3) &= \left(\overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^+}(\Im_3), \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_2^-}(\Im_3)\right) = (\{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1\}).
\end{aligned}$$

Step 4: *According to Definition 7.1, the* $*-$*ideal bipolar soft lower and UA matrices can be calculated as follow:*

$$[\underline{\mathcal{M}}] = \left(\begin{bmatrix}(1,1,1,0,0), \left(0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\right)\\(1,1,1,0,0), \left(0,0,0,\frac{1}{2},\frac{1}{2}\right)\end{bmatrix} \begin{bmatrix}(1,0,1,0,1), \left(0,\frac{1}{2},0,\frac{1}{2},0\right)\\(1,0,1,0,1), \left(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\end{bmatrix} \begin{bmatrix}(0,1,0,1,1), \left(\frac{1}{2},0,\frac{1}{2},0,0\right)\\(0,0,0,0,1), \left(\frac{1}{2},0,\frac{1}{2},0,0\right)\end{bmatrix}\right);$$

$$\left[\overline{\mathcal{M}}\right] = \left(\begin{bmatrix}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0\right), (0,0,0,0,1)\\\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0\right), (0,0,0,1,1)\end{bmatrix} \begin{bmatrix}\left(\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), (0,1,0,1,0)\\\left(\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}\right), (0,1,0,1,0)\end{bmatrix} \begin{bmatrix}\left(0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\right), (1,0,1,0,0)\\\left(0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2}\right), (1,0,0,0,0)\end{bmatrix}\right).$$

Step 5: *By Definition 7.2,* $\mathbb{V}^f$ *and* $\mathbb{V}^g$ *can be calculated as follow:*

$$\begin{aligned}
\mathbb{V}^f &= \bigoplus_{j=1}^{3}\bigoplus_{q=1}^{2}\left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^+}(\Im_j) \oplus \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^+}(\Im_j)\right) = (6,5,6,3,6),\\
\mathbb{V}^g &= \bigoplus_{j=1}^{3}\bigoplus_{q=1}^{2}\left(\underline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^-}(\Im_j) \oplus \overline{* - \mathfrak{SR}}^{\mathscr{L}}_{\beta_q^-}(\Im_j)\right) = (3,3.5,2.5,5,3.5).
\end{aligned}$$

Step 6: *By Definition 7.3, we get*

$$\mathbb{V}_d = \mathbb{V}^f - \mathbb{V}^g = (3, 1.5, 3.5, -2.5, 2.5)).$$

Step 7: *As* $\beth_{1\leq i\leq 5}\ \delta_i = \mathscr{C}_3 = 3.5$. *Therefore,* $\mathscr{C}_3$ *is the best member for that position of a senior faculty. Accordingly, we get the inclination arrange of those five candidates are as follows:*

$$\mathscr{C}_3 > \mathscr{C}_1 > \mathscr{C}_5 > \mathscr{C}_2 > \mathscr{C}_4.$$

A graphical representation of the candidate inclination arrange is appeared in Figure 2.

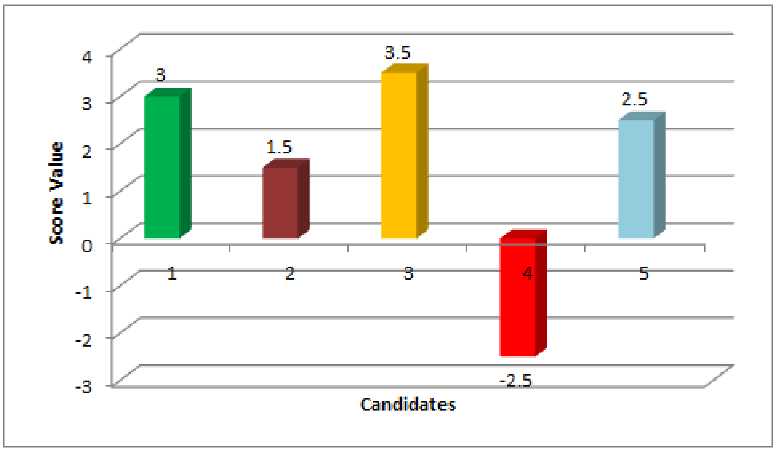

Figure 2: Preference order of the candidates in Example 7.1

8. Analytic comparison

8.1. The value of the proposed technique

The advantages of the proposed method upper the existing strategies are explained down.

(1) The proposed strategy considers positive and negative viewpoints of each elective within the shape of bipolar soft set. This crossover demonstrate is more generalized and appropriate for managing with aggressive DM.

(2) Utilizing the $*-$ideal bipolar soft lower and UAs, this approach gives another way to get the bunch inclination assessment based on the person inclination assessment for a considered MAGDM issue.

(3) Our proposed strategy successfully solves MAGDM issues when the weight data for the attribute is totally obscure.

(4) The proposed approach considers not as it were the point of views of DMs but too past encounters (essential assessments) by $*-$ideal bipolar soft lower and UAs in genuine scenarios. In this manner, it may be a more comprehensive approach for a higher translation of accessible data and hence makes choices utilizing artificial intelligence.

(5) In the event that we compare our proposed strategy with strategies displayed in [13, 38–40], we understand that those strategies are unable of identifying bipolarity within the DM process, which could be a key component of human considering and behaviour.

8.2. Comparing with other techniques

In this subsection, we reevaluate the finest DM method for the instability issue produced in Example 7.1 utilizing the calculation given by Shabir and Gul [41]. These results were compared with the DM strategy produced in the paper. To begin with, we utilized the calculations produced by Shabir and Gul [41] to solve Example 7.1. Upon these results, we got the inclination ordering of those candidates as takes after:

$$\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 = \mathscr{C}_4 = \mathscr{C}_5.$$

Table 2: Comparative analysis: A brief summary according to Example 7.1.

Different methods	Ranking of candidates	Optimal candidate
(Shabir and Gul, 2020) [41]	$\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 = \mathscr{C}_4 = \mathscr{C}_5$	Cannot handle
(Karaaslan and Çağman, 2018) [33]	$\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 > \mathscr{C}_5 > \mathscr{C}_5$	$\mathscr{C}_1$ or $\mathscr{C}_2$ or $\mathscr{C}_3$
(Gul et al.,2022) [34]	$\mathscr{C}_3 > \mathscr{C}_1 = \mathscr{C}_5 > \mathscr{C}_2 > \mathscr{C}_4$	$\mathscr{C}_3$
Suggested method	$\mathscr{C}_3 > \mathscr{C}_1 > \mathscr{C}_5 > \mathscr{C}_2 > \mathscr{C}_4$	$\mathscr{C}_3$

Stated differently, it was not possible to determine the candidates' preference order.

Using the technique described in Karaaslan and Çağman [33] on Example 7.1, we can now determine the candidates' preference order as follows:

$$\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 > \mathscr{C}_5 > \mathscr{C}_5.$$

Therefore, we cannot determine the best candidates among $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$. Applying the method described in Gul et al. [34] on Example 7.1, determine the candidates' preference order as follows:

$$\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 > \mathscr{C}_5 > \mathscr{C}_5.$$

Therefore, we cannot determine the best candidates between $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$. Table 2 demonstrates that our suggested approach is able to distinguish each candidate clearly and determine who is the best fit for a senior academic job. Given all of these advantages, we advise using the strategy presented in this study based on ideals in the DM process for uncertainty concerns.

9. Conclusion

The notion of ideals plays a pivotal role in topological spaces and serves as a fundamental tool for addressing various topological challenges. This paper presents a generalization and enhancement of three bipolar soft rough set models, introducing new approximation techniques referred to as $*-$ideal bipolar soft rough approximations. These methods expand the scope of previous approaches by incorporating unique properties and features. Two different techniques of $*-$ideal bipolar SA spaces defined with two ideals, called $*-$ideal bipolar SA spaces, were presented in Definitions 5.2 and 5.3. Additionally, these methodologies' comparisons are examined. Using n-ideals, this strategy can be expanded in analogously. Moreover, certain uncertainty measures related to the discussed $*-$ideal bipolar soft rough approximation spaces are also offered. To highlight the significance of this work, a generic framework for multi-attribute group decision-making (MAGDM) was proposed, based on the $*-$ideal bipolar soft rough approximations. This framework effectively enhances the decision-making process by improving the reliability of expert evaluations and facilitating the selection of optimal alternatives. Two primary advantages of the proposed decision-making algorithm were emphasized:

(1) Its ability to control uncertainty and bipolarity in the data.

(2) Its capacity to incorporate the opinions of multiple experts across various alternatives.

The validity of the proposed methodology was demonstrated through a real-world application of the MAGDM framework. A comparison study between the suggested approach and the existing methods confirmed the superiority of the $*-$ideal bipolar soft approximation technique. This generalization proved to be a more effective and reliable tool for addressing complex decision-making challenges, showcasing its potential to guide accurate and informed decisions in practical scenarios.

Applications and Future Directions. The practical applications of this research are vast and diverse, particularly in areas requiring enhanced decision-making under conditions of uncertainty. Key applications include multi-attribute decision-making in fields such as healthcare, financial risk assessment, and engineering project evaluations. In healthcare, for instance, the proposed framework can assist in diagnostic decision-making by integrating conflicting expert opinions and prioritizing alternatives. Similarly, in finance, it can be used to weigh the potential risks and returns of investment portfolios. Beyond these, the methodology has potential utility in artificial intelligence, where decision-making systems require the handling of nuanced and uncertain data. Looking forward, this research paves the way for several promising directions. One significant avenue lies in integrating the proposed techniques with machine learning models to enhance their robustness and scalability. Another key direction is the exploration of hybrid models that combine ideal bipolar soft rough sets with neural networks or deep learning frameworks to address high-dimensional and complex datasets. Additionally, expanding the application of these methods to dynamic systems, such as real-time decision-making in autonomous vehicles or adaptive control in robotics, represents a compelling opportunity. Further theoretical advancements, including the extension to n-dimensional and probabilistic spaces, could also enhance the versatility and impact of this innovative framework across a broader spectrum of scientific and industrial domains.

Conflicts of interest: The authors declare that there is no conflict of interest regarding this paper.

Data Availability Statement: The data sets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

References

[1] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[2] Daniel Paternain, Aranzazu Jurio, E Barrenechea, H Bustince, B Bedregal, and Eulalia Szmidt. An alternative to fuzzy methods in decision-making problems. *Expert Systems with Applications*, 39(9):7729–7735, 2012.

[3] Dmitriy Molodtsov. Soft set theory—first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[4] PK Maji, Akhil Ranjan Roy, and Ranjit Biswas. An application of soft sets in a decision making problem. *Computers & Mathematics with Applications*, 44(8-9):1077–1083, 2002.

[5] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[6] Serdar Enginoğlu, Naim Çağman, Serkan Karataş, and Tuğçe AYDIN. On soft topology. *El-Cezeri*, 2(3):23–38, 2015.

[7] Fatia Fatimah, Dedi Rosadi, RB Fajriya Hakim, and José Carlos R Alcantud. N-soft sets and their decision making algorithms. *Soft Computing*, 22:3829–3842, 2018.

[8] Muhammad Akram, Ghous Ali, José CR Alcantud, and Fatia Fatimah. Parameter reductions in n-soft sets and their applications in decision-making. *Expert Systems*, 38(1):e12601, 2021.

[9] Rana Muhammad Zulqarnain, Harish Garg, Imran Siddique, Rifaqat Ali, Abdelaziz Alsubie, Nawaf N Hamadneh, and Ilyas Khan. Algorithms for a generalized multipolar neutrosophic soft set with information measures to solve medical diagnoses and decision-making problems. *Journal of Mathematics*, 2021(1):6654657, 2021.

[10] Tareq M Al-shami, Ljubiša DR Kočinac, and Baravan A Asaad. Sum of soft topological spaces. *Mathematics*, 8(6):990, 2020.

[11] José Carlos R Alcantud and Gustavo Santos-García. A new criterion for soft set based

decision making problems under incomplete information. *International Journal of Computational Intelligence Systems*, 10(1):394–404, 2017.

[12] Kemal Taşköprü and Elif Karaköse. A soft set approach to relations and its application to decision making. *Mathematical Sciences and Applications E-Notes*, 11(1):1–13, 2023.

[13] Feng Feng. Soft rough sets applied to multicriteria group decision making. *Annals of Fuzzy Mathematics and Informatics*, 2(1):69–80, 2011.

[14] Shawkat Alkhazaleh and Emad A Marei. New soft rough set approximations. *International Journal of Fuzzy Logic and Intelligent Systems*, 21(2):123–134, 2021.

[15] Dragan Janković and TR Hamlett. New topologies from old via ideals. *The american mathematical monthly*, 97(4):295–310, 1990.

[16] Ali Kandil, Sobhy A El-Sheikh, Mona Hosny, and Mahmoud Raafat. Bi-ideal approximation spaces and their applications. *Soft Computing*, 24:12989–13001, 2020.

[17] Tareq M Al-Shami and Abdelwaheb Mhemdi. Approximation operators and accuracy measures of rough sets from an infra-topology view. *Soft Computing*, 27(3):1317–1330, 2023.

[18] Tareq M Al-Shami and Mona Hosny. Improvement of approximation spaces using maximal left neighborhoods and ideals. *IEEE Access*, 10:79379–79393, 2022.

[19] M Hosny, Tareq M Al-shami, and Abdelwaheb Mhemdi. Novel approaches of generalized rough approximation spaces inspired by maximal neighbourhoods and ideals. *Alexandria Engineering Journal*, 69:497–520, 2023.

[20] Rehab Alharbi, SE Abbas, E El-Sanowsy, HM Khiamy, KA Aldwoah, and Ismail Ibedou. New soft rough approximations via ideals and its applications. *AIMS Mathematics*, 9(4):9884–9910, 2024.

[21] Muhammad Shabir and Munazza Naz. On bipolar soft sets. *arXiv preprint arXiv:1303.1344*, 2013.

[22] Keon Myung Lee. Bipolar-valued fuzzy sets and their operations. In *Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 2000*, pages 307–312, 2000.

[23] Ghous Ali, Muhammad Akram, and José Carlos R Alcantud. Attributes reductions of bipolar fuzzy relation decision systems. *Neural computing and applications*, 32(14):10051–10071, 2020.

[24] Abdallah Al-Husban, Ala Amourah, and Jamil J Jaber. Bipolar complex fuzzy sets and their properties. *Italian Journal of Pure and Applied Mathematics*, 43:754–761, 2020.

[25] Muhammad Riaz and Syeda Tayyba Tehrim. Bipolar fuzzy soft mappings with application to bipolar disorders. *International Journal of Biomathematics*, 12(07):1950080, 2019.

[26] Munazza Naz and Muhammad Shabir. On fuzzy bipolar soft sets, their algebraic structures and applications. *Journal of Intelligent & Fuzzy Systems*, 26(4):1645–1656, 2014.

[27] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):4077–4087, 2017.

[28] Jin-Ying Wang, Yan-Ping Wang, and Lei Liu. Hesitant bipolar-valued fuzzy soft sets and their application in decision making. *Complexity*, 2020(1):6496030, 2020.

[29] Asghar Khan, Fawad Hussain, Asmat Hadi, and Sajjad Ahmad Khan. A decision making approach based on multi-fuzzy bipolar soft sets. *Journal of Intelligent & Fuzzy Systems*, 37(2):1879–1892, 2019.

[30] Muhammad Akram, Feng Feng, Arsham Borumand Saeid, and V Leoreanu-Fotea. A new multiple criteria decision-making method based on bipolar fuzzy soft graphs. *Iranian Journal of Fuzzy Systems*, 15(4):73–92, 2018.

[31] Saima Mustafa, Neelofar Safdar, Murrium Bibi, AF Sayed, Muhammad Ghaffar Khan, and Zabidin Salleh. A study of bipolar fuzzy soft sets and its application in decision-making

problems. *Mathematical Problems in Engineering*, 2021(1):5742288, 2021.
[32] Nosheen Malik and Muhammad Shabir. Rough fuzzy bipolar soft sets and application in decision-making problems. *Soft Computing*, 23:1603–1614, 2019.
[33] Faruk Karaaslan and Naim Çağman. Bipolar soft rough sets and their applications in decision making. *Afrika Matematika*, 29:823–839, 2018.
[34] Rizwan Gul, Muhammad Shabir, Wali Khan Mashwani, and Hayat Ullah. Novel bipolar soft rough-set approximations and their application in solving decision-making problems. *Int. J. Fuzzy Log. Intell. Syst.*, 22(3):303–324, 2022.
[35] Heba I Mustafa. Bipolar soft ideal rough set with applications in covid-19. *Turkish Journal of Mathematics*, 47(1):1–36, 2023.
[36] Günther Gediga and Ivo Düntsch. Rough approximation quality revisited. *Artificial Intelligence*, 132(2):219–234, 2001.
[37] Yi-Yu Yao. Notes on rough set approximations and associated measures. *Journal of zhejiang ocean university (natural science)*, 29(5):399–410, 2010.
[38] Salvatore Greco, Benedetto Matarazzo, and Roman Slowinski. The use of rough sets and fuzzy sets in mcdm. In *Multicriteria decision making: Advances in MCDM models, algorithms, theory, and applications*, pages 397–455. Springer, 1999.
[39] Salvatore Greco, Benedetto Matarazzo, and Roman Slowinski. Rough sets theory for multicriteria decision analysis. *European journal of operational research*, 129(1):1–47, 2001.
[40] Yıldıray Çelik and Sultan Yamak. Fuzzy soft set theory applied to medical diagnosis using fuzzy arithmetic operations. *Journal of inequalities and applications*, 2013:1–9, 2013.
[41] Muhammad Shabir and Rizwan Gul. Modified rough bipolar soft sets. *Journal of Intelligent & Fuzzy Systems*, 39(3):4259–4283, 2020.