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Hierarchy Elements in an Almost Distributive Lattice

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Abstract. In this paper, we introduce hierarchy elements in an almost distributive lattice with respect to a non-empty set and obtain some of their algebraic properties. We characterize initial segments, ideals, and maximal sets in almost distributive lattices in terms of hierarchy sets and prove that the class of hierarchy sets forms a distributive lattice, which is not an induced sublattice. Also, we characterize hierarchy sets using compatible sets in an almost distributive lattice.

2020 Mathematics Subject Classifications: 06D99, 06D75

Key Words and Phrases: Hierarchy elements, ideals, principal ideals, almost distributive lattices

1. Introduction

One of both the lattice theoretic and ring theoretic generalizations of a distributive lattice (Boolean algebra) [7], led by Swamy and Rao in 1981 and called an almost distributive lattice [8]. Almost distributive lattices is an algebraic structure $(L, \vee, \wedge, 0)$ that satisfies almost every axiom of a distributive lattice with zero, not including the three identities (the commutativity of $\wedge, \vee$ and the right distributivity of $\vee$ over $\wedge$). Each of these three identities is equivalent to each other in an almost distributive lattice, and an almost distributive lattice with any of the above identities gives a distributive lattice. The structure of an almost distributive lattice is not even distributive; the lattice and the associativity of $\vee$ are not yet to be known. Hence, it is difficult to deal with it. For example, fix an element x_0 in a non-empty set L. Given $x, y \in L$, define $x \wedge y = y, x_0 \wedge y = x_0, x \vee y = x$ and $x_0 \vee y = y$. Then $(L, \wedge, \vee, x_0)$ is an almost distributive lattice. This L is neither lattice nor distributive. It is called a discrete [8] almost distributive lattice. Given a, b in

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an almost distributive lattice L, we say that $a \leq b$ if $a \wedge b = a$ (or equivalently, $a \vee b = b$). Then, $\leq$ is a partial ordering on L. An element $m \in L$ is called a maximal element [8] if $m \wedge x = x$ for all $x \in L$.

The authors [3–5, 8, 9] developed the theory of initial segments, ideals, and filters in almost distributive lattices analogous to the concepts in distributive lattices. It is well-known that "every distributive lattice is a one-to-one correspondence with the class of principal ideals" in it [1]. The above-mentioned property does not hold in an almost distributive lattice L because "given $a, b \in L, (a] = (b]$ does not imply $a = b$" [8], where $(a] = \{a \wedge x \mid x \in L\}$ is the smallest ideal containing a. Given an initial segment $[0, a] = \{x \in L \mid 0 \leq x \leq a\}$ in an almost distributive lattice L, the authors [8] observed that given $b \in L, b \in [0, a]$ if and only if $a \wedge b = b \wedge a$. Further, the authors extended the above properties to the concept of principal ideals and stated that "given $b \in L, b \in (a]$ if and only if $a \wedge b = b$" [8]. In 2024, Noorbhasha et al. [2] proved some properties of prime σ-ideals of a normal almost distributive lattice topologically. Srikanth et al. [6] looked into why the binary operation ρ does not work in semi-Brouwerian almost distributive lattices. They found that it does not work associatively or commutatively.

With this motivation, we find an arbitrary non-empty set S in an almost distributive lattice L with the property that for any $h \in L, s \wedge h = h$, for some $s \in S$. We derive algebraic properties from the class of hierarchy elements with respect to a set S. Also, we characterize the class of hierarchy sets in an almost distributive lattice and provide sufficient counterexamples. Mainly, we observe that the class of hierarchy sets is a distributive lattice with respect to the operations $\cup$ and $\bigwedge$ where $S_1 \bigwedge S_2 = \{s_1 \wedge s_2 \mid s_1 \in S_1$ and $s_2 \in S_2\}$, for all non-empty subsets S_1, S_2 of L.

2. Hierarchy sets in almost distributive lattices

In this section, we define hierarchy elements with respect to a non-empty set in an almost distributive lattice. We prove several algebraic properties in the class of hierarchy elements and hierarchy sets. Finally, we obtain that the class of hierarchy sets forms a distributive lattice, which is not an induced sub-distributive lattice of the class of ideals of almost distributive lattices. Finally, we derive some equivalent conditions for a hierarchy set in an almost distributive lattice to become an ideal.

Definition 1. *An element $h \in L$ is said to be a hierarchy with respect to a non-empty subset S of L if $s \wedge h = h$, for some $s \in S$. The set of hierarchy elements with respect to S is denoted by H_S. It is easy to observe that $H_S \neq \emptyset$ (since $0 \in H_S$).*

Proposition 1. *For any $\emptyset \neq S \subseteq L$, we have*

(i) $S \subseteq H_S$.

(ii) H_S *is closed under* $\wedge$.

(iii) $H_L = L$ *and* $H_{\{0\}} = \{0\}$, *where* 1 *is the greatest element in* L.

(iv) *If* $1 \in H_S$*, then* $H_S = L$*.*

(v) *If* $m \in H_S$*, then* $H_S = L$*, where* m *is a maximal element in* L*.*

Proof. (i) Let $s \in S$. Now, $s \wedge s = s$. Therefore, $s \in H_S$. Hence, $S \subseteq H_S$.

(ii) Let $h_1, h_2 \in H_S$. Then we can find $s_1, s_2 \in S$ such that $s_1 \wedge h_1 = h_1$ and $s_2 \wedge h_2 = h_2$. Now, $s_1 \wedge (h_1 \wedge h_2) = (s_1 \wedge h_1) \wedge h_2 = h_1 \wedge h_2$. Therefore, $h_1 \wedge h_2 \in H_S$. Hence, H_S is closed under $\wedge$.

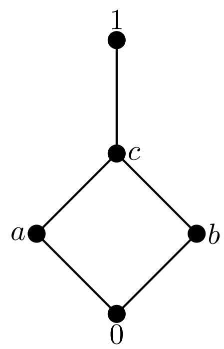
(iii) From (i), it is easy to observe that $H_L = L$. Let $x \in H_{\{0\}}$. Then $0 \wedge x = x$. Therefore, $x = 0$. Hence, $H_{\{0\}} = \{0\}$.

(iv) If $1 \in H_S$, then there is an element $s \in S$ such that $1 = s \wedge 1 = s \in S$. For this $1 \in S$, $1 \wedge x = x$ for all $x \in L$. Therefore, $L \subseteq H_S$. Hence, $H_S = L$.

(v) For $m \in H_S$, there is an element $s \in S$ such that $s \wedge m = m$. Let $x \in L$. Now, $s \wedge x = m \wedge (s \wedge x) = (m \wedge s) \wedge x = (s \wedge m) \wedge x = m \wedge x = x$. Therefore, $x \in H_S$. Hence, $L \subseteq H_S$. Thus, $H_S = L$.

Remark 1. *For any* $\emptyset \neq S \subseteq L$*,* H_S *need not be closed under* $\vee$ *by the following counterexample:*

Example 1. *Let* $L = \{0, a, b, c, 1\}$*, whose Hasse diagram is given below:*



For $S_1 = \{a, b\}$*, it is easy to verify that* $a \vee b = c \notin H_{S_1} = \{0, a, b\}$*. Therefore,* H_{S_1} *is not closed under* $\vee$*.*

Proposition 2. *For any* $\emptyset \neq S \subseteq L$*, and* $a, b, h \in L$*, we have*

(i) $a \leq b$ *implies* $H_a \subseteq H_b$*,*

(ii) $a \leq b$ *and* $b \in H_S$ *implies* $a \in H_S$*,*

(iii) $h \in H_S$ *implies* $(h] \subseteq H_S$*,*

(iv) H_a *is an ideal of* L*.*

Proof. (i) Assume that $a \leq b$. Let $x \in H_a$. Then $a \wedge x = x$. Now, $b \wedge x = b \wedge (a \wedge x) = (b \wedge a) \wedge x = a \wedge x = x$. Therefore, $x \in H_b$. Hence, $H_a \subseteq H_b$.

(ii) Assume that $a \leq b$ and $b \in H_S$. For this $b \in H_S$, there is an element $s \in S$ such that $s \wedge b = b$. Now, $s \wedge a = s \wedge (a \wedge b) = (s \wedge a) \wedge b = a \wedge (s \wedge b) = a \wedge b = a$. Therefore, $a \in H_S$.

(iii) Let $h \in H_S$. Then $s \wedge h = h$ for some $s \in S$. For any $x \in (h]$, $x = h \wedge x$. Now, $s \wedge x = s \wedge (h \wedge x) = (s \wedge h) \wedge x = h \wedge x = x$. Therefore, $x \in H_S$. Hence, $(h] \subseteq H_S$.

(iv) Let $x, y \in H_a$. Then $a \wedge x = x$ and $a \wedge y = y$. Now, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = x \vee y$. Therefore, $x \vee y \in H_a$. Let $l \in L$. Then $a \wedge (l \wedge x) = (a \wedge l) \wedge x = l \wedge (a \wedge x) = l \wedge x$. Therefore, $l \wedge x \in H_a$ and $x \wedge l \in H_a$. Thus, H_a is an ideal of L.

Remark 2. *For any non-empty subsets S of a discrete almost distributive lattice X, either $H_S = \{0\}$ or $H_S = L$. Suppose $H_S \neq \{0\}$. Then there exists a non-zero element h in H_S such that $s \wedge h = h$ for some non-zero element $s \in S$ (if $s = 0$, then $h = 0$). Let $y \in L$. Now, $s \wedge y = (s \vee h) \wedge y = (s \wedge y) \vee (h \wedge y) = (s \wedge y) \vee y = y$ (since $h \neq 0$). Therefore, $y \in H_S$. Hence, $L \subseteq H_S$. Thus, $H_S = L$.*

Proposition 3. *For any $\emptyset \neq S_1, S_2 \subseteq L$, we have*

(i) $S_1 \subseteq S_2$ *implies* $H_{S_1} \subseteq H_{S_2}$,

(ii) $H_{S_1 \cup S_2} = H_{S_1} \cup H_{S_2}$,

(iii) $H_{S_1 \cap S_2} \subseteq H_{S_1} \cap H_{S_2}$.

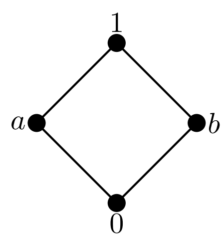
Proof. (i) Let $x \in H_{S_1}$. Then $s_1 \wedge x = x$ for some $s_1 \in S_1$. Since $S_1 \subseteq S_2$, $x \in H_{S_2}$. Hence, $H_{S_1} \subseteq H_{S_2}$.

(ii) Since $S_1, S_2 \subseteq S_1 \cup S_2$, we have $H_{S_1}, H_{S_2} \subseteq H_{S_1 \cup S_2}$. Therefore, $H_{S_1} \cup H_{S_2} \subseteq H_{S_1 \cup S_2}$. Let $h \in H_{S_1 \cup S_2}$. Then there is an element $s \in S_1 \cup S_2$ such that $s \wedge h = h$. If $s \in S_1$, then $h \in H_{S_1}$; if $s \in S_2$, then $h \in H_{S_2}$; and if $s \in S_1 \cap S_2$, then $h \in H_{S_1} \cap H_{S_2}$. Therefore, $h \in H_{S_1} \cup H_{S_2}$. So that $H_{S_1 \cup S_2} \subseteq H_{S_1} \cup H_{S_2}$. Hence, $H_{S_1 \cup S_2} = H_{S_1} \cup H_{S_2}$.

(iii) Let $h \in H_{S_1 \cap S_2}$. Then there is an element $s \in S_1 \cap S_2$ such that $s \wedge h = h$. Therefore, $h \in H_{S_1} \cap H_{S_2}$ (since $s \in S_1 \cap S_2$). Hence, $H_{S_1 \cap S_2} \subseteq H_{S_1} \cap H_{S_2}$.

Remark 3. *For any non-empty subsets S_1, S_2 of L, $H_{S_1 \cap S_2}$ need not be equal to $H_{S_1} \cap H_{S_2}$ by the following counterexample:*

Example 2. *Let $L = \{0, a, b, 1\}$, whose Hasse diagram is given below:*



For $S_1 = \{a, b\}$ and $S_2 = \{a, 1\}$, it is clearly to observe that $S_1 \cap S_2 = \{a\}$, $H_{S_1} = \{0, a, b\}$, $H_{S_2} = \{0, a, b, 1\}$, $H_{S_1 \cap S_2} = \{0, a\}$ and $H_{S_1} \cap H_{S_2} = \{0, a, b\}$. Hence, $H_{S_1 \cap S_2} \neq H_{S_1} \cap H_{S_2}$.

Theorem 1. *If $\emptyset \neq S \subseteq L$, is closed under $\vee$, then*

(i) H_S is closed under $\vee$,

(ii) H_S is a sub-almost distributive lattice of L,

(iii) H_S is an ideal of L,

(iv) H_S is the smallest ideal generated by $S (H_S = (S])$.

Proof. Suppose that S is closed under $\vee$.

(i) Let $h_1, h_2 \in H_S$. Then there exist $s_1, s_2 \in S$ such that $s_1 \wedge h_1 = h_1$ and $s_2 \wedge h_2 = h_2$. Now, $(s_1 \vee s_2) \wedge (h_1 \vee h_2) = (s_2 \vee s_1) \wedge (h_1 \vee h_2) = [(s_2 \vee s_1) \wedge h_1] \vee [(s_2 \vee s_1) \wedge h_2)] = [(s_2 \vee s_1) \wedge h_1] \vee [(s_1 \vee s_2) \wedge h_2)] = [(s_2 \wedge h_1) \vee (s_1 \wedge h_1] \vee [(s_1 \wedge h_2) \vee (s_2 \wedge h_2)] = [(s_2 \wedge h_1) \vee h_1] \vee [(s_1 \wedge h_2) \vee h_2] = h_1 \vee h_2$. Since S is closed under $\vee$, $h_1 \vee h_2 \in H_S$. Hence, H_S is closed under $\vee$.

(ii) By Proposition 1(ii) and Definition 1, we can clearly observe that the set H_S is a sub-almost distributive lattice of L.

(iii) Let $l \in L$ and $h \in H_S$. Then $s \wedge h = h$ for some $s \in S$. Now, $s \wedge (h \wedge l) = (s \wedge h) \wedge l = h \wedge l$ and $s \wedge (l \wedge h) = (s \wedge l) \wedge h = l \wedge (s \wedge h) = l \wedge h$. Therefore, $h \wedge l, l \wedge h \in H_S$. Hence, H_S is an ideal of L (since H_S is a sub-almost distributive lattice of L).

(iv) Since $S \subseteq H_S$ and H_S is an ideal of L, $(S] \subseteq H_S$. Let $h \in H_S$. Then $s \wedge h = h$ for some $s \in S$. Therefore, $h = s \wedge h \in (S]$. Hence, $H_S \subseteq (S]$. Thus, $H_S = (S]$.

Let us denote $S_1 \bigwedge S_2 = \{s_1 \wedge s_2 \in L \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}$, where S_1 and S_2 are two non-empty subsets of L.

Proposition 4. *For any $\emptyset \neq S_1, S_2, S_3 \subseteq L$, we have*

(i) $S_1 \bigwedge S_2 \neq \emptyset$,

(ii) $S_1 \subseteq S_1 \bigwedge S_1$,

(iii) $(S_1 \bigwedge S_2) \bigwedge S_3 = S_1 \bigwedge (S_2 \bigwedge S_3)$.

Proof. (i) For any $s_1 \in S_1$ and $s_2 \in S_2$, we have $s_1 \wedge s_2 \in L$. Therefore, $s_1 \wedge s_2 \in S_1 \bigwedge S_2$. Hence, $S_1 \bigwedge S_2 \neq \emptyset$.

(ii) For any $s_1 \in S_1$, $s_1 = s_1 \wedge s_1 \in S_1 \bigwedge S_1$. Therefore, $S_1 \subseteq S_1 \bigwedge S_1$.

(iii) Let $s \in (S_1 \bigwedge S_2) \bigwedge S_3$. Then $s = (s_1 \wedge s_2) \wedge s_3$ for some $s_1 \in S_1, s_2 \in S_2, s_3 \in S_3$. Since L is $\wedge$-associative, $s = s_1 \wedge (s_2 \wedge s_3)$. Therefore, $s \in S_1 \bigwedge (S_2 \bigwedge S_3)$. Hence, $(S_1 \bigwedge S_2) \bigwedge S_3 \subseteq S_1 \bigwedge (S_2 \bigwedge S_3)$. Similarly, we can prove the converse. Thus, $(S_1 \bigwedge S_2) \bigwedge S_3 = S_1 \bigwedge (S_2 \bigwedge S_3)$.

Remark 4. *For any $\emptyset \neq S \subseteq L$, S need not be idempotent under the operation $\bigwedge$. In Example 1, $S_1 \neq S_1 \bigwedge S_1$ (assume $S_1 = \{a, b\}$).*

Remark 5. *Any $\emptyset \neq S_1, S_2 \subseteq L$ need not be commutative under $\bigwedge$. In a discrete almost distributive lattice X, let $S_1 = \{x\}$ and $S_2 = \{y\}$, then it can be easy to verify that $S_1 \bigwedge S_2 \neq S_2 \bigwedge S_1$.*

Proposition 5. *For any $\emptyset \neq S_1, S_2, S_3 \subseteq L$, we have*

(i) $H_{S_1} \bigwedge H_{S_2} = H_{S_1 \bigwedge S_2}$,

(ii) $H_{S_1} \bigwedge H_{S_1} = H_{S_1}$ *(Idempotent law),*

(iii) $H_{S_1} \bigwedge H_{S_2} = H_{S_2} \bigwedge H_{S_1}$ *(Commutative law),*

(iv) $H_{S_1} \bigwedge (H_{S_2} \bigwedge H_{S_3}) = (H_{S_1} \bigwedge H_{S_2}) \bigwedge H_{S_3}$ *(Distributive law).*

Proof. (i) Let $h \in H_{S_1} \bigwedge H_{S_2}$. Then $h = h_1 \wedge h_2$ for some $h_1 \in H_{S_1}$ and $h_2 \in H_{S_2}$. For this $h_1 \in H_{S_1}$ and $h_2 \in H_{S_2}$, there exist $s_1 \in S_1, s_2 \in S_2$ such that $s_1 \wedge h_1 = h_1$ and $s_2 \wedge h_2 = h_2$. Now, $s_1 \wedge s_2 \wedge h = s_1 \wedge s_2 \wedge h_1 \wedge h_2 = s_1 \wedge h_1 \wedge s_2 \wedge h_2 = h_1 \wedge h_2 = h$ where $s_1 \wedge s_2 \in S_1 \bigwedge S_2$. Therefore, $h \in H_{S_1 \bigwedge S_2}$. Hence, $H_{S_1} \bigwedge H_{S_2} \subseteq H_{S_1 \bigwedge S_2}$. Let $h \in H_{S_1 \bigwedge S_2}$. Then there exist $s_1 \in S_1, s_2 \in S_2$ such that $s_1 \wedge s_2 \wedge h = h$. Now, $s_1 \wedge h = s_1 \wedge s_1 \wedge s_2 \wedge h = s_1 \wedge s_2 \wedge h = h$ and $s_2 \wedge h = s_2 \wedge (s_1 \wedge s_2 \wedge h) = s_1 \wedge s_2 \wedge h = h$. Therefore, $h \in H_{S_1}$ and $h \in H_{S_2}$. Hence, $h = h \wedge h = H_{S_1} \bigwedge H_{S_2}$. Thus, $H_{S_1} \bigwedge H_{S_2} = H_{S_1 \bigwedge S_2}$.

(ii) Let $h \in H_{S_1} \bigwedge H_{S_1} = H_{S_1 \bigwedge S_1}$ (from (i)). Then there exist $s_1, s_1' \in S_1$ such that $s_1 \wedge s_1' \wedge h = h$. Now, $s_1 \wedge h = s_1 \wedge (s_1 \wedge s_1' \wedge h) = s_1 \wedge s_1' \wedge h = h$. Therefore, $h \in H_{S_1}$. Hence, $H_{S_1} \bigwedge H_{S_1} \subseteq H_{S_1}$. By Proposition 4(ii), $S_1 \subseteq S_1 \bigwedge S_1$. Therefore, $H_{S_1} \subseteq H_{S_1} \bigwedge H_{S_1}$. Thus, $H_{S_1} = H_{S_1} \bigwedge H_{S_1}$.

(iii) Let $h \in H_{S_1} \bigwedge H_{S_2} = H_{S_1 \bigwedge S_2}$(from (i)). Then $s_1 \wedge s_2 \wedge h = h = s_2 \wedge s_1 \wedge h$ for some $s_1 \in S_1$ and $s_2 \in S_2$. Therefore, $h \in H_{S_2} \bigwedge H_{S_1}$. Hence, $H_{S_1} \bigwedge H_{S_2} \subseteq H_{S_2} \bigwedge H_{S_1}$. Similarly, we can prove the converse. Hence, $H_{S_1} \bigwedge H_{S_2} = H_{S_2} \bigwedge H_{S_1}$.

(iv) By Proposition 4(iii), we have $H_{S_1} \bigwedge (H_{S_2} \bigwedge H_{S_3}) = H_{S_1} \bigwedge H_{S_2 \bigwedge S_3} = H_{S_1 \bigwedge (S_2 \bigwedge S_3)}$ $= H_{(S_1 \bigwedge S_2) \bigwedge S_3} = H_{(S_1 \bigwedge S_2)} \bigwedge H_{S_3} = (H_{S_1} \bigwedge H_{S_2}) \bigwedge H_{S_3}$.

Proposition 6. *For any $\emptyset \neq S_1, S_2 \subseteq L$, we have*

(i) $H_{S_1} \cup H_{S_2} = H_{S_1 \cup S_2}$,

(ii) $H_{S_1} \cup H_{S_1} = H_{S_1}$ *(Idempotent),*

(iii) $H_{S_1} \cup H_{S_2} = H_{S_2} \cup H_{S_1}$ *(Commutative),*

(iv) $H_{S_1} \cup (H_{S_2} \cup H_{S_3}) = (H_{S_1} \cup H_{S_2}) \cup H_{S_3}$ *(Associative).*

Proposition 7. *For any $\emptyset \neq S \subseteq L$ and a is an element in L, we have*

(i) $H_a = (a]$,

(ii) $H_S = \bigcup_{s \in S} (s]$.

Proof. (i) By Proposition 2(iv), H_a is an ideal of L and $a \in H_a$. Then $(a] \subseteq H_a$. Let $h \in H_a$. Then $a \wedge h = h$. Therefore, $h = a \wedge h \in (a]$. So that $H_a \subseteq (a]$. Hence, $H_a = (a]$.

(ii) Let $h \in H_S$. Then, we have an element $s \in S$ such that $h = s \wedge h \in (s]$. Therefore, $H_S \subseteq \bigcup_{s \in S}(s]$. Let $y \in \bigcup_{s \in S}(s]$. Then $y \in (s]$ for some $s \in S$. Therefore, $y = s \wedge y$. So that $y \in H_S$. Hence, $\bigcup_{s \in S}(s] \subseteq H_S$. Thus, $H_S = \bigcup_{s \in S}(s]$.

Proposition 8. *For any $\emptyset \neq S \subseteq L$, $S \subseteq H_S \subseteq (S]$.*

Remark 6. *For any $\emptyset \neq S \subseteq L$, H_S need not be equal to $(S]$.*

In Example 2, let $S = \{a, b\}$, then $H_S = \{0, a, b\}$ and $(S] = \{0, a, b, 1\} = L$. Therefore, $H_S \subset (S]$ but $H_S \neq (S]$.

Proposition 9. *For any $\emptyset \neq S_1, S_2, S_3 \subseteq L$, we have*

(i) $S_1 \subseteq S_1 \cup (S_2 \bigwedge S_1)$,

(ii) $S_1 \subseteq S_1 \bigwedge (S_2 \cup S_1)$,

(iii) $S_1 \cup (S_2 \bigwedge S_3) \subseteq (S_1 \cup S_2) \bigwedge (S_1 \cup S_3)$,

(iv) $(S_1 \bigwedge S_2) \cup S_3 \subseteq (S_1 \cup S_3) \bigwedge (S_2 \cup S_3)$,

(v) $S_1 \bigwedge (S_2 \cup S_3) = (S_1 \bigwedge S_2) \cup (S_1 \bigwedge S_3)$ *(Distributive law),*

(vi) $(S_1 \cup S_2) \bigwedge S_3 = (S_1 \bigwedge S_3) \cup (S_2 \bigwedge S_3)$ *(Distributive law).*

Remark 7. *For any non-empty subsets S_1, S_2 of L, the following absorption laws need not be held:*

(i) $S_1 \cup (S_2 \bigwedge S_1) = S_1$,

(ii) $S_1 \bigwedge (S_2 \cup S_1) = S_1$.

In Example 1, (i) take $S_1 = \{a\}$ and $S_2 = \{b\}$. Then $S_2 \bigwedge S_1 = \{0\}$. Therefore, $S_1 \cup (S_2 \bigwedge S_1) = \{0, a\} \neq S_1$.

(ii) Take $S_2 \cup S_1 = \{a, b\}$. Then $S_1 \bigwedge (S_2 \cup S_1) = \{0, a\}$. Therefore, $S_1 \bigwedge (S_2 \cup S_1) \neq S_1$.

Remark 8. *For any $\emptyset \neq S_1, S_2, S_3 \subseteq L$, the following distributive laws need not be held:*

(i) $S_1 \cup (S_2 \bigwedge S_3) = (S_1 \cup S_2) \bigwedge (S_1 \cup S_3)$ *(Distributive law),*

(ii) $(S_1 \bigwedge S_2) \cup S_3 = (S_1 \cup S_3) \bigwedge (S_2 \cup S_3)$ *(Distributive law).*

In Example 1, (i) take $S_1 = \{a, b\}, S_2 = \{1\}$ and $S_3 = \{c\}$. Then $S_2 \bigwedge S_3 = \{c\}, S_1 \cup S_2 = \{a, b, 1\}$ and $S_1 \cup S_3 = \{a, b, c\}$. Now, $S_1 \cup (S_2 \bigwedge S_3) = \{a, b, c\}$ and $(S_1 \cup S_2) \bigwedge (S_1 \cup S_3) = \{0, a, b, c\}$. Therefore, $S_1 \cup (S_2 \bigwedge S_3) \neq (S_1 \cup S_2) \bigwedge (S_1 \cup S_3)$.

(ii) Take $S_1 = \{a\}, S_2 = \{1\}$ and $S_3 = \{b, c\}$. Then $S_1 \bigwedge S_2 = \{a\}, S_1 \cup S_3 = \{a, b, c\}$ and $S_2 \cup S_3 = \{b, c, 1\}$. Now, $(S_1 \bigwedge S_2) \cup S_3 = \{a, b, c\}$ and $(S_1 \cup S_3) \bigwedge (S_2 \cup S_3) = \{0, a, b, c\}$. Therefore, $(S_1 \bigwedge S_2) \cup S_3 \neq (S_1 \cup S_3) \bigwedge (S_2 \cup S_3)$.

Proposition 10. *For any $\emptyset \neq S_1, S_2, S_3 \subseteq L$, we have*

(i) $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1}) = H_{S_1}$ *(Absorption law),*

(ii) $H_{S_1} \bigwedge (H_{S_2} \cup H_{S_1}) = H_{S_1}$ *(Absorption law),*

(iii) $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_3}) = (H_{S_1} \cup H_{S_2}) \bigwedge (H_{S_1} \cup H_{S_3})$ *(Distributive law),*

(iv) $H_{S_1} \bigwedge (H_{S_2} \cup H_{S_3}) = (H_{S_1} \bigwedge H_{S_2}) \cup (H_{S_1} \bigwedge H_{S_3})$ *(Distributive law).*

Proof. (i) Let $x \in H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1}) = H_{S_1} \cup H_{S_2 \bigwedge S_1}$. If $x \in H_{S_1}$, then $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1}) \subseteq H_{S_1}$. If $x \in H_{S_2 \bigwedge S_1}$, then there exists $s = s_2 \wedge s_1 \in S_2 \bigwedge S_1$ such that $s \wedge x = s_2 \wedge s_1 \wedge x = x$ for some $s_2 \in S_2$ and $s_1 \in S_1$, then $s_1 \wedge x = s_1 \wedge s_2 \wedge s_1 \wedge x = s_2 \wedge s_1 \wedge x = x$. Therefore, $x \in H_{S_1}$. Hence, $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1}) \subseteq H_{S_1}$. By Proposition 9(i), $S_1 \subseteq S_1 \cup (S_2 \bigwedge S_1)$. Therefore, $H_{S_1} \subseteq H_{S_1 \cup (S_2 \bigwedge S_1)}$. Hence, $H_{S_1} \subseteq H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1})$. Thus, $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_1}) = H_{S_1}$.

(ii) Let $x \in H_{S_1} \bigwedge (H_{S_2} \cup H_{S_1}) = H_{S_1} \bigwedge (H_{S_2 \cup S_1}) = H_{S_1 \bigwedge (S_2 \cup S_1)}$. Then there exists $s \in S_1 \bigwedge (S_2 \cup S_1)$ such that $s \wedge x = x$. Therefore, for this $s \in S_1 \bigwedge (S_2 \cup S_1), s = s_1 \wedge s'$ for some $s_1 \in S_1$ and $s' \in S_2 \cup S_1$. Now, $s_1 \wedge x = s_1 \wedge (s \wedge x) = s_1 \wedge (s_1 \wedge s') \wedge x = s_1 \wedge s' \wedge x = s \wedge x = x$. Therefore, $x \in H_{S_1}$. Hence, $H_{S_1} \bigwedge (H_{S_2} \cup H_{S_1}) \subseteq H_{S_1}$. By Proposition 9(ii), $S_1 \subseteq S_1 \bigwedge (S_2 \cup S_1)$. Therefore, $H_{S_1} \subseteq H_{S_1 \bigwedge (S_2 \cup S_1)} = H_{S_1} \bigwedge H_{(S_2 \cup S_1)} = H_{S_1} \bigwedge (H_{S_2} \cup H_{S_1})$. Hence, $H_{S_1} \bigwedge (H_{S_2} \cup H_{S_1}) = H_{S_1}$.

(iii) Let $x \in (H_{S_1} \cup H_{S_2}) \bigwedge (H_{S_1} \cup H_{S_3}) = H_{(S_1 \cup S_2)} \bigwedge H_{(S_1 \cup S_3)} = H_{(S_1 \cup S_2) \bigwedge (S_1 \cup S_3)}$. Then there exists $s \in (S_1 \cup S_2) \bigwedge (S_1 \cup S_3)$ such that $s \wedge x = x$. For this $s \in (S_1 \cup S_2) \bigwedge (S_1 \cup S_3), s = a \wedge b$ for some $a \in (S_1 \cup S_2), b \in (S_1 \cup S_3)$.
If $a \in S_1$ and $b \in S_1$, then $a \wedge x = a \wedge s \wedge x = a \wedge a \wedge b \wedge x = a \wedge b \wedge x = s \wedge x = x$. Therefore, $x \in H_{S_1}$.
If $a \in S_2$ and $b \in S_1$, then $b \wedge x = b \wedge s \wedge x = b \wedge a \wedge b \wedge x = a \wedge b \wedge x = s \wedge x = x$. Hence, $x \in H_{S_1}$.
If $a \in S_2$ and $b \in S_3$, then $s = a \wedge b \in S_2 \bigwedge S_3$ and $s \wedge x = x$. Therefore, $x \in H_{S_2 \bigwedge S_3}$. Hence, $(H_{S_1} \cup H_{S_2}) \bigwedge (H_{S_1} \cup H_{S_3}) \subseteq H_{S_1} \cup (H_{S_2} \bigwedge H_{S_3})$. By Proposition 9(iii), $S_1 \cup (S_2 \bigwedge S_3) \subseteq (S_1 \cup S_2) \bigwedge (S_1 \cup S_3)$. Therefore, $H_{S_1 \cup (S_2 \bigwedge S_3)} \subseteq H_{(S_1 \cup S_2) \bigwedge (S_1 \cup S_3)}$. Hence, $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_3}) \subseteq (H_{S_1} \cup H_{S_2}) \bigwedge (H_{S_1} \cup H_{S_3})$. Thus, $H_{S_1} \cup (H_{S_2} \bigwedge H_{S_3}) = (H_{S_1} \cup H_{S_2}) \bigwedge (H_{S_1} \cup H_{S_3})$.

(iv) We can prove similarly by Proposition 9(v).

Let us denote the hierarchy sets of L by $\mathcal{H}_{\mathcal{S}} (= \{H_S \mid S \subseteq L \text{ and } S \neq \emptyset\})$. We have the final conclusion from Proposition 10, below.

Theorem 2. $(\mathcal{H}_{\mathcal{S}}, \bigwedge, \cup, \{0\}, L)$ *becomes a bounded distributive lattice.*

Proof. By the above Propositions 5, 6, and 10, we observe that the set of hierarchy sets $\mathcal{H}_{\mathcal{S}}$ is a distributive lattice with the operations $\bigwedge$ and $\cup$ and hence, $H_{\{0\}} = \{0\}$ is the least element and $H_L = L$ is the greatest element in $\mathcal{H}_{\mathcal{S}}$.

Theorem 3. *For any non-empty subsets S_0, S_1 of L, $H_{S_0} \bigwedge H_{S_1} = \{0\}$ and $H_{S_0} \cup H_{S_1} = L$ if and only if $S_0 \bigwedge S_1 = \{0\}$ and $S_0 \cup S_1$ contains a maximal element if and only if $S_0 = \{0\} \Rightarrow S_1$ contains a maximal element or $S_1 = \{0\} \Rightarrow S_0$ contains a maximal element.*

Theorem 4. *For any non-empty subset S of L, the following are equivalent:*

(i) H_S *is closed under* $\vee$,

(ii) H_S *is a subADL of* L,

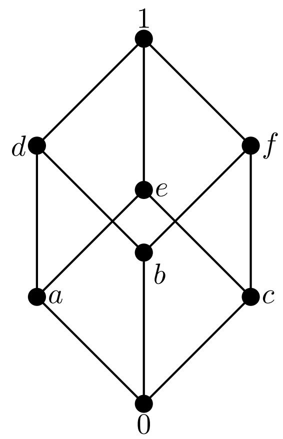
(iii) H_S *is an ideal of* L,

(iv) H_S *is a smallest ideal containing* S.

Lemma 1. *Let S be a non-empty subset and I be an ideal of L. If $H_S = H_I$, then $(S] = I$.*

Remark 9. *The converse of Lemma 1 need not be true. We can see the following counterexample:*

Example 3. *Let $L = \{0, a, b, c, d, e, f, 1\}$ be an ADL whose Hasse diagram is given below:*



Let $S_1 = \{a, b\}$. Then $(S_1] = \{0, a, b, d\} := I$. But $H_{S_1} = \{0, a, b\} \neq \{0, a, b, d\} = H_I$.

Lemma 2. *For any non-empty subset S of L, we have*

(i) $\bigcap_{S \subseteq L} H_S = \{0\}$,

(ii) $\bigcup_{S \subseteq L} H_S = L$,

(iii) $H_{H_S} = H_S$.

Let us denote the set S_M = set of maximal elements of S, and we can prove the following theorem on an ADL with maximal elements.

Theorem 5. *For any non-empty subset S of L, we have*

(i) $(S] = (S_M]$,

(ii) $\bigcup_{s_m \in S_M} H_{s_m} = H_S$.

Proof. (i) Let $h \in (S]$. Then $h = \left(\bigvee_{i=1}^{m} s_i\right) \wedge x$ for some $x \in L$ and $s_i \in S$ for all $1 \leq i \leq m$. Let $s_m \in S_M$. Then $s_m \wedge s_i = s_i$ for all $1 \leq i \leq m$. Therefore, $s_m \wedge s_i \in (S_M]$ for all $1 \leq i \leq m$. So that $\bigvee_{i=1}^{m} s_i = \bigvee_{i=1}^{m} (s_m \wedge s_i) \in (S_M]$. Hence, $h = \left(\bigvee_{i=1}^{m} s_i\right) \wedge x \in (S_M]$. Since $S_M \subseteq S, (S_M] \subseteq (S]$. Therefore, $(S] = (S_M]$.

(ii) Let $h \in \bigcup_{s_m \in S_M} H_{s_m}$. Then $h \in H_{s_m}$ for some $s_m \in S_M$ and $s_m \wedge h = h$. Since $S_M \subseteq S, h \in H_S$. Therefore, $\bigcup_{s_m \in S_M} H_{s_m} \subseteq H_S$. Let $h \in H_S$. Then there exists $s \in S$ such that $s \wedge h = h$. Let $s_m \in s_M$. Then $s_m \wedge s = s$. Now, $s_m \wedge h = s_m \wedge s \wedge h = s \wedge h = h$. Therefore, $h \in H_{s_m}$ for some $s_m \in S_M$. So that $h \in \bigcup_{s_m \in S_M} H_{s_m}$. Hence, $H_S \subseteq \bigcup_{s_m \in S_M} H_{s_m}$. Thus, $H_S = \bigcup_{s_m \in S_M} H_{s_m}$.

3. Characterization of hierarchy sets with respect to a compatible set

In this section, we characterize the class of hierarchy sets in terms of compatible sets, introduce a new class of sets in an almost distributive lattice, and study rigorously.

A non-empty subset S of L is said to be compatible if for each $s_1, s_2 \in S, s_1 \wedge s_2 = s_2 \wedge s_1$ or equivalently, $s_1 \vee s_2 = s_2 \vee s_1$.

Lemma 3. *Let $h \in L$ and a non-empty subset S of L. If there exists $s \in S$ such that $h \leq s$, then $h \in H_S$.*

Proof. If $h \leq s$ for some $s \in S$, then $s \wedge h = h = h \wedge s$. Therefore, $h \in H_S$.

Remark 10. *The converse of Lemma 3 need not be true. In a discrete ADL X, let $S = \{s\}$ for some $s \in X \setminus \{0\}$. Then $H_S = X$. Let $h \in X$. Then $h \in H_S$ and $h \wedge s = s$ and $s \wedge h = h$ for all $s \in S$. Therefore, $h \nleq s$.*

Remark 11. *If H_S is compatible, then the converse of Lemma 3 is true.*

Proof. Let $h \in H_S$ for some $h \in L$. Then, an element $s \in S$ exists such that $s \wedge h = h$. Since $S \subseteq H_S, s \wedge h = h \wedge s = h$. Hence, $h \leq s$.

Definition 2. *For any non-empty subset S of L, define a set $\widehat{S} = \{x \in L \mid x \leq s$, for some $s \in S\}$.*

Remark 12. *For any non-empty subset S of L, we can observe that $\widehat{S} \neq \emptyset$ and $S \subseteq \widehat{S}$.*

Lemma 4. *For any non-empty subset S of L, we have the following:*

(i) $\widehat{S}$ *is closed under* $\wedge$*,*

(ii) $h \wedge \widehat{s_1} \in \widehat{S}$*, for any* $h \in L$*,*

(iii) $\widehat{S} \subseteq H_S$*,*

(iv) $H_S = H_{\widehat{S}}$*.*

Proof. (i) Let $\widehat{s_1}, \widehat{s_2} \in \widehat{S}$. Then there exist $s_1, s_2 \in S$ such that $\widehat{s_1} \leq s_1$ and $\widehat{s_2} \leq s_2$. Now, $(\widehat{s_1} \wedge \widehat{s_2}) \wedge s_2 = \widehat{s_1} \wedge (\widehat{s_2} \wedge s_2) = \widehat{s_1} \wedge \widehat{s_2}$. Therefore, $\widehat{s_1} \wedge \widehat{s_2} \leq s_2$. So that $\widehat{s_1} \wedge \widehat{s_2} \in \widehat{S}$. Hence, $\widehat{S}$ is closed under $\wedge$.

(ii) Let $h \in L$ and $\widehat{s_1} \in \widehat{S}$. Then there exists $s_1 \in S$ such that $\widehat{s_1} \leq s_1$. Now, $(h \wedge \widehat{s_1}) \wedge s_1 = h \wedge (\widehat{s_1} \wedge s_1) = h \wedge \widehat{s_1}$. Then $h \wedge \widehat{s_1} \leq s_1$. Therefore, $h \wedge \widehat{s_1} \in \widehat{S}$.

(iii) Let $\widehat{s_1} \in \widehat{S}$. Then $\widehat{s_1} \wedge s_1 = s_1 \wedge \widehat{s_1} = \widehat{s_1}$. Therefore, $\widehat{s_1} \in H_S$ (since $s_1 \in S$). Hence, $\widehat{S} \subseteq H_S$.

(iv) By Definition 2, $S \subseteq \widehat{S}$. Then $H_S \subseteq H_{\widehat{S}}$. Let $h \in H_{\widehat{S}}$. Then $\widehat{s_1} \wedge h = h$ for some $\widehat{s_1} \in \widehat{S}$. For this $\widehat{s_1} \in \widehat{S}$, there exists $s_1 \in S$ such that $\widehat{s_1} \leq s_1$. Now, $s_1 \wedge h = s_1 \wedge (\widehat{s_1} \wedge h) = (s_1 \wedge \widehat{s_1}) \wedge h = \widehat{s_1} \wedge h = h$. Then $h \in H_S$ (since $s_1 \in S$). Hence, $H_{\widehat{S}} \subseteq H_S$. Thus, $H_S = H_{\widehat{S}}$.

Remark 13. $\widehat{S}$ *need not be closed under* $\vee$*.*

In Example 2, let $S = \{a, b\}$. Then $\widehat{S} = \{0, a, b\}$. Now, $a \vee b = c \notin \widehat{S}$. Hence, $\widehat{S}$ is not closed under $\vee$.

Remark 14. *For any* $h \in L$ *and* $\widehat{s_1} \in \widehat{S}, \widehat{s_1} \wedge h$ *need not be in* $\widehat{S}$*.*

In a discrete ADL X, let $S = \{a\}$ for some non-zero element $a \in X$. Then $\widehat{S} = \{0, a\}$. Let $x \in X$. Then $a \wedge x = x \notin \widehat{S}$.

Remark 15. *For any non-empty subset S of L,* $\widehat{S}$ *need not be equal to* $H_{\widehat{S}}$*.*

In a discrete ADL X, let $S = \{a\}$, for some non-zero element $a \in X$. Then $H_S = H_a = X$ and $\widehat{S} = \{0, a\}$. Therefore, $H_S \neq \widehat{S}$.

Lemma 5. *If H_S is compatible, then* $H_S = \widehat{S}$*, where $S \subseteq L$ and $S \neq \emptyset$.*

Proof. Let $h \in H_S$. Then, an element $s \in S$ exists such that $s \wedge h = h$. Since $S \subseteq H_S, s \wedge h = h \wedge s = h$. Therefore, $h \leq s$ and $s \in S$. So that $h \in \widehat{S}$. Hence, $H_S \subseteq \widehat{S}$. By Lemma 4(iii), $H_S = \widehat{S}$.

Theorem 6. *For any compatible subset S of L, we have the following:*

(i) $\widehat{S}$ *is compatible,*

(ii) For each $h \in H_S$*, there exists unique* $\widehat{s} \in \widehat{S}$ *such that* $\widehat{s} \wedge h = h$ *and* $h \wedge \widehat{s} = \widehat{s}$*.*

Proof. (i) Let $\widehat{s_1}, \widehat{s_2} \in \widehat{S}$. Then there exist $s_1, s_2 \in S$ such that $\widehat{s_1} \leq s_1$ and $\widehat{s_2} \leq s_2$. Now, $\widehat{s_1} \wedge \widehat{s_2} = (\widehat{s_1} \wedge s_1) \wedge (\widehat{s_2} \wedge s_2) = \widehat{s_2} \wedge \widehat{s_1} \wedge (s_1 \wedge s_2) = \widehat{s_2} \wedge \widehat{s_1} \wedge (s_2 \wedge s_1) = (\widehat{s_2} \wedge s_2) \wedge (\widehat{s_1} \wedge s_1) = \widehat{s_2} \wedge \widehat{s_1}$ (since S is compatible). Therefore, $\widehat{S}$ is compatible.

(ii) Let $h \in H_S$. By Lemma 4(iv), $h \in H_{\widehat{S}} = H_S$. For this $h \in H_{\widehat{S}}$, there exists $\widehat{s} \in \widehat{S}$ such that $\widehat{s} \wedge h = h$ and for this $\widehat{s} \in \widehat{S}$, there exists $s \in S$ such that $\widehat{s} \leq s$. Let $s^0 = h \wedge s$. Than $s^0 \wedge s = (h \wedge s) \wedge s = h \wedge s = s^0$. Therefore, $s^0 \leq s$. So that $s^0 \in \widehat{S}$. Now, $h \wedge s^0 = h \wedge h \wedge s = h \wedge s = s^0$ and $s^0 \wedge h = h \wedge s \wedge h = s \wedge h \wedge h = s \wedge h = h$. Let $\widehat{a} \in \widehat{S}$ such that $h \wedge \widehat{a} = \widehat{a}$ and $\widehat{a} \wedge h = h$. Now, $s^0 \wedge \widehat{a} = h \wedge s \wedge \widehat{a} = h \wedge \widehat{a} \wedge s = \widehat{a} \wedge h \wedge s = h \wedge s = s^0$ (since $\widehat{S}$ is compatible). Therefore, $s^0 \leq \widehat{a}$ and $\widehat{a} \wedge s^0 = \widehat{a} \wedge h \wedge s = h \wedge \widehat{a} \wedge s = h \wedge s \wedge \widehat{a} = s \wedge h \wedge \widehat{a} = h \wedge \widehat{a} = \widehat{a}$. Therefore, $\widehat{a} \leq s^0$. Hence, s^0 is unique.

Theorem 7. *Let S be a non-empty compatible subset of L. For any $h_1, h_2 \in H_S$, we have*

(i) $\widehat{(h_1 \vee h_2)} = \widehat{h_1} \vee \widehat{h_2}$,

(ii) $\widehat{(h_1 \wedge h_2)} = \widehat{h_1} \wedge \widehat{h_2}$,

(iii) $h \in H_S \Leftrightarrow h = \widehat{h}$, *for any* $\emptyset \neq S \subseteq L$,

(iv) $\widehat{0} = 0$.

Proof. Let $h_1, h_2 \in H_S$. By Theorem 6(ii), there exists $\widehat{h_1}, \widehat{h_2} \in \widehat{S}$ such that $h_1 \wedge \widehat{h_1} = \widehat{h_1}, h_2 \wedge \widehat{h_2} = \widehat{h_2}$ and $\widehat{h_1} \wedge h_1 = h_1, \widehat{h_2} \wedge h_2 = h_2$.

(i) Now,

$$\begin{aligned}
(\widehat{h_1} \vee \widehat{h_2}) \wedge (h_1 \vee h_2) &= [\widehat{h_1} \vee \widehat{h_2}) \wedge h_1] \vee [\widehat{h_1} \vee \widehat{h_2}) \wedge h_2] \\
&= [(\widehat{h_1} \wedge h_1) \vee (\widehat{h_2} \wedge h_1)] \vee [(\widehat{h_1} \wedge h_2) \vee (\widehat{h_2} \wedge h_2)] \\
&= [h_1 \vee (\widehat{h_2} \wedge h_1)] \vee [(\widehat{h_1} \wedge h_2) \vee h_2] \\
&= [(h_1 \vee (\widehat{h_2} \wedge h_1)] \vee h_2 \\
&= [(h_1 \vee \widehat{h_2}) \wedge h_1)] \vee h_2 \\
&= [(\widehat{h_2} \vee h_1) \wedge h_1] \vee h_2 \\
&= h_1 \vee h_2,
\end{aligned}$$

$$\begin{aligned}
(h_1 \vee h_2) \wedge (\widehat{h_1} \vee \widehat{h_2}) &= [(h_1 \vee h_2) \wedge \widehat{h_1}] \vee [(h_1 \vee h_2) \wedge \widehat{h_2}] \\
&= [(h_1 \wedge \widehat{h_1}) \vee (h_2 \wedge \widehat{h_1})] \vee [(h_1 \wedge \widehat{h_2}) \vee (h_2 \wedge \widehat{h_2})] \\
&= [\widehat{h_1} \vee (h_2 \wedge \widehat{h_1})] \vee [(h_1 \wedge \widehat{h_2}) \vee \widehat{h_2}] \\
&= [(\widehat{h_1} \vee h_2) \wedge \widehat{h_1}] \vee \widehat{h_2} \\
&= [(h_2 \vee \widehat{h_1}) \wedge \widehat{h_1}] \vee \widehat{h_2} \\
&= \widehat{h_1} \vee \widehat{h_2}.
\end{aligned}$$

Therefore, $\widehat{(h_1 \vee h_2)} = \widehat{h_1} \vee \widehat{h_2}$.

(ii) Now,

$$\begin{aligned}
(\widehat{h_1} \wedge \widehat{h_2}) \wedge h_1 \wedge h_2 &= \widehat{h_1} \wedge h_1 \wedge \widehat{h_2} \wedge h_2 \\
&= h_1 \wedge h_2,
\end{aligned}$$

$$
\begin{aligned}
h_1 \wedge h_2 \wedge (\widehat{h_1} \wedge \widehat{h_2}) &= h_1 \wedge \widehat{h_1} \wedge h_2 \wedge \widehat{h_2} \\
&= \widehat{h_1} \wedge \widehat{h_2}.
\end{aligned}
$$

Therefore, $(\widehat{h_1 \wedge h_2}) = \widehat{h_1} \wedge \widehat{h_2}$.

(iii) Let $h \in S$. Then $h \in H_S$. By Theorem 6(iii), there exists a unique $\widehat{h} \in \widehat{S}$ such that $h \wedge \widehat{h} = \widehat{h}$ and $\widehat{h} \wedge h = h$. Since $h, \widehat{h} \in \widehat{S}$ and $\widehat{S}$ is compatible, $h = \widehat{h}$. The converse is trivial.

(iv) It's clear from (iii).

Theorem 8. *Let S be a compatible subset of L and $H_S = L$. Then the following are equivalent:*

(i) $H_S = L \Rightarrow \widehat{S}$ has large element,

(ii) L has a maximal element.

Proof. $(i) \Rightarrow (ii)$ Suppose that $H_S = L$ implies $\widehat{S}$ has the largest element. Let l be the largest element in $\widehat{S}$. Let $x \in L = H_S$. By Theorem 6(ii), there exists $\widehat{x} \in \widehat{S}$ such that $\widehat{x} \wedge x = x$ and $x \wedge \widehat{x} = x$. Now, $l \wedge x = l \wedge (\widehat{x} \wedge x) = (l \wedge \widehat{x}) \wedge x = \widehat{x} \wedge x = x$. Therefore, l has a maximal element of L.

$(ii) \Rightarrow (i)$ Suppose that L has a maximal element, say m. If $H_S = L$, for some $S \subseteq L$ and $S \neq \emptyset$, then $m \in H_S$. By Theorem 6(ii), there exists $\widehat{m} \in \widehat{S}$ such that $\widehat{m} \wedge m = m$ and $m \wedge \widehat{m} = \widehat{m}$. Now, for any $\widehat{s} \in \widehat{S}$,

$$
\begin{aligned}
\widehat{s} \wedge \widehat{m} &= \widehat{m} \wedge \widehat{s} \quad \text{(since } \widehat{S} \text{ is compatible)} \\
&= (\widehat{m \wedge s}) \\
&= \widehat{s}.
\end{aligned}
$$

Therefore, $\widehat{s} \leq \widehat{m}$. Hence, $\widehat{S}$ has the largest element $\widehat{m}$.

4. Conclusions and Future Work

We identify a distributive lattice structure in an almost distributive lattice that is not induced with respect to the operations in the almost distributive lattice. It helps us to discuss (extend) several properties of a distributive lattice via hierarchy elements in an almost distributive lattice.

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References

[1] G. Birkhoff. *Lattice theory.* American Mathematical Society Colloquium Publications XXV, Providence, U.S.A., 1967.

[2] R. Noorbhasha, R. Bandaru, and A. Iampan. σ-prime spectrum of almost distributive lattices. *Eur. J. Pure Appl. Math.*, 17(2):1094–1112, 2024.

[3] Y.S. Pawar and I.A. Shaikh. On prime, minimal prime and annihilator ideals in an almost distributive lattice. *Eur. J. Pure Appl. Math.*, 6(1):107–118, 2013.

[4] G.C. Rao and S. Ravi Kumar. Minimal prime ideals in almost distributive lattices. *Int. J. Contemp. Math. Sci.*, 4(9-12):475–484, 2009.

[5] G.C. Rao, N. Rafi, and R. Bandaru. s-linear almost distributive lattices. *Eur. J. Pure Appl. Math.*, 3(4):704–716, 2010.

[6] V.V.V.S.S.P.S. Srikanth, S. Ramesh, M.V. Ratnamani, R. Bandaru, and A. Iampan. Associative types in a semi-brouwerian almost distributive lattice with respect to the binary operation ρ. *Int. J. Anal. Appl.*, 22:Article no. 13, 2024.

[7] M.H. Stone. The theory of representation for boolean algebras. *Trans. Am. Math. Soc.*, 40:37–111, 1936.

[8] U.M. Swamy and G.C. Rao. Almost distributive lattices. *J. Aust. Math. Soc., Ser. A*, 31:77–91, 1981.

[9] R. Vasu Babu and B. Venkateswarallu. Initial and final segments in almost distributive lattices. *Southeast Asian Bull. Math.*, 41:127–131, 2017.