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LOW REGULARITY OF NON-$L^2(\mathbb{R}^n)$ LOCAL SOLUTIONS TO GMHD-α SYSTEMS

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Abstract. The Magneto-Hydrodynamic (MHD) system of equations governs viscous fluids subject to a magnetic field and is derived via a coupling of the Navier-Stokes equations and Maxwell's equations. Recently it has become common to study generalizations of fluids-based differential equations. Here we consider the generalized Magneto-Hydrodynamic alpha (gMHD-α) system, which differs from the original MHD system by including an additional non-linear terms (indexed by α), and replacing the Laplace operators by more general Fourier multipliers with symbols of the form $-|\xi|^\gamma/g(|\xi|)$. In [8], the problem was considered with initial data in the Sobolev space $H^{s,2}(\mathbb{R}^n)$ with $n \geq 3$. Here we consider the problem with initial data in $H^{s,p}(\mathbb{R}^n)$ with $n \geq 3$ and $p > 2$. Our goal is to minimizing the regularity required for obtaining uniqueness of a solution.

1. Introduction

This article concerns the generalized Magneto-Hydrodynamic alpha (gMHD-α) system of equations, reported in its full generality as,

$$\partial_t v + (u \cdot \nabla) v + \sum_{i=1}^{n} v_i \nabla u_i - \nu_1 \mathcal{L}_1 v + \frac{1}{2} \nabla |B|^2 = -\nabla p + (B \cdot \nabla) B, \tag{1.1}$$

$$\partial_t B + (u \cdot \nabla) B - (B \cdot \nabla) u - \nu_2 \mathcal{L}_2 B = 0, \tag{1.2}$$

$$v = (1 - \alpha^2 \mathcal{L}_3) u, \tag{1.3}$$

$$\operatorname{div} u = \operatorname{div} B = 0, \tag{1.4}$$

$$u(0,x) = u_0(x), \quad B(0,x) = B_0(x), \quad x \in \mathbb{R}^n. \tag{1.5}$$

Since these equations govern the motion of fluids subject to a magnetic field, its terms have specific physical meaning: u the fluid velocity, B the magnetic field, and p the scalar-valued pressure of the fluid, $\nu_1 > 0$ is the fluid viscosity, $\nu_2 > 0$ the magnetic diffusion, and $\alpha > 0$ a constant coming from varying the Hamiltonian that originally gave rise to the standard MHD equations (see [4]). Finally, the $\mathcal{L}_i$ terms are Fourier multipliers with symbol $-|\xi|^{\gamma_i}/g_i(|\xi|)$, where g_i is a positive scalar function and $\gamma_i > 0$.

The standard MHD system is the special case obtained when setting $\alpha = 0$, $g_1 = g_2 = 1$, and $\gamma_1 = \gamma_2 = 2$, so that $v = u$ and $\mathcal{L}_1 = \mathcal{L}_2 = \Delta$. The existence of

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a global solution up to initial conditions is a classic result, at least in two dimensions. Unfortunately, the presence of nonlinear terms makes the MHD equations particularly complex to solve in arbitrary dimensions, and so a common strategy has been to study modified versions of them.

One modification, termed Lagrangian Averaged MHD-α after the Lagrangian Averaged Navier-Stokes equation, is obtained from Equations (1.1)-(1.5) by setting $\mathcal{L}_i = \Delta$ for $i = 1, 2, 3$ ($\gamma_i = 2$ and $g_i = 1$). Linshiz and Titi proved the existence of a global solution for smooth initial data in three dimension [4]. Another version is obtained by setting $\alpha = 0$ and $g_1 = g_2 = 1$ and leaving the γ_i's unspecified. Zhao and Zhu used these generalized operators to guarantee a global solution to Equations (1.1)-(1.5) in the case of $n = 3$, provided that $g_1 = g_2 = g_3 = 1$, $\gamma_1 = \gamma_2 = n/2$, and $\gamma_3 = 2$ [14].

The first incorporation of a non-constant value for any g_i appeared in [9], where Tao proved the existence of a unique global solution to the generalized Navier-Stokes equation ($B_0 = \alpha = 0$) when $\gamma_1 = n/2 + 1$ and when g_1 is a radial non-decreasing function bounded below satisfying

$$\int_1^\infty \frac{ds}{s g_1(s)^4} = \infty, \tag{1.6}$$

the prototypical example of which is essentially a logarithm.

Wu obtained a similar result for the generalized MHD system in [12], specifically showing that there is a unique global solution provided that $u_0, B_0 \in H^{r,2}(\mathbb{R}^n)$ with $r > n/2 + 1$; $\gamma_1 \geq n/2 + 1$, $\gamma_2 > 0$, and $\gamma_1 + \gamma_2 \geq 1$; and g_1, g_2 are non-decreasing, bounded below by 1, and satisfy

$$\int_1^\infty \frac{ds}{s(g_1(s) + g_2(s))^2} = \infty. \tag{1.7}$$

This work was ultimately extended to the gMHD-α system in [13], where Yamazaki obtained a unique global solution in three dimensions provided that $\gamma_1+\gamma_2+\gamma_3 \geq 5$, $\min\{\gamma_1, \gamma_3\} > \gamma_2 > 0$, $\gamma_3 + 2\gamma_1 > 3$, and the g_i satisfy

$$\int_1^\infty \frac{ds}{s g_1(s)^2 g_2(s) g_3(s)^2} = \infty. \tag{1.8}$$

In [8], one of the authors considered a generalization of the equations in [14] with the incorporation of non-constant g_i, $i = 1, 2, 3$, while still leaving $\mathcal{L}_3 = \Delta$, and guaranteed a unique global solution.

In this article we will extend those results for the case of $\gamma_3 \neq 2$ and non-constant g_3. We will particularly focus on the case of low-regularity initial data in a non-$L^2(\mathbb{R}^n)$ setting to then obtain, in the future, global $L^p(\mathbb{R}^n)$ solutions using an interpolation technique, the details of which can be found in [5] and [1].

The rest of this article is organized as follows. Section 2 is devoted to explaining the notation we will use and some supporting results necessary for the algorithm. Section 3 contains the main result (Theorem 3.1) of this paper and its proof. We end this section with two important spacial cases of Theorem 3.1.

Theorem 1.1. *Let $g_1, g_2, g_3 : [0, \infty) \to \mathbb{R}$ be non-decreasing functions bounded below by 1, satisfying*

$$g_i^{(k)}(s) \leq C s^{-k} \tag{1.9}$$

for $i = 1, 2, 3$ and $0 \leq k \leq n/2 + 1$. Moreover, assume $0 \leq \gamma_3 \leq 1$ and $p, q \geq n$ with $2p > q$. Then, for any divergence-free $u_0 \in L^p(\mathbb{R}^n)$ and $B_0 \in L^q(\mathbb{R}^n)$, there *exists a unique local solution* (u, B) *to the generalized MHD-α system* (1.1)-(1.5) *provided that*

$$\gamma_1^- > 6 - \gamma_3,$$
$$\gamma_2^- > 1 + \frac{n}{p}.$$

Condition (1.9) is a modification of the condition in the Mikhlin multiplier theorem that is necessary for supporting estimates in Proposition (2.4) and (2.5). The functions that satisfy it are still essentially logarithms, the same type of functions that satisfy (1.6)-(1.8).

Theorem 1.2. *Let* $\gamma_3^- - 1 \leq \frac{n}{2p} \leq \gamma_3^-$, $\frac{n}{2q} - 1 + \gamma_3^- \leq \frac{n}{2p}$, *and let* $p, q \geq n$ *with* $q < 3p/2$. *Moreover, assume that* g_1, g_2, g_3 *satisfy the inequality* (1.9). *Then for each divergence-free* $u_0 \in H^{n/2p,p}(\mathbb{R}^n)$ *and* $B_0 \in H^{n/2q,q}(\mathbb{R}^n)$, *there exists a unique local solution* (u, B) *to the generalized MHD system from Equations* (1.1)-(1.5) *provided that*

$$\gamma_1^- > 6 - \gamma_3^- - \frac{n}{p},$$
$$\gamma_2^- > 1 + \frac{n}{2p}.$$

Note that in the statement of Theorem 1.1 and 1.2, and in what follows, we use $x^- = x - \varepsilon$ for some positive ε, i.e. x^- denotes a number arbitrarily close to, but strictly smaller than, x.

2. Notation and supporting facts

We let $H^{r,p}(\mathbb{R}^n)$ be the usual Sobolev space, and we write $\|f\|_{r,p}$ to mean $\|f\|_{H^{r,p}(\mathbb{R}^n)}$ and $\|f\|_p$ for $\|f\|_{L^p(\mathbb{R}^n)}$. Because of the nature of the procedure we use, we require that the solutions live in an auxiliary continuous-in-time space $C^T_{a;r,p}(\mathbb{R}^n)$ defined by

$$C^T_{a;r,p}(\mathbb{R}^n) := \big\{f \in C((0,T), H^{r,p}(\mathbb{R}^n)) : \|f\|_{a;r,p} < \infty\big\},$$

where $T > 0$, $a \geq 0$, $C(X, Y)$ is the space of continuous maps $X \to Y$, and

$$\|f\|_{a;r,p} := \sup_{(0,T)} t^a \|f(t)\|_{r,p}.$$

Finally, we denote by $\dot{C}^T_{a;r,p}(\mathbb{R}^n)$ the subspace of $C^T_{a;r,p}(\mathbb{R}^n)$ consisting of functions f such that $\lim_{t\to 0^+} t^a f(t) = 0$ and by $BC(X, Y) \subset C(X, Y)$ the subspace of bounded continuous maps $X \to Y$.

The following are supporting propositions that we will use throughout this paper. The first proposition is a product estimate, its proof can be found in [10, Chapter 2].

Proposition 2.1. *If* $r \geq 0$ *and* $1 < p \leq \infty$, *then*

$$\|fg\|_{r,p} \leq C\big(\|f\|_{p_1}\|g\|_{r,p_2} + \|f\|_{r,q_1}\|g\|_{q_2}\big),$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$$

and $p_1, p_2, q_1, q_2 \in [1, \infty]$.

The following is a useful Sobolev embedding which is a straightforward extension of a result from [11, Chapter 13].

Proposition 2.2. *Let $s \geq r$ and $(s-r)p < n$. Then*
$$\|f\|_{r,q} \leq C\|f\|_{s,p}$$
provided that
$$\frac{1}{q} - \frac{r}{n} = \frac{1}{p} - \frac{s}{n}.$$

Our next result follows from a simple calculus exercise.

Proposition 2.3. *If $0 < a, b \in \mathbb{R}$, then*
$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-a} s^{-b}\, ds \leq CT^{1-a-b},$$
provided that $a + b < 1$.

Our final two propositions consist of an estimate for the semigroup $e^{t\mathcal{L}_i}$ analogous to similar results for the heat kernel $e^{t\Delta}$ and an estimate for the operator $(1-\mathcal{L}_i)^{-1}$. The proofs of both propositions can be found in [7]. We recall that x^- is a number arbitrarily close to, but strictly smaller than, x.

Proposition 2.4. *Let $1 < p_1 \leq p_2 < \infty$, $r_1 \leq r_2$, $g(x)$ be a non-decreasing function bounded below by 1, satisfying $|g^{(k)}(x)| \leq C|x|^{-k}$ for $1 \leq k \leq n/2 + 1$. Then $e^{t\mathcal{L}_i} : H^{r_1,p_1}(\mathbb{R}^n) \to H^{r_2,p_2}(\mathbb{R}^n)$ and*
$$\|e^{t\mathcal{L}_i} f\|_{r_2,p_2} \leq t^{-(r_2-r_1+n/p_1-n/p_2)/\gamma_i^-} \|f\|_{r_1,p_1}.$$

Note that this proposition necessitates the requirements on the g_i's.

Proposition 2.5. *Let $1 < p < \infty$, $r \in \mathbb{R}$, $g(x)$ be a non-decreasing function bounded below by 1, satisfying $|g^{(k)}(x)| \leq C|x|^{-k}$ for all $1 \leq k \leq n/2 + 1$. Then*
$$\|(1-\mathcal{L}_i)^{-1} f\|_{r,p} \leq C\|f\|_{r-\gamma_i^-,p}.$$

3. Main result and its proof

In this section we state the most general form of the theorem and then proceed with its proof.

Theorem 3.1. *Let $g_1, g_2, g_3 : [0,\infty) \to \mathbb{R}$ be non-decreasing functions bounded below by 1, satisfying*
$$g_i^{(k)}(s) \leq Cs^{-k}$$
for $i = 1, 2, 3$ and $0 \leq k \leq n/2 + 1$. Let $r_0, r_1, r_2 \geq 0$ and let $p_0, p_1, p_2 \geq n$ with $p_0 \leq p_1$ and $p_2 < 2p_0$. Moreover, assume that
$$\gamma_3^- - 1 \leq r_0 \leq \gamma_3^- \leq r_1,$$
$$r_2 - 1 + \gamma_3^- \leq r_0,$$
$$r_2 \leq r_0 < \frac{n}{p_1},$$
$$2r_1 \geq \max\Big\{2, 1 + \gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1}\Big\},$$
$$r_2 < \min\Big\{\frac{n}{p_2}, \frac{2n}{p_2} - \frac{n}{p_0}\Big\}.$$

Then, for any divergence-free $u_0 \in H^{r_0,p_0}(\mathbb{R}^n)$ *and* $B_0 \in H^{r_2,p_2}(\mathbb{R}^n)$*, there exists a unique local solution* (u, B) *to the generalized MHD-*α *system* (1.1)-(1.5) *provided that*

$$\begin{gathered}\gamma_1^- > 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1},\\ \gamma_1^- > 1 - 2r_2 + r_1 - \gamma_3^- - \frac{n}{p_1} + \frac{2n}{p_2},\\ \gamma_2^- > 1 - r_0 + \frac{n}{p_0}.\end{gathered}$$

Note that Theorem 1.1 may be recovered by setting $r_0 = r_2 = 0$, $r_1 = 2$, $p := p_0 = p_1$, and $q := p_2$, while Theorem 1.2 may be recovered by setting $r_0 = n/(2p)$, $r_1 = 2$, $r_2 = n/(2q)$, $p := p_0 = p_1$, and $q := p_2$.

Proof of Theorem 3.1. For the sake of clarity and to highlight some technical details, the proof will be divided in subsections. We first write the generalized MHD-α system in a more helpful form. Without loss of generality, we set $\alpha = \nu_1 = \nu_2 = 1$. We pass to divergence-free vector fields by applying the Hodge operator P to equations (1.1) and (1.2) (more information about the Hodge operator can be found in [3, Chapter 11]), and then we apply $(1 - \mathcal{L}_3)^{-1}$ to equation (1.1). By noting that P, $(1 - \mathcal{L}_3)^{-1}$, and ∂_t all commute since they are Fourier multipliers, we obtain

$$\begin{aligned}&\partial_t u + P(1 - \mathcal{L}_3)^{-1}\Big((u \cdot \nabla)v + \sum_{i=1}^n v_i \nabla u_i - (B \cdot \nabla)B\Big) - \mathcal{L}_1 u\\ &= P\Big(-\nabla p - \frac{1}{2}\nabla |B|^2\Big) = 0.\end{aligned}$$

An application of the divergence-free condition in (1.4) allows us to rewrite the terms of the form $(x \cdot \nabla)y$ as $\operatorname{div}(x \otimes y)$. Note that $x \otimes y$ is the matrix whose (i, j) entry is $x_i y_j$, so that the product estimate in Proposition 2.1 applies to $x \otimes y$. We then have the system

$$\begin{gathered}\partial_t u + P(1 - \mathcal{L}_3)^{-1}\Big(\operatorname{div}(u \otimes v) + \sum_{i=1}^n v_i \nabla u_i - \operatorname{div}(B \otimes B)\Big) - \mathcal{L}_1 u = 0,\\ \partial_t B + P\Big(\operatorname{div}(u \otimes B) - \operatorname{div}(B \otimes u)\Big) - \mathcal{L}_2 B = 0,\\ v = (1 - \mathcal{L}_3)u,\\ \operatorname{div} u = \operatorname{div} B = 0,\\ u(x, 0) = u_0(x), \quad B(0, x) = B_0(x), \quad x \in \mathbb{R}^n.\end{gathered}$$

An application of Duhamel's principle shows that (u, B) is a solution to the system if and only if (u, B) is a fixed point of the map $\Phi(u, B) := (\Phi_1(u, B), \Phi_2(u, B))$ defined by

$$\begin{aligned}\Phi_1(u, B) &:= e^{t\mathcal{L}_1} u_0 - \int_0^t e^{(t-s)\mathcal{L}_1}\left(W_1(u, v) + W_2(u, v) - W_1(B, B)\right) ds,\\ \Phi_2(u, B) &:= e^{t\mathcal{L}_2} B_0 - \int_0^t e^{(t-s)\mathcal{L}_2}\left(W_3(u, B) - W_3(B, u)\right) ds,\end{aligned}$$

where

$$W_1(x, y) = P(1 - \mathcal{L}_3)^{-1} \operatorname{div}(x \otimes y),$$

$$
\begin{aligned}
W_2(x,y) &= P(1-\mathcal{L}_3)^{-1}\Big(\sum_{i=1}^{n} y_i \nabla x_i\Big),\\
W_3(x,y) &= P\operatorname{div}(x\otimes y).
\end{aligned}
$$

By the contraction mapping theorem, it suffices to show that Φ is a contraction on the space $X_{T,M}\times Y_{T,M}$, where

$$
\begin{aligned}
X_{T,M} := \Big\{ & f\in BC\big([0,T),H^{r_0,p_0}(\mathbb{R}^n)\big)\cap \dot{C}_{a_1,r_1,p_1}(\mathbb{R}^n):\\
& \sup_{(0,T)}\big(\|f(t)-e^{t\mathcal{L}_1}u_0\|_{r_0,p_0}+\|f(t)\|_{a_1;r_1,p_1}\big)<M\Big\}
\end{aligned}
$$

and

$$
Y_{T,M} := \Big\{ f\in BC([0,T),H^{r_2,p_2}(\mathbb{R}^n)) : \sup_{(0,T)}\|f(t)-e^{t\mathcal{L}_2}B_0\|_{r_2,p_2}<M\Big\}
$$

for some $0<T<1$ and $M>0$.

Following the methods in [6, 2], we complete the proof by showing that

$$
\begin{aligned}
I_1 &= \sup_{(0,T)} t^{a_1}\|e^{t\mathcal{L}_1}u_0\|_{r_1,p_1}<M/4,\\
I_2 &= \sup_{(0,T)}\Big\|\int_0^t e^{(t-s)\mathcal{L}_1}\left(W_1(u,v)+W_2(u,v)-W_1(B,B)\right)ds\Big\|_{r_0,p_0}<M/4,\\
I_3 &= \sup_{(0,T)} t^{a_1}\Big\|\int_0^t e^{(t-s)\mathcal{L}_1}\left(W_1(u,v)+W_2(u,v)-W_1(B,B)\right)ds\Big\|_{r_1,p_1}<M/4,\\
I_4 &= \sup_{(0,T)}\Big\|\int_0^t e^{(t-s)\mathcal{L}_2}\left(W_3(u,B)-W_3(B,u)\right)ds\Big\|_{r_2,p_2}<M/4.
\end{aligned}
$$

We start with I_1. If φ is in the Schwartz space, we have

$$
\begin{aligned}
I_1 &= \sup_{(0,T)} t^{a_1}\|e^{t\mathcal{L}_1}(u_0-\varphi+\varphi)\|_{r_1,p_1}\\
&\le \sup_{(0,T)} t^{a_1}\|e^{t\mathcal{L}_1}(u_0-\varphi)\|_{r_1,p_1}+\sup_{(0,T)} t^{a_1}\|e^{t\mathcal{L}_1}\varphi\|_{r_1,p_1}\\
&\le \sup_{(0,T)} t^{a_1}t^{-a_1}\|u_0-\varphi\|_{r_0,p_0}+\sup_{(0,T)} t^{a_1}\|\varphi\|_{r_1,p_1}\\
&\le \|u_0-\varphi\|_{r_0,p_0}+T^{a_1}\|\varphi\|_{r_1,p_1},
\end{aligned}
$$

provided that (by Proposition 2.4)

$$
0\le a_1=\frac{r_1-r_0+\frac{n}{p_0}-\frac{n}{p_1}}{\gamma_1^-}<1
$$

and $p_0\le p_1$. We can choose φ so that $\|u_0-\varphi\|_{r_0,p_0}$ is arbitrarily small, and then we can choose T small enough to reduce $T^{a_1}\|\varphi\|_{r_1,p_1}$ so that the sum of the two is bounded by $M/4$.

3.1. I_2 **and** I_3**.** Minkowski's inequality gives us

$$
\begin{aligned}
I_2 &\le J_1+J_2+J_3,\\
I_3 &\le K_1+K_2+K_3,
\end{aligned}
$$

where

$$\begin{aligned} J_1 &:= \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_1(u,v)\|_{r_0,p_0} ds, \\ K_1 &:= \sup_{(0,T)} t^{a_1} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_1(u,v)\|_{r_1,p_1} ds, \\ J_2 &:= \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_2(u,v)\|_{r_0,p_0} ds, \\ K_2 &:= \sup_{(0,T)} t^{a_1} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_2(u,v)\|_{r_1,p_1} ds, \\ J_3 &:= \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_1(B,B)\|_{r_0,p_0} ds, \\ K_3 &:= \sup_{(0,T)} t^{a_1} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_1(B,B)\|_{r_1,p_1} ds. \end{aligned}$$

We will show that each term is bounded above by CM^2T^k for various values of $k > 0$, which, since $T < 1$, will imply $I_2, I_3 < M/4$ provided that $CM^2 < M/4$.

We begin our algorithm with J_1 and K_1, showing the details of the calculations and highlighting the choices of parameters. The argument for the other two pairs of integrals is very similar, and the details will be omitted.

3.2. J_1 **and** K_1**.** By Proposition 2.4, J_1 is bounded by

$$J_1 \le \sup_{(0,T)} \int_0^t (t-s)^{-(r_0-(\gamma_3^- -1)+n/\pi_1 - n/p_0)/\gamma_1^-} \|W_1(u,v)\|_{\gamma_3^- -1,\pi_1} ds$$

provided that $\gamma_3^- - 1 \le r_0$ and where π_1 is an intermediate parameter that will be specified later. Now we work towards bounding $W_1(u,v)$, and an application of Proposition 2.5 gives us

$$\|W_1(u,v)\|_{\gamma_3^- -1,\pi_1} = \|P(1-\mathcal{L}_3)^{-1} \operatorname{div}(u \otimes v)\|_{\gamma_3^- -1,\pi_1} \le C\|u \otimes v\|_{\pi_1}.$$

We chose a regularity of $\gamma_3^- - 1$ when performing the semigroup estimate in order to end up with a product in zero regularity, so that we can apply Holder's inequality: by Proposition 2.1, if

$$\frac{1}{\pi_1} = \frac{1}{p'} + \frac{1}{p_1}, \tag{3.1}$$

we have

$$\|u \otimes v\|_{\pi_1} \le C\|u\|_{p'}\|v\|_{p_1} \le C\|u\|_{p'}\|u\|_{\gamma_3^-,p_1}.$$

Note that (3.1) specifies the required value of π_1. We obtain $\|u\|_{p'} \le \|u\|_{r_0,p_0}$ by Proposition 2.2 if

$$r_0 < \frac{n}{p_0} \quad \text{and} \quad \frac{1}{p'} = \frac{1}{p_0} - \frac{r_0}{n}; \tag{3.2}$$

we also get $\|u\|_{\gamma_3^-,p_1} \le \|u\|_{r_1,p_1}$ by requiring that $r_1 \ge \gamma_3^-$. Combining the two bounds gives us

$$\|W_1(u,v)\|_{\gamma_3^- -1,\pi_1} \le C\|u \otimes v\|_{\pi_1} \le C\|u\|_{r_0,p_0}\|u\|_{r_1,p_1}. \tag{3.3}$$

Note that (3.1) and (3.2) give us

$$\frac{1}{\pi_1} = \frac{1}{p_0} + \frac{1}{p_1} - \frac{r_0}{n}.$$

With this new bound on $W_1(u,v)$, we come back to J_1 and see that

$$\begin{aligned} J_1 &\le \sup_{(0,T)} \int_0^t (t-s)^{-(r_0-(\gamma_3^- -1)+n/\pi_1-n/p_0)/\gamma_1^-} \|W_1(u,v)\|_{\gamma_3^- -1,\pi_1} ds \\ &\le C \sup_{(0,T)} \int_0^t (t-s)^{-(r_0-(\gamma_3^- -1)+n/\pi_1-n/p_0)/\gamma_1^-} \|u\|_{r_0,p_0}\|u\|_{r_1,p_1} ds \\ &= C \sup_{(0,T)} \int_0^t (t-s)^{-(r_0-(\gamma_3^- -1)+n/\pi_1-n/p_0)/\gamma_1^-} s^{-a_1}\|u\|_{r_0,p_0} s^{a_1}\|u\|_{r_1,p_1} ds \\ &\le C\|u\|_{0;r_0,p_0}\|u\|_{a_1;r_1,p_1} \sup_{(0,T)} \int_0^t (t-s)^{-(r_0-(\gamma_3^- -1)+n/\pi_1-n/p_0)/\gamma_1^-} s^{-a_1} ds \\ &< CM^2 T^{1-(r_0-(\gamma_3^- -1)+n/\pi_1-n/p_0)/\gamma_1^- -a_1}, \end{aligned}$$

where the last inequality holds by Proposition 2.3, if

$$\begin{aligned} \gamma_1^- &> r_0 - (\gamma_3^- - 1) + \frac{n}{\pi_1} - \frac{n}{p_0} + \gamma_1 a_1 \\ &= r_0 - (\gamma_3^- - 1) + n\Big(\frac{1}{p_0} + \frac{1}{p_1} - \frac{r_0}{n}\Big) - \frac{n}{p_0} + r_1 - r_0 + \frac{n}{p_0} - \frac{n}{p_1} \\ &= 1 - r_0 + r_1 - \gamma_3^- + \frac{n}{p_0}. \end{aligned} \tag{3.4}$$

We further note that the requirement in (3.4) also guarantees that the exponent on T is positive, as desired.

We now turn our attention to K_1. Proposition 2.4 guarantees that, if $p_1 \ge \pi_1'$, then

$$K_1 \le \sup_{(0,T)} \int_0^t (t-s)^{-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_1)/\gamma_1^-} \|W_1(u,v)\|_{r_1-r_0+\gamma_3^- -1,\pi_1'} ds.$$

This time, we chose $r_1 - r_0 + \gamma_3^- - 1$ in order to match the previous "jump" in regularity from r_0 to $\gamma_3^- - 1$. Propositions 2.1 and 2.5 give us

$$\begin{aligned} \|W_1(u,v)\|_{r_1-r_0+\gamma_3^- -1,\pi_1'} &= \|P(1-\mathcal{L}_3)^{-1}\operatorname{div}(u\otimes v)\|_{r_1-r_0+\gamma_3^- -1,\pi_1'} \\ &\le C\|u\otimes v\|_{r_1-r_0,\pi_1'} \\ &\le C\left(\|u\|_{r_1-r_0,p'}\|v\|_{p''} + \|v\|_{r_1-r_0,q'}\|u\|_{q''}\right), \end{aligned}$$

provided that

$$\frac{1}{\pi_1'} = \frac{1}{p'} + \frac{1}{p''} = \frac{1}{q'} + \frac{1}{q''}.$$

Four applications of Proposition 2.2 lead us to the following bounds:

$$\|u\|_{r_1-r_0,p'} \le \|u\|_{r_1,p_1} \quad \text{if } r_0 < \frac{n}{p_1} \text{ and } \frac{1}{p'} = \frac{1}{p_1} - \frac{r_0}{n}, \tag{3.5}$$

$$\|v\|_{p''} \le \|v\|_{r_0-\gamma_3^-,p_0} \quad \text{if } r_0 < \frac{n}{p_0} + \gamma_3^- \text{ and } \frac{1}{p''} = \frac{1}{p_0} - \frac{r_0-\gamma_3^-}{n}, \tag{3.6}$$

$$\|v\|_{r_1-r_0,q'} \le \|v\|_{r_1-\gamma_3^-,p_1} \quad \text{if } r_0 < \frac{n}{p_0} + \gamma_3^- \text{ and } \frac{1}{q'} = \frac{1}{p_1} - \frac{r_0-\gamma_3^-}{n}, \tag{3.7}$$

$$\|u\|_{q''} \le \|u\|_{r_0,p_0} \qquad \text{if } r_0 < \frac{n}{p_0} \text{ and } \frac{1}{q''} = \frac{1}{p_0} - \frac{r_0}{n}. \tag{3.8}$$

Combining the parameters specified by (3.5)-(3.8), we obtain

$$\frac{1}{\pi_1'} = \frac{1}{p_0} + \frac{1}{p_1} - \frac{2r_0 - \gamma_3^-}{n}$$

and

$$\|W_1(u,v)\|_{r_1-r_0+\gamma_3^- -1,\pi_1'} \le C\|u\otimes v\|_{r_1-r_0,\pi_1'} \le C\|u\|_{r_0,p_0}\|u\|_{r_1,p_1}.$$

Moreover, the integrability requirement from Proposition 2.4 necessitates

$$\frac{1}{p_1} \le \frac{1}{\pi_1'} = \frac{1}{p_0} + \frac{1}{p_1} - \frac{2r_0 - \gamma_3^-}{n},$$

and so

$$r_0 \le \frac{1}{2}\Big(\frac{n}{p_0} + \gamma_3^-\Big). \tag{3.9}$$

We can finally plug this bound into K_1:

$$\begin{aligned}
K_1 &\le \sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_1)/\gamma_1^-} \\
&\quad \times \|W_1(u,v)\|_{r_1-r_0+\gamma_3^- -1,\pi_1'}\,ds \\
&\le C\sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_1)/\gamma_1^-}\|u\|_{r_0,p_0}\|u\|_{r_1,p_1}ds \\
&= C\sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_1)/\gamma_1^-} s^{-a_1} \\
&\quad \times \|u\|_{r_0,p_0} s^{a_1}\|u\|_{r_1,p_1}ds \\
&\le C\|u\|_{0;r_0,p_0}\|u\|_{a_1;r_1,p_1}\sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_1)/\gamma_1^-} \\
&\quad \times s^{-a_1}ds \\
&< CM^2 T^{1-(r_1-(r_1-r_0+\gamma_3^- -1)+n/\pi_1'-n/p_0)/\gamma_1^-},
\end{aligned}$$

where the last inequality follows by Proposition 2.3 if

$$\begin{aligned}
\gamma_1^- &> r_1 - \big(r_1 - r_0 + \gamma_3^- - 1\big) + \frac{n}{\pi_1'} - \frac{n}{p_1} + \gamma_1^- a_1 \\
&= r_0 - \gamma_3^- + 1 + n\Big(\frac{1}{p_0} + \frac{1}{p_1} - \frac{2r_0 - \gamma_3^-}{n}\Big) - \frac{n}{p_1} + r_1 - r_0 + \frac{n}{p_0} - \frac{n}{p_1} \\
&= 1 - 2r_0 + r_1 + \frac{2n}{p_0} - \frac{n}{p_1}\,.
\end{aligned} \tag{3.10}$$

Note that, once again, the requirement that Proposition 2.3 hold is sufficient to guarantee that the exponent on T be positive.

To summarize, here is the list of inequalities needed to obtain the desired bounds on J_1 and K_1:

$$\begin{aligned}
&r_0 \le \gamma_3^- && \text{(assumption)},\\
&\gamma_3^- - 1 \le r_0 && \text{(semigroup estimate for } J_1),\\
&r_1 - r_0 + \gamma_3^- - 1 \le r_1 && \text{(semigroup estimate for } K_1)
\end{aligned}$$

$$\begin{aligned}
&r_0 < \frac{n}{p_0} && \text{(by (3.2))},\\
&r_0 < \frac{n}{p_1} && \text{(by (3.5))},\\
&r_0 \le \frac{1}{2}\Big(\frac{n}{p_0} + \gamma_3^-\Big) && \text{(by (3.9))},\\
&r_0 < \frac{n}{p_0} + \gamma_3^- && \text{(by (3.12))},\\
&r_1 \ge \gamma_3^- && \text{(bound on } \|u\|_{\gamma_3^-,p_1} \text{ in } J_1),\\
&\gamma_1^- > 1 - r_0 + r_1 - \gamma_3^- + \frac{n}{p_0} && \text{(by (3.4))},\\
&\gamma_1^- > 1 - 2r_0 + r_1 + \frac{2n}{p_0} - \frac{n}{p_1} && \text{(by (3.10))}.
\end{aligned}$$

After some obvious simplifications and after noting that (3.10) implies (3.4) since

$$\begin{aligned}
&\underbrace{\Big(1 - 2r_0 + r_1 + \frac{2n}{p_0} - \frac{n}{p_1}\Big)}_{\text{RHS of (3.10)}} - \underbrace{\Big(1 - r_0 + r_1 - \gamma_3^- + \frac{n}{p_0}\Big)}_{\text{RHS of (3.4)}}\\
&= \gamma_3^- - r_0 + \frac{n}{p_0} - \frac{n}{p_1} \ge 0,
\end{aligned}$$

the list reduces to

$$\begin{gathered}
\gamma_3^- - 1 \le r_0 \le \gamma_3^- \le r_1,\\
r_0 < \frac{n}{p_1},\\
\gamma_1^- > 1 - 2r_0 + r_1 + \frac{2n}{p_0} - \frac{n}{p_1}.
\end{gathered}$$

3.3. J_2 and K_2**.** We have

$$J_2 = \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_2(u,v)\|_{r_0,p_0} ds \le \sup_{(0,T)} \int_0^t \|W_2(u,v)\|_{r_0,p_0} ds.$$

We now work towards bounding $W_2(u,v)$. We immediately see, thanks to Proposition 2.5, that

$$\begin{aligned}
\|W_2(u,v)\|_{r_0,p_0} &= \Big\| P(1-\mathcal{L}_3)^{-1} \sum_{i=1}^n v_i \nabla u_i \Big\|_{r_0,p_0}\\
&\le C \Big\| \sum_{i=1}^n v_i \nabla u_i \Big\|_{r_0 - \gamma_3^-, p_0}\\
&\le C \sum_{i=1}^n \|v_i \nabla u_i\|_{p_0}
\end{aligned}$$

since $r_0 \le \gamma_3^-$. Now the product estimate is nothing more than Holder's inequality, so if

$$\frac{1}{p_0} = \frac{1}{p'} + \frac{1}{p''}$$

we obtain

$$\begin{aligned}\|W_2(u,v)\|_{r_0,p_0} &\le C\sum_{i=1}^n \|v_i\|_{p'}\|\nabla u_i\|_{p''}\\ &\le C\sum_{i=1}^n \|v\|_{p'}\|\nabla u\|_{p''}\\ &\le C\|u\|_{\gamma_3^-,p'}\|u\|_{1,p''}\,.\end{aligned}$$

By Proposition 2.2 we have

$$\|u\|_{\gamma_3^-,p'} \le C\|u\|_{\gamma_3^-+\beta,p_1} \le C\|u\|_{r_1,p_1} \quad \text{and} \quad \|u\|_{1,p''} \le C\|u\|_{r_1,p_1}\,,$$

where the first set of inequalities requires that

$$0 \le \beta < \frac{n}{p_1}, \quad \frac{1}{p'} = \frac{1}{p_1} - \frac{\beta}{n}, \quad r_1 \ge \gamma_3^- + \beta, \tag{3.11}$$

and the second inequality requires that

$$\frac{1}{p''} = \frac{1}{p_1} - \frac{r_1-1}{n} \quad \text{and} \quad r_1 \ge 1. \tag{3.12}$$

We finally obtain

$$\|W_2(u,v)\|_{r_0,p_0} \le C\|u\|_{\gamma_3^-,p'}\|u\|_{1,p''} \le C\|u\|^2_{r_1,p_1}.$$

We pause here to note that, without the presence of the space $\dot{C}_{a_1;r_1,p_1}(\mathbb{R}^n)$ in the definition of $X_{T,M}$, we would not be able to bound this W_2 term.

Returning to J_2, we have

$$\begin{aligned}J_2 &\le \sup_{(0,T)}\int_0^t \|W_2(u,v)\|_{r_0,\pi_2}ds\\ &\le C\sup_{(0,T)}\int_0^t \|u\|^2_{r_1,p_1}ds\\ &= C\sup_{(0,T)}\int_0^t s^{-2a_1}s^{a_1}\|u\|_{r_1,p_1}s^{a_1}\|u\|_{r_1,p_1}ds\\ &\le C\|u\|^2_{a_1;r_1,p_1}\sup_{(0,T)}\int_0^t s^{-2a_1}ds\\ &< CM^2T^{1-2a_1},\end{aligned}$$

provided $2a_1 > 1$, which is equivalent to

$$\gamma_1^- > 2r_1 - 2r_0 + \frac{2n}{p_0} - \frac{2n}{p_1} \tag{3.13}$$

and we recall that

$$\frac{n}{p_0} = \frac{n}{p'} + \frac{n}{p''} = \frac{2n}{p_1} - \beta - r_1 + 1.$$

We choose β to be exactly

$$\beta = 1 - r_1 - \frac{n}{p_0} + \frac{2n}{p_1}, \tag{3.14}$$

and so the two requirements in (3.11) become

$$r_1 \ge 1 - \frac{n}{p_0} + \frac{n}{p_1} \quad \text{and} \quad 2r_1 \ge 1 + \gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1}\,. \tag{3.15}$$

Turning to K_2, noting that we go down to γ_3^- instead of r_0, we have

$$\begin{aligned}
K_2 &= \sup_{(0,T)} t^{a_1} \int_0^t \|e^{(t-s)\mathcal{L}_1} W_2(u,v)\|_{r_1,p_1} ds \\
&\le \sup_{(0,T)} t^{a_1} \int_0^t (t-s)^{-(r_1-\gamma_3^-+n/p_0-n/p_1)/\gamma_1^-} \|W_2(u,v)\|_{\gamma_3^-,p_0} ds \\
&\le C \sup_{(0,T)} t^{a_1} \int_0^t (t-s)^{-(r_1-\gamma_3^-+n/p_0-n/p_1)/\gamma_1^-} \|u\|_{r_1,p_1} \|u\|_{r_1,p_1} ds \\
&= C \sup_{(0,T)} t^{a_1} \int_0^t (t-s)^{-(r_1-\gamma_3^-+n/p_0-n/p_1)/\gamma_1^-} s^{-2a_1} s^{a_1} \|u\|_{r_1,p_1} s^{a_1} \|u\|_{r_1,p_1} ds \\
&\le C\|u\|^2_{a_1;r_0,p_0} \sup_{(0,T)} t^{a_1} \int_0^t (t-s)^{-(r_1-\gamma_3^-+n/p_0-n/p_1)/\gamma_1^-} s^{-2a_1} ds \\
&< CM^2 T^{1-(r_1-\gamma_3^-+n/\pi_2-n/p_1)/\gamma_1^- - a_1},
\end{aligned}$$

where, by Proposition 2.3, the last inequality holds if

$$\begin{aligned}
\gamma_1^- &> r_1 - \gamma_3^- + \frac{n}{p_0} - \frac{n}{p_1} + 2\gamma_1^- a_1 \\
&= 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1}.
\end{aligned} \tag{3.16}$$

Here is a summary of the inequalities needed to obtain the required bounds on J_2 and K_2:

$$\begin{aligned}
& r_1 \ge 1 && \text{(by (3.12))}, \\
& r_1 \ge 1 - \frac{n}{p_0} + \frac{n}{p_1} && \text{(by(3.15))}, \\
& 2r_1 \ge 1 + \gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1} && \text{(by (3.15))}, \\
& \gamma_1^- > 2r_1 - 2r_0 + \frac{2n}{p_0} - \frac{2n}{p_1} && \text{(by (3.13))}, \\
& \gamma_1^- > 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1} && \text{(by (3.16))}.
\end{aligned}$$

By noting that

$$\begin{aligned}
&\underbrace{\Big(3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1}\Big)}_{\text{RHS of (3.16)}} - \underbrace{\Big(2r_1 - 2r_0 + \frac{2n}{p_0} - \frac{2n}{p_1}\Big)}_{\text{RHS of (3.13)}} \\
&= r_1 - \gamma_3^- + \frac{n}{p_0} - \frac{n}{p_1} \ge 0
\end{aligned}$$

we conclude that (3.16) implies (3.13), and so the list reduces to

$$\begin{gathered}
2r_1 \ge \max\Big\{2, 1 + \gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1}\Big\}, \\
\gamma_1^- > 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1}.
\end{gathered}$$

3.4. J_3 and K_3. Provided that $r_2-1+\gamma_3^- \le r_0$ and $\frac{1}{\pi_3}\ge\frac{1}{p_0}$, Proposition 2.4 gives us

$$\begin{aligned}
J_3 &= \sup_{(0,T)}\int_0^t \|e^{(t-s)\mathcal{L}_1}W_1(B,B)\|_{r_0,p_0}ds\\
&\le \sup_{(0,T)}\int_0^t (t-s)^{-(r_0-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_0)/\gamma_1^-}\|W_1(B,B)\|_{r_2-1+\gamma_3^-,\pi_3}ds.
\end{aligned}$$

Once again, applying Propositions 2.1 and 2.5 gives us

$$\begin{aligned}
\|W_1(B,B)\|_{r_2-1+\gamma_3^-,\pi_3} &= \|P(1-\mathcal{L}_3)^{-1}\operatorname{div}(B\otimes B)\|_{r_2-1+\gamma_3^-,\pi_3}\\
&\le C\|B\otimes B\|_{r_2,\pi_3}\\
&\le C\|B\|_{r_2,p_2}\|B\|_{p'},
\end{aligned}$$

where the product estimate requires that $r_2\ge 0$ and

$$\frac{1}{\pi_3}=\frac{1}{p_2}+\frac{1}{p'}.$$

Provided that

$$r_2<\frac{n}{p_2}\quad\text{and}\quad \frac{1}{p'}=\frac{1}{p_2}-\frac{r_2}{n}, \tag{3.17}$$

which combines with the previous equation to give

$$\frac{1}{\pi_3}=\frac{2}{p_2}-\frac{r_2}{n},$$

we can bound $\|B\|_{p'}$ by $\|B\|_{r_2,p_2}$ thanks to Proposition 2.2. Thus,

$$\|W_1(B,B)\|_{r_2-1+\gamma_3^-,\pi_3}\le C\|B\|_{r_2,p_2}^2.$$

Moreover, Proposition (2.4) requires that

$$\frac{1}{p_0}\le\frac{1}{\pi_3}=\frac{2}{p_2}-\frac{r_2}{n},$$

which can be restated as

$$r_2\le\frac{2n}{p_2}-\frac{n}{p_0}. \tag{3.18}$$

Plugging the bound for $W_1(B,B)$ back into the integral gives us

$$\begin{aligned}
J_3 &\le \sup_{(0,T)}\int_0^t (t-s)^{-(r_0-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_0)/\gamma_1^-}\|W_1(B,B)\|_{r_2-1+\gamma_3^-,\pi_3}ds\\
&\le C\sup_{(0,T)}\int_0^t (t-s)^{-(r_0-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_0)/\gamma_1^-}\|B\|_{r_2,p_2}^2ds\\
&\le C\|B\|_{0;r_2,p_2}^2\sup_{(0,T)}\int_0^t (t-s)^{-(r_0-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_0)/\gamma_1^-}ds\\
&< CM^2T^{1-(r_0-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_0)/\gamma_1^-},
\end{aligned}$$

where once again the last inequality holds if

$$\begin{aligned}
\gamma_1^- &> r_0-(r_2-1+\gamma_3^-)+\frac{n}{\pi_3}-\frac{n}{p_0}\\
&= r_0-2r_2-\gamma_3^-+1+\frac{2n}{p_2}-\frac{n}{p_0}.
\end{aligned}\tag{3.19}$$

The same bounds for $W_1(B,B)$ work in the case of K_3, so that

$$\begin{aligned}
K_3 &= \sup_{(0,T)} t^{a_1}\int_0^t \|e^{(t-s)\mathcal{L}_1}W_1(B,B)\|_{r_1,p_1}ds\\
&\le \sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_1)/\gamma_1^-}\|W_1(B,B)\|_{r_2-1+\gamma_3^-,\pi_3}ds\\
&\le C\sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_1)/\gamma_1^-}\|B\|_{r_2,p_2}^2 ds\\
&\le C\|B\|_{0;r_2,p_2}^2\sup_{(0,T)} t^{a_1}\int_0^t (t-s)^{-(r_1-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_1)/\gamma_1^-}ds\\
&< CM^2T^{1-(r_1-(r_2-1+\gamma_3^-)+n/\pi_3-n/p_1)/\gamma_1^-+a_1},
\end{aligned}$$

which holds provided that

$$\begin{aligned}
\gamma_1^- &> r_1-(r_2-1+\gamma_3^-)+\frac{n}{\pi_3}-\frac{n}{p_1}\\
&= r_1-2r_2-\gamma_3^-+1-\frac{n}{p_1}+\frac{2n}{p_2}.
\end{aligned} \tag{3.20}$$

What follows is the list of inequalities needed to bound J_3 and K_3 as desired:

$$\begin{aligned}
&r_2-1+\gamma_3^-\le r_0 && \text{(semigroup estimate for } J_3),\\
&r_2-1+\gamma_3^-\le r_1 && \text{(semigroup estimate for } K_3),\\
&r_2\ge 0 && \text{(product estimate)},\\
&r_2<\min\Big\{\frac{n}{p_2},\frac{2n}{p_2}-\frac{n}{p_0}\Big\} && \text{(by (3.17)-(3.18))},\\
&\gamma_1^- > r_0-2r_2-\gamma_3^-+1-\frac{n}{p_0}+\frac{2n}{p_2} && \text{(by (3.19))},\\
&\gamma_1^- > r_1-2r_2-\gamma_3^-+1+\frac{2n}{p_2}-\frac{n}{p_1} && \text{(by (3.20))}.
\end{aligned}$$

We see that (3.20) suffices for (3.19) since

$$\begin{aligned}
&\underbrace{\Big(r_1-2r_2-\gamma_3^-+1+\frac{2n}{p_2}-\frac{n}{p_1}\Big)}_{\text{RHS of (3.20)}}-\underbrace{\Big(r_0-2r_2-\gamma_3^-+1-\frac{n}{p_0}+\frac{2n}{p_2}\Big)}_{\text{RHS of (3.19)}}\\
&= r_1-r_0+\frac{n}{p_0}-\frac{n}{p_1}\ge 0,
\end{aligned}$$

and so the list reduces to

$$\begin{gathered}
r_2-1+\gamma_3^-\le r_0,\\
0\le r_2<\min\Big\{\frac{n}{p_2},\frac{2n}{p_2}-\frac{n}{p_0}\Big\},\\
\gamma_1^- > r_1-2r_2-\gamma_3^-+1+\frac{2n}{p_2}-\frac{n}{p_1}.
\end{gathered}$$

3.5. **Bounding I_4.** Applying Minkowski's inequality to I_4 gives

$$I_4\le L_1+L_2,$$

where

$$L_1 := \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_2} W_3(u,B)\|_{r_2,p_2} ds$$
$$L_2 := \sup_{(0,T)} \int_0^t \|e^{(t-s)\mathcal{L}_2} W_3(B,u)\|_{r_2,p_2} ds$$

We can immediately note that, since W_3 is not symmetric, $L_1 \neq L_2$, but our techniques will give the same bound for each. So, we set $L := L_1$ and proceed to bound only L_1. Proposition 2.4 gives us

$$L \le \sup_{(0,T)} \int_0^t (t-s)^{-(r_2-(r_2-1)+n/\pi_4-n/p_2)/\gamma_2^-} \|W_3(u,B)\|_{r_2-1,\pi_4} ds,$$

provided that $1/\pi_4 \ge 1/p_2$.

Continuing with $W_3(u,B)$, we obtain

$$\|W_3(u,B)\|_{r_2-1,\pi_4} = \|P\operatorname{div}(u\otimes B)\|_{r_2-1,\pi_4} \le C\|u\otimes B\|_{r_2,\pi_4};$$

an application of Proposition 2.1 gives us

$$\|u\otimes B\|_{r_2,\pi_4} \le C\Big(\|u\|_{r_2,p'}\|B\|_{p''} + \|B\|_{r_2,p_2}\|u\|_{q''}\Big)$$

as long as

$$\frac{1}{\pi_4} = \frac{1}{p'} + \frac{1}{p''} = \frac{1}{p_2} + \frac{1}{q''}.$$

We want to bound $\|u\otimes B\|_{r_2,\pi_4}$ by $\|u\|_{r_0,p_0}\|B\|_{r_2,p_2}$, which requires three applications of Proposition 2.2. First, we obtain $\|u\|_{r_2,p'} \le \|u\|_{r_0,p_0}$ if

$$0 \le r_0 - r_2 < \frac{n}{p_0} \quad \text{and} \quad \frac{1}{p'} - \frac{r_2}{n} = \frac{1}{p_0} - \frac{r_0}{n}. \tag{3.21}$$

We further obtain $\|u\|_{q''} \le \|u\|_{r_0,p_0}$ provided that

$$r_0 < \frac{n}{p_0} \quad \text{and} \quad \frac{1}{q''} = \frac{1}{p_0} - \frac{r_0}{n}. \tag{3.22}$$

The last embedding, $\|B\|_{p''} \le \|B\|_{r_2,p_2}$, requires

$$r_2 < \frac{n}{p_2} \quad \text{and} \quad \frac{1}{p''} = \frac{1}{p_2} - \frac{r_2}{n}. \tag{3.23}$$

Combining Equations (3.21)-(3.23) gives us

$$\frac{1}{\pi_4} = \frac{1}{p_0} + \frac{1}{p_2} - \frac{r_0}{n},$$

which is required to satisfy

$$\frac{1}{p_2} \le \frac{1}{\pi_4} = \frac{1}{p_0} + \frac{1}{p_2} - \frac{r_0}{n} \implies r_0 \le \frac{n}{p_0}. \tag{3.24}$$

This is the bound we were looking for:

$$\|W_3(u,B)\|_{r_2-1,\pi_4} \le C\Big(\|u\|_{r_2,p'}\|B\|_{p''} + \|B\|_{r_2,q'}\|u\|_{q''}\Big) \le C\|u\|_{r_0,p_0}\|B\|_{r_2,p_2}.$$

We can plug the above into L and obtain

$$L \le \sup_{(0,T)} \int_0^t (t-s)^{-(1+n/\pi_4-n/p_2)/\gamma_2^-} \|W_3(u,B)\|_{r_2-1,\pi_4} ds$$

$$
\begin{aligned}
&\le C\sup_{(0,T)}\int_0^t (t-s)^{-(1+n/\pi_4-n/p_2)/\gamma_2^-}\|u\|_{r_0,p_0}\|B\|_{r_2,p_2}ds\\
&\le C\|u\|_{0;r_0,p_0}\|B\|_{0;r_2,p_2}\sup_{(0,T)}\int_0^t (t-s)^{-(1+n/\pi_4-n/p_2)/\gamma_2^-}ds\\
&\le CM^2T^{1-(1+n/\pi_4-n/p_2)/\gamma_2^-},
\end{aligned}
$$

which holds if

$$
\gamma_2^- > 1+\frac{n}{\pi_4}-\frac{n}{p_2} = 1-r_0+\frac{n}{p_0}. \tag{3.25}
$$

The list of inequalities necessary to bound L is thus

$$
\begin{aligned}
&0\le r_0-r_2<\frac{n}{p_0} &&\text{(by (3.21))},\\
&r_2<\frac{n}{p_2} &&\text{(by (3.23))},\\
&r_0\le\frac{n}{p_0} &&\text{(by (3.24))},\\
&\gamma_2^- > 1-r_0+\frac{n}{p_0} &&\text{(by (3.25))}.
\end{aligned}
$$

3.6. **Wrapping up.** On one final note, we point out that since

$$
\begin{aligned}
&\underbrace{\Big(3r_1-2r_0-\gamma_3^-+\frac{3n}{p_0}-\frac{3n}{p_1}\Big)}_{\text{RHS of (3.16)}}-\underbrace{\Big(1-2r_0+r_1+\frac{2n}{p_0}-\frac{n}{p_1}\Big)}_{\text{RHS of (3.10)}}\\
&=2r_1-1-\gamma_3^-+\frac{n}{p_0}-\frac{2n}{p_1}\ge 0
\end{aligned}
$$

we have that (3.16) implies (3.10), and so the following is the definitive list containing all the inequalities needed for I_2, I_3, and I_4:

$$
\begin{gathered}
\gamma_3^- -1\le r_0\le\gamma_3^-\le r_1,\\
r_2-1+\gamma_3^-\le r_0,\\
r_2\le r_0<\frac{n}{p_1},\\
2r_1\ge\max\Big\{2,1+\gamma_3^- -\frac{n}{p_0}+\frac{2n}{p_1}\Big\},\\
r_2<\min\Big\{\frac{n}{p_2},\frac{2n}{p_2}-\frac{n}{p_0}\Big\},\\
\gamma_1^- > 3r_1-2r_0-\gamma_3^-+\frac{3n}{p_0}-\frac{3n}{p_1},\\
\gamma_1^- > 1-2r_2+r_1-\gamma_3^- -\frac{n}{p_1}+\frac{2n}{p_2},\\
\gamma_2^- > 1-r_0+\frac{n}{p_0}.
\end{gathered}
$$

The above inequalities coincide with those in Theorem 3.1, and so the proof is complete. □

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