Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 83, pp. 1–26.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.83

HÖLDER REGULARITY OF WEAK SOLUTIONS TO NONLOCAL p-LAPLACIAN TYPE SCHRÖDINGER EQUATIONS WITH A_1^p-MUCKENHOUPT POTENTIALS

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Abstract. In this article, using the De Giorgi-Nash-Moser method, we obtain an interior Hölder continuity of weak solutions to nonlocal p-Laplacian type Schrödinger equations given by an integro-differential operator L_K^p $(p > 1)$,

$$\begin{aligned} L_K^p u + V|u|^{p-2}u = 0 \quad &\text{in } \Omega, \\ u = g \quad &\text{in } \mathbb{R}^n \backslash \Omega. \end{aligned}$$

Where $V = V_+ - V_-$ with $(V_-, V_+) \in L^1_{\rm loc}(\mathbb{R}^n) \times L^q_{\rm loc}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ and $0 < s < 1$ is a potential such that $(V_-, V_+^{b,i})$ belongs to the (A_1, A_1)-Muckenhoupt class and $V_+^{b,i}$ is in the A_1-Muckenhoupt class for all $i \in N$. Here, $V_+^{b,i} := V_+ \max\{b, 1/i\}/b$ for an almost everywhere positive bounded function b on $\mathbb{R}^n$ with $V_+/b \in L^q_{\rm loc}(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. In addition, we prove local boundedness of weak subsolutions of the nonlocal p-Laplacian type Schrödinger equations. Also we obtain the logarithmic estimate of the weak supersolutions which play a crucial role in proving Hölder regularity of the weak solutions. We note that all the above results also work for a nonnegative potential in $L^q_{\rm loc}(\mathbb{R}^n)$ $(q > \frac{n}{ps} > 1,\ 0 < s < 1)$.

1. Introduction

The research on nonlocal partial differential equations has been performed not only in pure mathematics, but also in scientific areas that necessitate its concrete applications. This kind of problems appear in various applications such as continuum mechanics, phase transition phenomena related to a nonlocal version of classical Allen-Cahn equation, population dynamics, nonlocal minimal surfaces, a nonlocal version of Schrödinger equations for standing waves (see [3, 5, 6]), game theory and also constrained variational problems with fractional diffusion arising in the quasi-geostrophic flow model, anomalous diffusions and American options with jump processes (see [7, 8, 31]).

For $q > \frac{n}{ps} > 1$ $(p > 1,\ 0 < s < 1)$, let $\mathcal{P}_q^{s,p}(\mathbb{R}^n)$ be the class of potentials $V = V_+ - V_-$ such that

(i) $V_- \in L^1_{\rm loc}(\mathbb{R}^n)$,
(ii) $V_+ \in L^q_{\rm loc}(\mathbb{R}^n)$,
(iii) there is an almost everywhere positive bounded function b on $\mathbb{R}^n$ so that $V_+/b \in L^q_{\rm loc}(\mathbb{R}^n)$, $(V_-, V_+^{b,i})$ belongs to the (A_1, A_1)-Muckenhoupt class and $V_+^{b,i}$ is in the A_1-Muckenhoupt class for all $i \in \mathbb{N}$, where

$$V_+^{b,i} := \frac{\max\{b, 1/i\}}{b}\, V_+.$$

2020 *Mathematics Subject Classification.* 47G20, 45K05, 35J60, 35B65, 35D10, 60J75.
Key words and phrases. Hölder regularity; nonlocal p-Laplacian type Schrödinger equation; A_1^p-Muckenhoupt potential; De Giorgi-Nash-Moser method.

Submitted February 21, 2025. Published August 8, 2025.

If $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1,\ 0 < s < 1$), then we say that V is a A^p_1-*Muckenhoupt potential*. When $p = 2$, we call it A_1-*Muckenhoupt potential*.

The aim of this paper is to obtain an interior Hölder regularity of weak solutions of nonlocal p-Laplacian type Schrödinger equations with A^p_1-Muckenhoupt potentials and to additionally obtain the local boundedness of weak subsolutions of the nonlocal equation.

For $p > 1$, let $\mathcal{K}_p$ be the family of all positive symmetric kernels satisfying the uniformly ellipticity assumption

$$\frac{c_{n,p,s}\lambda}{|y|^{n+ps}} \le K(y) = K(-y) \le \frac{c_{n,p,s}\Lambda}{|y|^{n+ps}}, \quad 0 < s < 1,\ y \in \mathbb{R}^n\backslash\{0\}, \tag{1.1}$$

where $c_{n,p,s} > 0$ is the normalization constant given by

$$c_{n,p,s} = \frac{\Gamma(\frac{n+p}{2})\,p\,(1-s)}{\pi^{\frac{n-1}{2}}\,\Gamma(\frac{p+1}{2})}.$$

For $K \in \mathcal{K}_p$ $(p > 1)$, we consider integro-differential operators L^p_K given by

$$L^p_K u(x,t) = \text{p. v.} \int_{\mathbb{R}^n} H_p(u(x) - u(y))K(x-y)\,dy \tag{1.2}$$

where $H_p(t) = |t|^{p-2}t$ for $t \in \mathbb{R}$. If $p = 2$, then we write $L^p_K = L_K$. In particular, if $K(y) = c_{n,p,s}|y|^{-n-ps}$, then $L^p_K = (-\Delta)^s_p$ is the fractional p-Laplacian and it is well-known [20] that

$$\lim_{s\to 1^-} (-\Delta)^s_p u = -\Delta_p u$$

for any function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the classical p-Laplacian.

We are interested in the Dirichlet problem for the nonlocal p-Laplacian type Schrödinger equation

$$\begin{gathered} L^p_K u + V|u|^{p-2}u = 0 \quad \text{in } \Omega \\ u = g \quad \text{in } \mathbb{R}^n\backslash\Omega \end{gathered} \tag{1.3}$$

where $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$ and $0 < s < 1$), $g \in W^{s,p}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. The existence and uniqueness of weak solution to the above nonlocal equation was obtained in [24] by applying standard technique of calculus of variations. More precisely speaking about the problem, by employing the De Giorgi-Nash-Moser theory we obtain an interior Hölder continuity of weak solutions to the nonlocal p-Laplacian type Schrödinger equations with A^p_1-Muckenhoupt potentials, and we obtain the local boundedness of weak subsolutions of the nonlocal equation. Here, we note that the boundary condition is imposed on $\mathbb{R}^n\backslash\Omega$ with nonlocality. In fact, from the probabilistic point of view, it conforms to the natural phenomenon that a discontinuous Lévy process on the domain Ω can exit Ω for the first time jumping to any point in $\mathbb{R}^n\backslash\Omega$.

When $p = 2$, the research on the nonlocal equations was strongly motivated by the study of standing wave solutions of the form

$$\Psi(x,t) = e^{-i\omega t}u(x)$$

of the time-dependent nonlocal Schrödinger equations

$$i\frac{\partial\Psi}{\partial t} = L_K\Psi + V(x)\Psi$$

which is a fundamental equation of fractional quantum mechanics and fractional quantum physics. This equation was used for the first time in the literature by Laskin [25].

It turns out in Section 3 that any potential V in $\mathcal{P}^{s,p}_q(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} |\varphi(y)|\,V_-(y)\,dy \le \int_{\mathbb{R}^n} |\varphi(y)|\,V_+(y)\,dy\,, \tag{1.4}$$

$$\int_{\mathbb{R}^n} |\varphi(y)|^p \, V_-(y)\, dy \le \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_+(y)\, dy \tag{1.5}$$

for every $\varphi \in Y_0^{s,p}(\Omega)$, whenever $q > \frac{n}{ps} > 1$ ($p > 1$ and $0 < s < 1$). As a matter of fact, the inequality (1.4) is a useful tool for the proof of nonlocal Caccioppoli type inequality to be given in Theorem 1.5, and also the inequality (1.5) makes it possible to prove in Lemma 2.3 below that $Y_0^{s,p}(\Omega)$ is a quasi-Banach space.

Notation.
- For $r > 0$, $x_0 \in \mathbb{R}^n$ and $s \in (0,1)$, let us denote by $B_r^0 = B_r(x_0)$, $B_r = B_r(0)$.
- For two quantities a and b, we write $a \lesssim b$ (resp. $a \gtrsim b$) if there is a universal constant $C > 0$ (depending only on $\lambda, \Lambda, n, p, s$ and Ω) such that $a \le Cb$ (resp. $b \le Ca$).
- For $a, b \in \mathbb{R}$, we denote by

$$a \vee b = \max\{a, b\} \quad \text{and} \quad a \wedge b = \min\{a, b\}.$$

- Let $\mathcal{F}^n$ be the family of all real-valued Lebesgue measurable functions on $\mathbb{R}^n$.
- For $u \in C(B_r^0)$, we consider the norm

$$\|u\|_{C(B_r^0)} = \sup_{x \in B_r^0} |u(x)|.$$

For $\gamma \in (0,1)$, the γ^{th} *Hölder seminorm* of u on B_r^0 is defined by

$$[u]_{C^\gamma(B_r^0)} = \sup_{x,y \in B_r^0,\, x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\gamma}$$

and the γ^{th} *Hölder norm* of u on B_r^0 is defined by

$$\|u\|_{C^\gamma(B_r^0)} = \|u\|_{C(B_r^0)} + [u]_{C^\gamma(B_r^0)}.$$

- For $x_0 \in \Omega$, $p > 1$ and $r > 0$ with $B_r^0 \subset \Omega$, the *nonlocal tails* of the function u in $B_r^0 \subset \Omega$ is defined by

$$\mathcal{T}_r(u; x_0) = \Big(r^{ps} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(y)|^{p-1}}{|y - x_0|^{n+ps}}\, dy \Big)^{\frac{1}{p-1}}. \tag{1.6}$$

We now state one of our main results which is called the local boundedness of weak subsolutions to the nonlocal p-Laplacian type Schrödinger equation (1.3), as follows.

Theorem 1.1. *Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $s \in (0,1)$) and $B_{2r}^0 \subset \Omega$. If $u \in Y_g^{s,p}(\Omega)^-$ is a weak subsolution of nonlocal p-Laplacian type Schrödinger equation (1.3), then there is a constant $C_0 > 0$ depending only on $n, s, p, \lambda, \Lambda$ and Ω such that*

$$\sup_{B_r^0} u \le \delta\, \mathcal{T}_r(u_+; x_0) + C_0\, \delta^{-\frac{(p-1)n}{sp^2}} \Big(\fint_{B_{2r}^0} u_+^p \, dx \Big)^{1/p}$$

for all $\delta \in (0,1]$.

1.1. **Remarks.** (a) If $u \in Y_g^{s,p}(\Omega)^-$ is a weak subsolution of the nonlocal p-Laplacian type Schrödinger equation (1.3) and $g \in W^{s,p}(\mathbb{R}^n)$ for $s \in (0,1)$, then we see that $u \in L^p(\Omega)$ and $u \le g$ on $\mathbb{R}^n \setminus \Omega$, and thus $u_+ \le g_+$ there. Then it follows from Hölder's inequality and fractional Sobolev inequalities (2.4), (2.5) below that

$$\begin{aligned}[\mathcal{T}_r(u_+; x_0)]^{p-1} &\le r^{ps} \Big(\int_{\mathbb{R}^n \setminus \Omega} \frac{g_+^{p-1}(y)}{|y - x_0|^{n+ps}}\, dy + \int_{\Omega \setminus B_r(x_0)} \frac{|u(y)|^{p-1}}{|y - x_0|^{n+ps}}\, dy \Big) \\ &\lesssim \frac{r^{-n(1-\frac{1}{p})}}{((p-1)n + p^2 s)^{1/p}} \big(\|g\|_{W^{s,p}(\mathbb{R}^n)}^{p-1} + \|u\|_{L^p(\Omega)}^{p-1} \big) < \infty. \end{aligned}$$

(b) If $u \in Y_g^{s,p}(\Omega)^+$ is a weak supersolution of the nonlocal p-Laplacian type Schrödinger equation (1.3) and $g \in W^{s,p}(\mathbb{R}^n)$ for $s \in (0,1)$, then $-u$ is its weak subsolution and $u \ge g$ on $\mathbb{R}^n \setminus \Omega$, and so $u_- \le g_-$ there. Then, as in the above (a), we obtain that

$$[\mathcal{T}_r(u_-; x_0)]^{p-1} \lesssim \frac{r^{-n(1-\frac{1}{p})}}{((p-1)n + p^2 s)^{1/p}} \big(\|g\|_{W^{s,p}(\mathbb{R}^n)}^{p-1} + \|u\|_{L^p(\Omega)}^{p-1} \big) < \infty.$$

Then, from Theorem 1.1, we easily have

$$-\inf_{B_r^0} u = \sup_{B_r^0}(-u) \le \delta \mathcal{T}_r(u_-;x_0) + C_0\, \delta^{-\frac{(p-1)n}{sp^2}} \Big(\fint_{B_{2r}^0} u_-^p \, dx \Big)^{1/p}$$

for all $\delta \in (0,1]$.

(c) If $u \in Y_g^{s,p}(\Omega)^+$ is a weak solution of the nonlocal p-Laplacian type Schrödinger equation (1.3) and $g \in W^{s,p}(\mathbb{R}^n)$ for $s \in (0,1)$, then it follows from (a), (b) and Theorem 1.1 that

$$\operatorname{osc}_{B_r^0} u \le 2\delta\, \mathcal{T}_r(u;x_0) + 2C_0\, \delta^{-\frac{(p-1)n}{sp^2}} \Big(\fint_{B_{2r}^0} |u|^p \, dx \Big)^{1/p}$$

for all $\delta \in (0,1]$.

The next logarithmic estimate plays a crucial role in proving Hölder regularity of weak solutions to the nonlocal p-Laplacian type Schrödinger equation and in showing that the logarithm of its weak solution is a function with locally bounded mean oscillation. In a different way from that of [10], we obtain the logarithmic estimate.

Theorem 1.2. *Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$ and $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$ and $0 < s < 1$). If $u \in Y_g^{s,p}(\Omega)^+$ is a weak supersolution of nonlocal p-Laplacian type Schrödinger equation* (1.3) *with $u \ge 0$ in $B_R^0 \subset \Omega$, then there is a constant $c_0 > 0$ depending only on $n, s, p, \lambda, \Lambda$ and Ω such that*

$$\iint_{B_r^0 \times B_r^0} \Big| \ln\Big(\frac{u(x)+b}{u(y)+b} \Big) \Big|^p d_K(x,y) \le c_0 r^{n-ps} \Big[\frac{1}{b^{p-1}} \Big(\frac{r}{R} \Big)^{ps} [\mathcal{T}_R(u_-;x_0)]^{p-1} + \big(1 + \|V_+\|_{L^q(\Omega)}\big) \Big]$$

for any $b \in (0,1)$ and $r \in (0, R/2)$, where $d_K(x,y) = K(x-y)\,dx\,dy$.

Employing the De Giorgi-Nash-Moser theory and using Theorems 1.1 and 1.2, we obtain the following Hölder continuity of weak solutions to the nonlocal p-Laplacian type Schrödinger equation, and also we can easily derive Corollary 1.4 as a natural by-product of Theorem 1.3.

Theorem 1.3. *Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$), and let $B_{2R}^0 \subset \Omega$. If $u \in Y_g^{s,p}(\Omega)$ is a weak solution of the nonlocal p-Laplacian type Schrödinger equation* (1.3)*, then there exist constants $\eta_0^- \in (0, \frac{ps}{2(p-1)})$ and $\eta_0^+ \in (\frac{ps}{2(p-1)}, \frac{ps}{p-1})$ such that u is locally η-Hölder continuous in Ω for any $\eta \in (0,\eta_0^-] \cup [\eta_0^+, \frac{ps}{p-1})$. Furthermore, for each $x_0 \in \Omega$ and for each $\eta \in (0,\eta_0^-] \cup \big[\eta_0^+, \frac{ps}{p-1}\big)$, we have*

$$\operatorname{osc}_{B_r^0} u \lesssim \Big(\frac{r}{R} \Big)^{\eta} \Big[\mathcal{T}_R(u;x_0) + \Big(\fint_{B_{2R}^0} |u(x)|^p \, dx \Big)^{1/p} \Big] \tag{1.7}$$

for any $r \in (0, R/2)$. Here it turns out that there exist universal constants $c_0, c_* > 0$ such that

$$\eta_0^{\pm} = \frac{\ln\Big(\frac{1 \pm \sqrt{1 - 4\,\delta^{\frac{ps}{p-1}}}}{2} \Big)}{\ln \delta} \quad \text{for } \delta = e^{-(c_0/c_*)(1+\|V_+\|_{L^q(\Omega)})^{1/p}} \wedge \Big(\frac{1}{4}\Big)^{\frac{p-1}{ps}}.$$

The next corollary can easily be obtained by using Theorem 1.3 and employing the interpolation on Hölder spaces between $C^{\eta_0^-}(B_r^0)$ and $C^{\eta_0^+}(B_r^0)$, which eventually fill up an interior η-Hölder continuity of u in Ω for all $\eta \in (\eta_0^-, \eta_0^+)$.

Corollary 1.4. *Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$), and let $B_{2R}^0 \subset \Omega$. If $u \in Y_g^{s,p}(\Omega)$ is a weak solution of the nonlocal p-Laplacian type Schrödinger equation* (1.3)*, then we have the estimate*

$$\sup_{r \in (0,R/2)} \|u\|_{C^{\eta}(B_r^0)} \lesssim \frac{1}{R^{\eta}} \Big[\mathcal{T}_R(u;x_0) + \Big(\fint_{B_{2R}^0} |u(x)|^p \, dx \Big)^{1/p} \Big] \tag{1.8}$$

for all $\eta \in \big(0, \frac{ps}{p-1}\big)$.

If $\frac{p-1}{p} < s < 1$, then we can expect the better regularity, i.e. $C^{1,\alpha}$-estimate for some $\alpha \in (0,1)$. As a basic tool for our main results, we show that any weak subsolution of the nonlocal p-Laplacian type Schrödinger equations enjoys the following *nonlocal Caccioppoli type inequality*.

Theorem 1.5. *Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $s \in (0,1)$), and let $B_{2r}^0 \subset \Omega$. If $u \in Y_g^{s,p}(\Omega)^-$ is a weak subsolution of the nonlocal p-Lapacian type Schrödinger equation* (1.3)*, then for any nonnegative $\zeta \in C_c^\infty(B_r^0)$ we have the estimate*

$$\begin{aligned}
&\int_{B_r^0} [w(y)\zeta(y)]^p V(y)\,dy + \iint_{B_r^0\times B_r^0} |\zeta(x)w(x) - \zeta(y)w(y)|^p\,d_K(x,y)\\
&\le 2^{2p+1}\big(\frac{1}{4} + c_p\big) \iint_{B_r^0\times B_r^0} [w(x)\vee w(y)]^p |\zeta(x) - \zeta(y)|^p\,d_K(x,y)\\
&\quad + 2^{p+2}\Big(\sup_{x\in \operatorname{supp}(\zeta)} \int_{\mathbb{R}^n\setminus B_r^0} w^{p-1}(y)K(x-y)\,dy\Big)\|w\zeta^p\|_{L^1(B_r^0)}
\end{aligned}$$

where $w = (u-M)_+$ for $M \in (0,\infty)$ and $c_p = \frac{1}{2}[2(p-1)]^{p-1}$.

Remark 1.6. (a) In case that $p = 2$ and $s = 1$, the study of the classical Schrödinger operator, i.e. local Schrödinger operator $-\Delta + V$ has been ongoing actively and widely in analysis area in Mathematics and Mathematical Physics (refer to [1, 4, 12, 29, 30, 32]).

(b) When $p = 2$ and $V \in L_{\rm loc}^q(\mathbb{R}^n)$ with $q > \frac{n}{2s}$ ($0 < s < 1$) is nonnegative, it is known in [5] that a fundamental solution for nonlocal Schrödinger operator $L_K + V$ exists and its decay can be obtained. Under an additional restiction that the potential V is in a reverse Hölder class RH^γ for $\gamma > \frac{n}{2s} > 1$ ($0 < s < 1$), the $L^\alpha - L^\beta$ estimate for the Schrödinger operator $L_K + V$ was obtained inside certain trapezoidal region $\mathcal{Z}$ which is consist of $(\frac{1}{\alpha}, \frac{1}{\beta})$ and also the weak type $L^\alpha - L^\beta$ estimate was partially obtained on the boundary of the region $\mathcal{Z}$ (see [6]).

(c) In case that $p = 2$, $0 < s < 1$ and V is an A_1-Muckenhoupt potential, it was shown in [24] that a fundamental solution for nonlocal Schrödinger operator $L_K + V$ exists and its decay can be obtained. Moreover, Hölder continuity and nonlocal Harnack inequalities for $L_K + V$ were obtained in [22] and [23].

(d) When $V = 0$ and $0 < s < 1$, the nice result of this problem was obtained by Di Castro, Kuusi and Palatucci [9]; as a matter of fact, when $p \in (1,\infty)$, they proved nonlocal Harnack inequalities for elliptic nonlocal p-Laplacian equations there. Also they obtained Hölder regularity in [10].

(e) In case that $p = 2$ and $0 < s < 1$, nonlocal Harnack inequalities for (locally nonnegative in Ω) weak solutions of nonlocal heat equations was obtained in [21] by applying the De Giorgi-Nash-Moser theory and the Krylov-Safonov covering theorem [27].

(f) When the nonlocal equation (1.3) with the forcing term $f \in L^\infty(\Omega)$ and $V = g = 0$ is considered for $0 < s < 1 < p < \infty$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$ boundary, Iannizzotto, Mosconi and Squassina obtained the first *global Hölder regularity* for its weak solutions in [19], i.e. there exist some $\alpha \in (0,s]$ and $C > 0$ depending only on n, p, s and Ω such that

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\,\|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$$

for any weak solution $u \in W_0^{s,p}(\Omega)$ of the nonlocal equation. If the nonlocal equation mentioned just in the above is considered for $0 < s < 1$, $p \ge 2$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with $C^{1,1}$ boundary, then they also established the first *fine boundary regularity* for its weak solutions in [18], i.e. there exist some $\alpha \in (0,s]$ and $C > 0$ depending only on n, p, s and Ω such that

$$\Big\|\frac{u}{\mathrm{d}_\Omega^s}\Big\|_{C^\alpha(\overline{\Omega})} \le C\|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$$

for any weak solution $u \in W_0^{s,p}(\Omega)$ of the nonlocal equation, where $\mathrm{d}_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

The article is organized as follows. In Section 2, we furnish the function spaces and the definition of weak solutions of the nonlocal p-Laplacian type Schrödinger equation given in (1.3), and mention a well-known lemma which is very useful in applying the De Giorgi-Nash-Moser theory. In Section 3, we give a brief introduction about weighted norm inequalities and the A_p-Muckenhoupt class. Additionally, we furnish several examples about sign-changing potentials which is in the class $\mathcal{P}_q^{s,p}(\mathbb{R}^n)$ ($q > \frac{n}{ps} > 1$, $p > 1$, $0 < s < 1$). In Section 4, we obtain a sort of *nonlocal Caccioppoli type inequality* and several useful local properties of weak solutions to the nonlocal p-Laplacian type Schrödinger equation by using it. In Section 5, we show that the logarithm of a weak solution to the nonlocal p-Laplacian type Schrödinger equation with A^p_1-Muckenhoupt potentials becomes a function with locally bounded mean oscillation. In Section 6, we obtain an interior Hölder continuity of weak solutions to the nonlocal p-Laplacian type Schrödinger equation by applying the results obtained in Sections 4 and 5.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $K \in \mathcal{K}_p$ for $p > 1$. For $p > 1$ and $0 < s < 1$, let $X^{s,p}(\Omega)$ be the linear function space of all Lebesgue measurable functions $v \in \mathcal{F}^n$ such that $v|_\Omega \in L^p(\Omega)$ and

$$\iint_{\mathbb{R}^{2n}_\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,dx\,dy < \infty$$

where $\mathbb{R}^{2n}_S := \mathbb{R}^{2n} \backslash (S^c \times S^c)$ for a set $S \subset \mathbb{R}^n$. We also set

$$X^{s,p}_0(\Omega) = \{v \in X^{s,p}(\Omega) : v = 0 \quad \text{a.e. in } \mathbb{R}^n \backslash \Omega\} \tag{2.1}$$

Since $C^2_0(\Omega) \subset X^{s,p}_0(\Omega)$, we see that $X^{s,p}(\Omega)$ and $X^{s,p}_0(\Omega)$ are nonempty. Then we see that $(X^{s,p}(\Omega), \|\cdot\|_{X^{s,p}(\Omega)})$ is a normed space with the norm $\|\cdot\|_{X^{s,p}(\Omega)}$ given by

$$\|v\|_{X^{s,p}(\Omega)} = \|v\|_{L^p(\Omega)} + \Big(\iint_{\mathbb{R}^{2n}_\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\Big)^{1/p} < \infty \tag{2.2}$$

for $v \in X^{s,p}(\Omega)$. For $p \ge 1$, we denote by $W^{s,p}(\Omega)$ the usual fractional Sobolev space with the norm

$$\|v\|_{W^{s,p}(\Omega)} := \|v\|_{L^p(\Omega)} + [v]_{W^{s,p}(\Omega)} < \infty \tag{2.3}$$

with the seminorm

$$[v]_{W^{s,p}(\Omega)} = \Big(\iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\Big)^{1/p}.$$

When $\Omega = \mathbb{R}^n$ in (2.3), similarly we define the spaces $W^{s,p}(\mathbb{R}^n)$ for $p \ge 1$ and $s \in (0,1)$.

If $p \ge 1$ and $s \in (0,1)$ satisfy $ps < n$, then it is well-known [11] that there exists a constant $c = c(n,p,s,\Omega) > 0$ such that

$$\|f\|_{L^\tau(\Omega)} \le c\,\|f\|_{W^{s,p}(\Omega)} \tag{2.4}$$

for all $f \in W^{s,p}(\Omega)$ and $\tau \in [p, p_*]$, where p_* is the Sobolev exponent

$$p_* = \frac{pn}{n-ps}.$$

Moreover, there is a constant $c = c(n,p,s) > 0$ such that

$$\begin{gathered} \|f\|_{L^\tau(\mathbb{R}^n)} \le c\|f\|_{W^{s,p}(\mathbb{R}^n)}, \quad \forall \tau \in [p, p_*], \\ \|f\|_{L^{p_*}(\mathbb{R}^n)} \le c\,[f]_{W^{s,p}(\mathbb{R}^n)} \quad \forall f \in W^{s,p}(\mathbb{R}^n). \end{gathered} \tag{2.5}$$

Using (2.5), we easily see that there exists a constant $c > 1$ depending only on n, p, s and Ω such that

$$\|u\|_{X^{s,p}_0(\Omega)} \le \|u\|_{X^{s,p}(\Omega)} \le c\,\|u\|_{X^{s,p}_0(\Omega)} \quad \forall u \in X^{s,p}_0(\Omega), \tag{2.6}$$

where

$$\|u\|_{X^{s,p}_0(\Omega)} := \Big(\iint_{\mathbb{R}^{2n}_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\Big)^{1/p}. \tag{2.7}$$

Thus $\|\cdot\|_{X^{s,p}_0(\Omega)}$ is a norm on $X^{s,p}_0(\Omega)$ equivalent to (2.2). By using the change of variables, we can easily derive the following version of the fractional Sobolev inequality (2.4).

Proposition 2.1. *Let B_R be a ball with radius $R > 0$. If $s \in (0,1)$ and $p \in [1,\infty)$ with $sp < n$, then there is a constant $c = c(n,p,s) > 0$ such that*

$$\|f\|_{L^\tau(B_R)} \le c\,R^{-n(\frac{1}{p}-\frac{1}{\tau})}\|f\|_{L^p(B_R)} + cR^{-n(\frac{1}{p}-\frac{1}{\tau})+s}[f]_{W^{s,p}(B_R)} \quad \forall \tau \in [p, p_*].$$

In particular, if $\tau = p_* := \frac{pn}{n-ps}$, we have

$$\|f\|_{L^{p_*}(B_R)} \le c\,R^{-s}\|f\|_{L^p(B_R)} + c\,[f]_{W^{s,p}(B_R)}. \tag{2.8}$$

For $g \in W^{s,p}(\mathbb{R}^n)$, we consider the convex subsets of $X^{s,p}(\Omega)$ defined by

$$X_g^{s,p}(\Omega)^\pm = \{v \in X^{s,p}(\Omega) : (g-v)_\pm \in X_0^{s,p}(\Omega)\},$$
$$X_g^{s,p}(\Omega) := X_g^{s,p}(\Omega)^+ \cap X_g^{s,p}(\Omega)^- = \{v \in X^{s,p}(\Omega) : g - v \in X_0^{s,p}(\Omega)\}.$$

For $g \in W^{s,p}(\mathbb{R}^n)$ and a potential $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$ with $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$), let

$$Y^{s,p}(\Omega) = X^{s,p}(\Omega) \cap L_V^p(\Omega) \quad \text{and} \quad Y_g^{s,p}(\Omega) = X_g^{s,p}(\Omega) \cap L_V^p(\Omega)$$

where $L_V^p(\Omega)$ is the weighted L^p class of all real-valued measurable functions u on $\mathbb{R}^n$ satisfying

$$-\infty < \|u\|_{L_V^p(\Omega)}^p := \int_\Omega |u(y)|^p \, V_+(y)\, dy - \int_\Omega |u(y)|^p \, V_-(y)\, dy := \|u\|_{L_{V_+}^p(\Omega)}^p - \|u\|_{L_{V_-}^p(\Omega)}^p < \infty.$$

That is, we see that $u \in L_V^p(\Omega)$ if and only if $u \in L_{V_+}^p(\Omega) \cap L_{V_-}^p(\Omega)$. Here, we note that $\|u\|_{L_V^p(\Omega)}^p$ need not be nonnegative, and so the class $L_V^p(\Omega)$ is not always a normed space.

Also we consider function spaces $Y_g^{s,p}(\Omega)^+$ and $Y_g^{s,p}(\Omega)^-$ defined by

$$Y_g^{s,p}(\Omega)^\pm = \{u \in Y^{s,p}(\Omega) : (g-u)_\pm \in Y_0^{s,p}(\Omega)\}.$$

Then we see that

$$Y_g^{s,p}(\Omega) = Y_g^{s,p}(\Omega)^+ \cap Y_g^{s,p}(\Omega)^-.$$

If $u = g = 0$ in $\mathbb{R}^n \setminus \Omega$, then we easily know that $Y_0^{s,p}(\Omega) = X_0^{s,p}(\Omega) \cap L_V^p(\Omega)$ need not be a Banach space. However, if $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$), then it turns out in Lemma 3.2 below that the class $Y_0^{s,p}(\Omega)$ is a quasi-Banach space with the quasinorm $\|\cdot\|_{Y_0^{s,p}(\Omega)}$ given by

$$\|u\|_{Y_0^{s,p}(\Omega)}^p := \|u\|_{X_0^{s,p}(\Omega)}^p + \int_\Omega |u(y)|^p V(y)\, dy, \ u \in Y_0^{s,p}(\Omega),$$

$Y_0^{s,p}(\Omega) = X_0^{s,p}(\Omega)$ and they are quasinorm-equivalent.

To define weak solutions of the nonlocal equation (1.3), we consider a bilinear form $\langle \cdot, \cdot \rangle_{H_p,K} : X^{s,p}(\Omega) \times X^{s,p}(\Omega) \to \mathbb{R}$ defined by

$$\langle u, v \rangle_{H_p,K} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} H_p(u(x) - u(y))(v(x) - v(y))\, d_K(x,y)$$

where $d_K(x,y) := K(x-y)\, dx\, dy$.

Definition 2.2. Let $V \in \mathcal{P}_q^{s,p}(\mathbb{R}^n)$ and $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$). Then we say that a function $u \in Y_g^{s,p}(\Omega)^-$ ($u \in Y_g^{s,p}(\Omega)^+$) is a *weak subsolution (weak supersolution)* of the nonlocal p-Laplacian type Schrödinger equation (1.3), if it satisfies

$$\langle u, \varphi \rangle_{H_p,K} + \int_{\mathbb{R}^n} V(x)|u(x)|^{p-2}u(x)\varphi(x)\, dx \le 0 \quad (\ge 0) \tag{2.9}$$

for all nonnegative $\varphi \in Y_0^{s,p}(\Omega)$. Also, we say that a function u is a *weak solution* of the nonlocal equation (1.3), if it is both a weak subsolution and a weak supersolution, i.e.

$$\langle u, \varphi \rangle_{H_p,K} + \int_{\mathbb{R}^n} V(x)|u(x)|^{p-2}u(x)\varphi(x)\, dx = 0 \quad \forall \varphi \in Y_0^{s,p}(\Omega). \tag{2.10}$$

To prove our results, we need a well-known lemma [15] that is useful in applying the De Giorgi-Nash-Moser method.

Lemma 2.3. *Let $\{N_k\}_{k=0}^\infty \subset \mathbb{R}$ be a sequence of positive numbers such that*

$$N_{k+1} \le d_0\, e_0^k N_k^{1+\eta} \quad \forall k \in \mathbb{N} \cup \{0\},$$

where $d_0, \eta > 0$ and $e_0 > 1$. If $N_0 \le d_0^{-1/\eta} e_0^{-1/\eta^2}$, then we have $N_k \le e_0^{-k/\eta}\, N_0$ for any $k = 0, 1, \dots$ and moreover $\lim_{k\to\infty} N_k = 0$.

We need several elementary inequalities which are useful in proving Theorems 1.2 and 1.5.

Lemma 2.4 ([24]). *(a) If $a, b \in \mathbb{R}$ and $A, B \geq 0$, then we have the inequality*

$$|b-a|^{p-2}(b-a)(bB^p - aA^p) \geq -c_p \left(|a|+|b|\right)^p |B-A|^p$$

for all $p > 1$, where $c_p = \frac{(p-1)^{p-1}}{2}$.

(b) If $a, b \in \mathbb{R}$ with $b \geq a$ and $A, B \geq 0$, then we have the inequality

$$(b-a)^{p-1}(bB^p - aA^p) \geq \frac{1}{4}(b-a)^p(A^p+B^p) - d_p\big(|a|+|b|\big)^p |B-A|^p$$

for all $p > 1$, where $d_p = \frac{1}{2}[2(p-1)]^{p-1}$.

(c) If $A \geq B \geq 0$ and $p > 1$, then $(A-B)^{p-1} \geq b_p A^{p-1} - B^{p-1}$ for $p > 1$, where $b_p = \mathbf{1}_{(1,2]}(p) + 2^{-(p-1)}\mathbf{1}_{(2,\infty)}(p)$.

Lemma 2.5 ([10]). *If $p \geq 1$, $\varepsilon \in (0,1]$ and $a, b \geq 0$, then*

$$a^p \leq b^p + c_p\varepsilon\, b^p + (1+c_p\varepsilon)\varepsilon^{1-p}|a-b|^p,$$

where $c_p = (p-1)\Gamma(1 \vee (p-2))$ for the standard Gamma function Γ.

3. Weighted norm inequalities on the A_p-Muckenhoupt class

In this section, we briefly introduce the A_p-Muckenhoupt class for $p \geq 1$ and we prove that any potential V in $\mathcal{P}^{s,p}_q(\mathbb{R}^n)$ ($q > \frac{n}{ps} > 1$, $p > 1$, $0 < s < 1$) satisfies certain weighted norm inequalities related with (V_-, V_+).

By a *weight* ω on $\mathbb{R}^n$ given by the Lebesgue measure, we mean a locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ almost everywhere. For $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) = \sup_{Q^x} \fint_{Q^x} |f(y)|\, dy,$$

where the supremum is taken over all cubes Q^x with center x. For a pair (υ, ω) of weights, the quantity $[\upsilon,\omega]_{(A_p,A_p)}$ is defined by

$$[\upsilon,\omega]_{(A_p,A_p)} = \begin{cases} \sup_Q \big(\fint_Q \upsilon(y)\,dy\big)\big(\fint_Q \omega(y)^{-\frac{1}{p-1}}\,dy\big)^{p-1}, & 1 < p < \infty, \\ \sup_Q \big(\fint_Q \upsilon(y)\,dy\big)\|\omega^{-1}\|_{L^\infty(Q)}, & p = 1, \end{cases}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $1 \leq p < \infty$, we say that $(\upsilon,\omega) \in (A_p, A_p)$ if $[\upsilon,\omega]_{(A_p,A_p)} < \infty$, and $\omega \in A_p$ if $[\omega]_{A_p} := [\omega,\omega]_{(A_p,A_p)} < \infty$. For $1 \leq p < \infty$, the facts that $(A_1, A_1) \subset (A_p, A_p)$, $A_1 \subset A_p$ and $(\upsilon,\omega) \in (A_p, A_p)$ is equivalent to the mapping property that

$$\mathcal{M} : L^p_\omega(\mathbb{R}^n) \to L^{p,\infty}_\upsilon(\mathbb{R}^n)$$

is bounded, i.e. there is a universal constant $C_{n,p} > 0$ such that

$$\sup_{t>0}\big[t\,\upsilon\big(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}\big)^{1/p}\big] \leq C_{n,p}\Big(\int_{\mathbb{R}^n} |f(y)|^p \omega(y)\,dy\Big)^{1/p} \tag{3.1}$$

for any $f \in L^p_\omega(\mathbb{R}^n)$, are well-known in [16]. Here, we denote by

$$\upsilon(E) = \int_E \upsilon(y)\,dy$$

for a set $E \subset \mathbb{R}^n$. The reader can refer to [16] for these stuffs in Fourier analysis.

We shall now furnish several examples in the class $\mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$) mentioned in the above introduction (see also [24]):

(a) If $g_\alpha(x) = |x|^\alpha$ for $\alpha \in \mathbb{R}$, then it is easy to check that

$$[g_\eta]_{A_1} < \infty \quad \text{if and only if} \quad -n < \alpha \leq 0.$$

If we consider a sign-changing potential

$$V_1(x) = |x|^{\alpha/q}\cos(|x|)$$

with $\alpha \in (-n, 0]$ and $q > \frac{n}{ps} > 1$ ($p > 1$, $0 < s < 1$), then we can easily check that $V_1 \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ with $b(x) = \cos^+(|x|)$.

(b) Let ν be a Borel measure on $\mathbb{R}^n$ satisfying that

$$\mathfrak{M}\nu(x) := \sup_{Q^x} \frac{\nu(Q^x)}{|Q^x|} \le C \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q^x with center $x \in \mathbb{R}^n$. Then it is known in [14] that $[h_\gamma]_{A_1} < \infty$ for $h_\gamma(x) = [\mathfrak{M}\nu(x)]^\gamma$, $\gamma \in (0,1)$. We consider the following sign-changing potential

$$V_2(x) = h_\gamma(x) \sin(1/|x|)$$

for $\gamma \in (0,1)$. If $q > \frac{n}{ps} > 1$ for $p > 1$ and $0 < s < 1$, then it is easy to check that $V_2 \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ with $b(x) = \sin^+(1/|x|)$.

(c) Let $\upsilon(x) = \ln(1/|x|)\mathbf{1}_{B(0;e^{-1})} + \mathbf{1}_{\mathbb{R}^n \setminus B(0;e^{-1})}$. Then it is easy to check that $\upsilon \in A_1$. We consider the following sign-changing potential

$$V_3(x) = \upsilon(x) \cos(1/|x|).$$

If $q > \frac{n}{ps} > 1$ for $p > 1$ and $0 < s < 1$, then it follows from properties of the Gamma function that $V_3 \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ with $b(x) = \cos^+(1/|x|)$.

Next, we derive several fundamental lemmas which are useful in proving Theorem 3.3.

Lemma 3.1. *If $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$, $p > 1$ and $0 < s < 1$, then*

$$\int_{\mathbb{R}^n} |\varphi(y)|^p \, V_-(y)\, dy \le \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_+(y)\, dy$$

for all $\varphi \in Y^{s,p}_0(\Omega)$.

Proof. In the exactly same way as the proof of [23, Theorem 3.6], we see that

$$\int_{\mathbb{R}^n} |\varphi(y)|^p \, V(y)\, dy \ge 0 \tag{3.2}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n)$. Take any $\varphi \in Y^{s,p}_0(\Omega)$. Since $C^\infty_c(\Omega)$ is dense in $X^{s,p}_0(\Omega)$ (see [13] and [17, Theorem 1.4.2.2]), we can take a sequence $\{\varphi_k\} \subset C^\infty_c(\Omega)$ such that

$$\varphi_k \to \varphi \quad \text{in } X^{s,p}_0(\Omega).$$

Since $V \in \mathbb{P}^{s,p}_q(\mathbb{R}^n)$, there is a nonnegative bounded function b on $\mathbb{R}^n$ such that

$$V^b_+ := \frac{V_+}{b} \in L^q_{\rm loc}(\mathbb{R}^n), (V_-, V^{b,i}_+) \in (A_1, A_1) \text{ and } V^{b,i}_+ \in A_1 \tag{3.3}$$

for all $i \in \mathbb{N}$. Then we claim that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_k(y)|^p \, V_+(y)\, dy = \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_+(y)\, dy; \tag{3.4}$$

indeed, we note that $\|\varphi_k\|_{X^{s,p}_0(\Omega)} \le 2\|\varphi\|_{X^{s,p}_0(\Omega)}$ for all sufficiently large $k \in \mathbb{N}$, and also we see that, for any $y \in \mathbb{R}^n$,

$$|\varphi_k(y)|^p - |\varphi(y)|^p = \int_{|\varphi(y)|}^{|\varphi_k(y)|} \frac{d}{d\tau}\tau^p \, d\tau \le p\big(|\varphi_k(y)| - |\varphi(y)|\big)\big(|\varphi_k(y)| \vee |\varphi(y)|\big)^{p-1}.$$

Thus it follows from the fractional Sobolev inequality and Hölder's inequality that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} |\varphi_k(y)|^p \, V_+(y)\, dy - \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_+(y)\, dy\Big| \\
&\le \int_{\mathbb{R}^n} \big||\varphi_k(y)|^p - |\varphi(y)|^p\big| \, V_+(y)\, dy \\
&\le p\Big(\int_\Omega \big||\varphi_k(y)| - |\varphi(y)|\big|^p V_+(y)\, dy\Big)^{1/p} \Big(\int_\Omega \big(|\varphi_k(y)| \vee |\varphi(y)|\big)^p \, V_+(y)\, dy\Big)^{\frac{p-1}{p}} \\
&\lesssim \|\varphi\|_{X^{s,p}_0(\Omega)} \|\varphi_k - \varphi\|_{X^{s,p}_0(\Omega)} \|V_+\|_{L^q(\Omega)} \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{3.5}$$

Also, we claim that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_k(y)|^p V_-(y)\, dy = \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_-(y)\, dy; \tag{3.6}$$

indeed, we have

$$V_-(y) \le [V_-, V_+^{b,1}]_{(A_1,A_1)} V_+^{b,1}(y) \quad \text{a.e. } y \in \mathbb{R}^n,$$

because $(V_-, V_+^{b,1}) \in (A_1, A_1)$ by (3.3). Since $V_+^b \in L^q_{\rm loc}(\mathbb{R}^n)$ by (3.3), as in (3.5), we obtain that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} |\varphi_k(y)|^p \, V_-(y)\, dy - \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_-(y)\, dy \Big| \\
&\le \int_{\mathbb{R}^n} \big| |\varphi_k(y)|^p - |\varphi(y)|^p \big| \, V_-(y)\, dy \\
&\le \big(\|b\|_{L^\infty(\mathbb{R}^n)} \vee 1\big) [V_-, V_+^{b,1}]_{(A_1,A_1)} \times \int_\Omega \big| |\varphi_k(y)|^p - |\varphi(y)|^p \big| V_+^b(y)\, dy \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Thus, by (3.2), (3.4) and (3.6), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\varphi(y)|^p \, V_-(y)\, dy &= \lim_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_k(y)|^p \, V_-(y)\, dy \\
&\le \lim_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_k(y)|^p \, V_+(y)\, dy \\
&= \int_{\mathbb{R}^n} |\varphi(y)|^p \, V_+(y)\, dy.
\end{aligned}$$

The proof is complete. □

Lemma 3.2. *If $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ $(p > 1,\ 0 < s < 1)$, then $Y^{s,p}_0(\Omega)$ is a quasi-Banach space with the quasinorm $\|\cdot\|_{Y^{s,p}_0(\Omega)}$ given by*

$$\|u\|^p_{Y^{s,p}_0(\Omega)} := \|u\|^p_{X^{s,p}_0(\Omega)} + \int_\Omega |u(y)|^p V(y)\, dy, \ u \in Y^{s,p}_0(\Omega).$$

Moreover, $Y^{s,p}_0(\Omega) = X^{s,p}_0(\Omega)$ and they are quasinorm-equivalent.

Proof. It follows from Lemma 3.1 and (2.5) that

$$\|u\|^p_{X^{s,p}_0(\Omega)} \le \|u\|^p_{Y^{s,p}_0(\Omega)} \le \|u\|^p_{X^{s,p}_0(\Omega)} + \|u\|^p_{L^p_{V_+}(\Omega)} \le (1 + |\Omega|^{\frac{ps}{n} - \frac{1}{q}}) \|u\|^p_{X^{s,p}_0(\Omega)}.$$

Since $X^{s,p}_0(\Omega)$ is a Banach space, this implies the required results. □

Theorem 3.3. *If $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ and $0 < s < 1$, then*

$$\int_{\mathbb{R}^n} |\varphi(y)| V_-(y)\, dy \le \int_{\mathbb{R}^n} |\varphi(y)| \, V_+(y)\, dy \tag{3.7}$$

for all $\varphi \in Y^{s,p}_0(\Omega)$.

Proof. Take any $\varphi \in Y^{s,p}_0(\Omega)$. Since $C^\infty_c(\Omega)$ is dense in $X^{s,p}_0(\Omega)$ (see [17] and [13]), $Y^{s,p}_0(\Omega) = X^{s,p}_0(\Omega)$ and they are quasinorm-equivalent by Lemma 3.2, we can take a sequence $\{\varphi_i\} \subset C^\infty_c(\Omega)$ such that $\varphi_i \to \varphi$ in $Y^{s,p}_0(\Omega)$. So by (2.5) we also have $\varphi_i \to \varphi$ in $L^{p_*}(\Omega)$, where $p_* = \frac{pn}{n-ps}$. So we can choose a subsequence $\{\varphi_{i_k}\}$ such that

$$\varphi_{i_k} \to \varphi \quad \text{a..e. in } \Omega. \tag{3.8}$$

Also we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_{i_k}(y)| \, V_+(y)\, dy = \int_{\mathbb{R}^n} |\varphi(y)| \, V_+(y)\, dy; \tag{3.9}$$

indeed, it follows from the fractional Sobolev inequality and Hölder's inequality that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^n} |\varphi_{i_k}(y)| \, V_+(y)\, dy - \int_{\mathbb{R}^n} |\varphi(y)| \, V_+(y)\, dy \Big| \\
&\le \int_\Omega \big| |\varphi_{i_k}(y)| - |\varphi(y)| \big| \, V_+(y)\, dy \\
&\le \Big(\int_\Omega \big| |\varphi_{i_k}(y)| - |\varphi(y)| \big|^p \, V_+(y)\, dy \Big)^{1/p} \Big(\int_\Omega V_+(y)\, dy \Big)^{\frac{p-1}{p}}
\end{aligned}$$

$$\lesssim |\Omega|^{\frac{s}{n}+\frac{1}{q'}-\frac{1}{p}} \|\varphi_{i_k}-\varphi\|_{Y_0^{s,p}(\Omega)} \|V_+\|_{L^q(\Omega)} \to 0 \quad \text{as } k\to\infty$$

where q' is the dual exponent of q. Hence, by Fatou's lemma, Lemma 3.1, [23, Lemma 3.5], (3.2), (3.4), (3.8) and (3.9), we conclude that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\varphi(y)|\, V_-(y)\, dy &= \lim_{p\to 1^+} \int_{\mathbb{R}^n} |\varphi(y)|^p\, V_-(y)\, dy \\
&\le \lim_{p\to 1^+} \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_{i_k}(y)|^p\, V_-(y)\, dy \\
&\le \lim_{p\to 1^+} \liminf_{k\to\infty} \int_{\mathbb{R}^n} |\varphi_{i_k}(y)|^p\, V_+(y)\, dy \\
&\le \lim_{p\to 1^+} \int_{\mathbb{R}^n} |\varphi(y)|^p\, V_+(y)\, dy \\
&= \int_{\mathbb{R}^n} |\varphi(y)|\, V_+(y)\, dy.
\end{aligned}$$

Therefore the proof is complete. □

4. Local properties of weak subsolutions

In this section, we shall obtain certain local properties for weak subsolutions to the nonlocal p-Laplacian type Schrödinger equation. These results play a crucial role in establishing an interior Hölder continuity for weak solutions to the nonlocal equation (1.3). To establish the result, we need several steps.

Lemma 4.1. *For $p>1$ and $N>0$, let $h(t)=t^{p-1}(t-N)_+ - (t-N)_+^p \ge 0$. Then h is Lipschitz continuous on $\mathbb{R}$.*

Proof. We note that $h(t)=0$ for $t<N$, h is in $C^1(N,\infty)$,

$$\lim_{t\to N^+} \frac{h(t)-h(N)}{t-N} = 0 \quad \text{and} \quad \lim_{t\to N^-} \frac{h(t)-h(N)}{t-N} = N^{p-1}.$$

Also it is easy to check that $\lim_{t\to\infty} h(t)=0$, because

$$\lim_{t\to\infty} \frac{t^{p-1}(t-N)_+}{(t-N)_+^p} = 1.$$

Moreover, in order to check the differentiability of h at the infinity, we set $g(t)=h(1/t)$. Then we have

$$\lim_{t\to 0^+} \frac{g(t)-g(0)}{t} = \lim_{t\to 0^+} \Big[\frac{1}{t^p}\big(\frac{1}{t}-N\big) - \frac{1}{t}\big(\frac{1}{t}-N\big)^p\Big] = 0,$$

because

$$\lim_{t\to 0^+} \Big[\frac{1}{t^p}\Big(\frac{1}{t}-N\Big)/\frac{1}{t}\big(\frac{1}{t}-N\big)^p\Big] = \lim_{t\to 0^+} \frac{1-tN}{(1-tN)^p} = 1.$$

This implies the required result. □

Corollary 4.2. *If $N>0$ and $u\in X^{s,p}(\Omega)$ for $p>1$ and $0<s<1$, then*

(a) *$|u|^{p-2}u(u-N)_+ - (u-N)_+^p \in X^{s,p}(\Omega)$ and it is nonnegative in $\mathbb{R}^n$, and*

(b) *$\big[|u|^{p-2}u(u-N)_+ - (u-N)_+^p\big]\zeta^p \in X_0^{s,p}(\Omega)$ for any nonnegative $\zeta\in C_c^\infty(\Omega)$.*

Proof. (a) Note that $|u|^{p-2}u(u-N)_+ - (u-N)_+^p = h\circ u$ for the function h given in Lemma 4.1. Since h is Lipschitz continuous on $\mathbb{R}$ by Lemma 4.1, it is obvious that $0\le h\circ u\in X^{s,p}(\Omega)$.

(b) By the mean value theorem, we have

$$\begin{aligned}
|\zeta^p(x)-\zeta^p(y)| &= \Big|\int_{\zeta(y)}^{\zeta(x)} \frac{d}{d\tau}\tau^p\, d\tau\Big| \\
&\le p\big(\zeta^{p-1}(x)\vee\zeta^{p-1}(y)\big)|\zeta(x)-\zeta(y)| \\
&\le 2p\|\zeta\|_{L^\infty(\mathbb{R}^n)}^{p-1}|\zeta(x)-\zeta(y)|
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. This means that ζ^p is Lipschitz continuous on $\mathbb{R}^n$. So we can easily derive the result. The proof is complete. □

Next we need the nonlocal Caccioppoli type inequality that is Theorem 1.5. This is a very useful tool in proving local boundedness of weak supersolutions to the nonlocal p-Laplacian type Schrödinger equation.

Proof of Theorem 1.5. For simplicity, we assume that $x_0 = 0$. Let $w = (u-M)_+$ for $M \in [0,\infty)$ and take any nonnegative $\zeta \in C_c^\infty(B_r)$. We use $\varphi = w\zeta^p$ as a testing function in the weak formulation of the equation. Then we have

$$\langle u, \varphi\rangle_{H_p,K} + \int_{\mathbb{R}^n} |u(y)|^{p-2}u(y)\varphi(y)V(y)\,dy \le 0, \tag{4.1}$$

where $d_K(x,y) = K(x-y)\,dx\,dy$ and

$$\langle u, \varphi\rangle_{H_p,K} = \iint_{\mathbb{R}^n\times\mathbb{R}^n} H_p(u(x)-u(y))(\varphi(x)-\varphi(y))\,d_K(x,y) \quad \text{for } p \ge 1.$$

The first term in the left-hand side of the above inequality can be decomposed into two parts as follows:

$$\begin{aligned}\langle u, \varphi\rangle_{H_p,K} &= \iint_{B_r\times B_r} H_p(u(x)-u(y))(\varphi(x)-\varphi(y))\,d_K(x,y)\\ &\quad + 2\iint_{(\mathbb{R}^n\setminus B_r)\times B_r} H_p(u(x)-u(y))\,\varphi(x)\,d_K(x,y)\\ &:= I_1 + 2I_2.\end{aligned} \tag{4.2}$$

For estimating I_1, without loss of generality we assume that $u(x) \ge u(y)$. Then we first observe that $w(x) \ge w(y)$ and

$$H_p(u(x)-u(y))(\varphi(x)-\varphi(y)) \ge (w(x)-w(y))^{p-1}(\varphi(x)-\varphi(y)) \tag{4.3}$$

whenever $x, y \in B_r$; indeed, it can easily be checked by considering three possible occasions (i) $u(x), u(y) > M$, (ii) $u(x) > M$, $u(y) \le M$, and (iii) $u(y) \le u(x) \le M$. Also we observe that

$$\begin{aligned}|\zeta(x)w(x)-\zeta(y)w(y)|^p &\le 2^{p-1}|w(x)-w(y)|^p(\zeta^p(x)+\zeta^p(y))\\ &\quad + 2^{p-1}(w^p(x)+w^p(y))|\zeta(y)-\zeta(x)|^p.\end{aligned} \tag{4.4}$$

By (b) of Lemma 2.4, (4.3) and (4.4), we have

$$\begin{aligned}&H_p(u(x)-u(y))(\varphi(x)-\varphi(y))\\ &\ge \frac{1}{4}|w(x)-w(y)|^p(\zeta^p(x)+\zeta^p(y)) - d_p(w^p(x)+w^p(y))|\zeta(y)-\zeta(x)|^p\\ &\ge \frac{1}{2^{p+1}}|\zeta(x)w(x)-\zeta(y)w(y)|^p - \big(\frac{1}{4}+d_p\big)(w^p(x)+w^p(y))|\zeta(y)-\zeta(x)|^p.\end{aligned} \tag{4.5}$$

Thus it follows that

$$\begin{aligned}I_1 &\ge \frac{1}{2^{p+1}}\iint_{B_r\times B_r}|\zeta(x)w(x)-\zeta(y)w(y)|^p\,d_K(x,y)\\ &\quad - 2^p\big(\frac{1}{4}+d_p\big)\iint_{B_r\times B_r}(w^p(x)+w^p(y))|\zeta(y)-\zeta(x)|^p\,d_K(x,y).\end{aligned} \tag{4.6}$$

For the estimate of I_2, we note that

$$\begin{aligned}H_p(u(x)-u(y))\varphi(x) &\ge -(u(y)-u(x))_+^{p-1}(u(x)-M)_+\zeta^p(x)\\ &\ge -(u(y)-M)_+^{p-1}(u(x)-M)_+\zeta^p(x)\\ &= -w^{p-1}(y)w(x)\zeta^p(x)\end{aligned}$$

and thus we have

$$\begin{aligned} I_2 &\geq -\iint_{(\mathbb{R}^n\setminus B_r)\times B_r} w^{p-1}(y)w(x)\zeta^p(x)\,d_K(x,y) \\ &\geq -\Big(\sup_{x\in\operatorname{supp}(\zeta)}\int_{\mathbb{R}^n\setminus B_r} w^{p-1}(y)\,K(x-y)\,dy\Big)\int_{B_r} w(x)\zeta^p(x)\,dx. \end{aligned} \tag{4.7}$$

Finally, we claim that

$$\int_{\mathbb{R}^n} |u(y)|^{p-2}u(y)\varphi(y)V(y)\,dy \geq \int_{\mathbb{R}^n} w^p(y)\zeta^p(y)V(y)\,dy. \tag{4.8}$$

This is equivalent to the inequality

$$\begin{aligned} &\int_{\mathbb{R}^n}\big[|u(y)|^{p-2}u(y)w(y)-w^p(y)\big]\zeta^p(y)V_+(y)\,dy \\ &\geq \int_{\mathbb{R}^n}\big[|u(y)|^{p-2}u(y)w(y)-w^p(y)\big]\zeta^p(y)V_-(y)\,dy, \end{aligned}$$

whose proof is just a direct application of Lemma 3.2, Theorem 3.3 and Corollary 4.2. Hence the required inequality can be obtained from (4.1), (4.2), (4.6), (4.7) and (4.8). □

Next, we shall obtain the local boundedness of such weak subsolutions which is Theorem 1.1 and a relation between the nonlocal tail terms of the positive part and the negative part of weak solutions of the nonlocal p-Laplacian type Schrödinger equation (1.3) in the following theorems.

Proof of Theorem 1.1. Take any $\zeta\in C_c^\infty(B_r^0)$ such that $|\nabla\zeta|\leq c/r$ on $\mathbb{R}^n$. Let $w=(u-M)_+$ for $M\in(0,\infty)$. By Lemma 3.2, Theorem 1.5 and the mean value theorem, we have

$$\begin{aligned} &\iint_{B_r^0\times B_r^0}|\zeta(x)w(x)-\zeta(y)w(y)|^p\,d_K(x,y) \\ &\lesssim r^{p-ps}\|\nabla\zeta\|^p_{L^\infty(B_r^0)}\|w\|^p_{L^p(B_r^0)}+\mathcal{A}(w,\zeta,r,s)\,\|w\|_{L^1(B_r^0)} \end{aligned} \tag{4.9}$$

where

$$\mathcal{A}(w,\zeta,r,s)=\sup_{x\in\operatorname{supp}(\zeta)}\int_{\mathbb{R}^n\setminus B_r^0} w^{p-1}(y)\,K(x-y)\,dy.$$

Applying Proposition 2.1 to (4.9), we obtain

$$\Big(\fint_{B_r^0}|w\zeta|^{p\gamma}\,dx\Big)^{1/\gamma}\lesssim\big(r^{p-ps}\|\nabla\zeta\|^p_{L^\infty(B_r^0)}+r^{-ps}\big)r^{ps}\fint_{B_r^0}|w|^p\,dx+\mathcal{A}(w,\zeta,r,s)\,r^{ps}\fint_{B_r^0}w\,dx \tag{4.10}$$

where $\gamma=\frac{n}{n-ps}>1$. For $k=0,1,2,\dots$, we set

$$\begin{gathered} r_k=(1+2^{-k})r,\quad r_k^*=\frac{r_k+r_{k+1}}{2}, \\ M_k=M+(1-2^{-k})M_*,\quad M_k^*=\frac{M_k+M_{k+1}}{2}, \end{gathered}$$

$w_k=(u-M_k)_+$ and $w_k^*=(u-M_k^*)_+$ for a constant $M_*>0$ to be determined later. In (4.9), for $k=0,1,\dots$, we choose a function $\zeta_k\in C_c^\infty(B^0_{r_k^*})$ with $\zeta_k|_{B^0_{r_{k+1}}}\equiv 1$ such that $0\leq\zeta_k\leq 1$ and

$$|\nabla\zeta_k|\leq c\,2^{k+2}/r\quad\text{in }\mathbb{R}^n.$$

For $k=0,1,2,\dots$, we set

$$N_k=\Big(\fint_{B^0_{r_k}}|w_k|^p\,dx\Big)^{1/p}.$$

Since $w_k^*\geq w_{k+1}$ and

$$w_k^*(x)\geq M_{k+1}-M_k^*=2^{-k-2}M_*$$

whenever $u(x)\geq M_{k+1}$, we then have

$$N_{k+1}\lesssim\Big(\frac{1}{|B^0_{r_k}|}\int_{B^0_{r_{k+1}}}\frac{w^p_{k+1}(w_k^*)^{p(\gamma-1)}}{(M_{k+1}-M_k^*)^{p(\gamma-1)}}\,dx\Big)^{1/p}\lesssim\Big(\frac{2^k}{M_*}\Big)^{\gamma-1}\Big(\fint_{B^0_{r_k}}|w_k^*\,\zeta_k|^{p\gamma}\,dx\Big)^{1/p}. \tag{4.11}$$

Since $\zeta_k \in C_c^\infty(B^0_{r_k})$, $w_k^* \le w_0$ for all k and

$$|y-x| \ge |y-x_0| - |x-x_0| \ge \Big(1-\frac{r_k^*}{r_k}\Big)|y-x_0| \ge 2^{-k-2}|y-x_0|$$

for any $x \in B^0_{r_k^*}$ and $y \in \mathbb{R}^n \setminus B^0_{r_k}$, we easily obtain that

$$\mathcal{A}(w_k^*, \zeta_k, r_k, s) \le c\, 2^{k(n+ps)}\, r^{-ps}\, [\mathcal{T}_r(w_0; x_0)]^{p-1}. \tag{4.12}$$

Since $0 \le w_k^* \le w_k$ and $w_k(x) \ge M_k^* - M_k = 2^{-k-2}M_*$ if $u(x) \ge M_k^*$, it follows from (4.10)–(4.12) that

$$\begin{aligned}
\Big(\frac{2^k}{M_*}\Big)^{-\frac{p(\gamma-1)}{\gamma}} N_{k+1}^{\frac{p}{\gamma}} &\le c2^{pk} \fint_{B^0_{r_k}} |w_k^*|^p\,dx + c2^{k(n+ps)}\Big(\frac{r_k}{r}\Big)^{ps}[\mathcal{T}_r(w_0;x_0)]^{p-1} \fint_{B^0_{r_k}} w_k^*\,dx \\
&\le c2^{pk}N_k^p + c[\mathcal{T}_r(w_0;x_0)]^{p-1}\,2^{k(n+ps)} \fint_{B^0_{r_k}} \frac{w_k^* w_k^{p-1}}{(M_k^*-M_k)^{p-1}}\,dx \\
&\le c\Big(2^{pk} + 2^{k(n+ps)}\Big(\frac{2^k}{M_*}\Big)^{p-1}[\mathcal{T}_r(w_0;x_0)]^{p-1}\Big)\,N_k^p.
\end{aligned}$$

Taking M^* in the above so that $M_* \ge \delta\mathcal{T}_r(w_0;x_0)$ for $\delta \in (0,1]$, we obtain that

$$\frac{N_{k+1}}{M_*} \le d_0\, a^k\, \Big(\frac{N_k}{M_*}\Big)^{1+\eta}$$

where $d_0 = c^{\frac{\gamma}{p}}\delta^{-\frac{p-1}{p}\gamma} > 0$, $a = 2^{\frac{\gamma}{p}(n+2s+p-1)+\frac{ps}{n-ps}} > 1$ and $\eta = \gamma - 1 > 0$.

If $N_0 \le d_0^{-\frac{1}{\eta}} a^{-\frac{1}{\eta^2}} M_*$, then we set

$$M_* = \delta\,\mathcal{T}_r(w_0;x_0) + c_0\,\delta^{-\frac{(p-1)n}{sp^2}}\, a^{\frac{(n-ps)^2}{p^2s^2}} N_0$$

where $c_0 = c^{\frac{n}{sp^2}}$. By Lemma 2.3, we conclude that

$$\sup_{B^0_r} u \le M + M_* \le M + \delta\mathcal{T}_r(w_0;x_0) + c_0\delta^{-\frac{(p-1)n}{sp^2}}\, a^{\frac{(n-ps)^2}{p^2s^2}} \Big(\fint_{B^0_{2r}} (u-M)_+^p\,dx\Big)^{1/p}.$$

Hence, taking $M \downarrow 0$ in the above estimate, we obtain the required result. □

The third estimate is a lemma which furnishes a relation between the nonlocal tails of the positive and negative part of weak solutions to the nonlocal p-Laplacian type Schrödinger equation.

Lemma 4.3. *Let $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$, $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $s \in (0,1)$). If $u \in Y^{s,p}_g(\Omega)$ is a weak solution of the nonlocal p-Laplacian type Schrödinger equation* (1.3) *such that $u \ge 0$ in $B^0_R \subset \Omega$, then we have the estimate*

$$\mathcal{T}_r(u_+;x_0) \lesssim \big(1 + \|V_+\|_{L^q(\Omega)}\big) \sup_{B^0_r} u + \Big(\frac{r}{R}\Big)^{\frac{ps}{p-1}} \mathcal{T}_R(u_-;x_0) \quad \forall r \in (0,R).$$

Proof. Without loss of generality, we assume that $x_0 = 0$. Let $M = \sup_{B_r} u$ and $\varphi(x) = w(x)\zeta^p(x)$ where $w(x) = u(x) - 2M$ and $\zeta \in C_c^\infty(B_{3r/4})$ is a function satisfying that $\zeta|_{B_{r/2}} \equiv 1$, $0 \le \zeta \le 1$ and $|\nabla\zeta| \le c/r$ in $\mathbb{R}^n$. Then we have

$$\begin{aligned}
0 &= \iint_{B_r\times B_r} H_p(u(x)-u(y))(\varphi(x)-\varphi(y))\,d_K(x,y) \\
&\quad + 2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} H_p(u(x)-u(y))(u(x)-2M)\zeta^p(x)\,d_K(x,y) \\
&\quad + \int_{\mathbb{R}^n} V(x)|u(x)|^{p-2}u(x)(u(x)-2M)\zeta^p(x)\,dx \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{4.13}$$

Since

$$-2M \le w(x) := u(x) - 2M \le -M \quad \forall x \in B_r, \tag{4.14}$$

by Lemma 2.4(a) we have

$$H_p(w(x)-w(y))(w(x)\zeta^p(x)-w(y)\zeta^p(y)) \geq -c_p 4^p M^p(\zeta(x)-\zeta(y))^p$$

for any $x, y \in B_r$, it follows from simple calculation that

$$J_1 \geq -c_p 4^p M^p \iint_{B_r\times B_r} (\zeta(x)-\zeta(y))^p \, d_K(x,y) \gtrsim -M^p r^{-ps}|B_r|. \tag{4.15}$$

The lower estimate on J_2 can be split as follows

$$\begin{aligned} J_2 &\geq 4\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} M(u(y)-M)_+^{p-1}\,\zeta^p(x)\,d_K(x,y) \\ &\quad - 4M\int_{E_M}\int_{B_r}(u(x)-u(y))_+^{p-1}\,\zeta^p(x)\,d_K(x,y) \\ &:= J_{2,1}-J_{2,2}, \end{aligned}$$

where $E_M=\{y\in\mathbb{R}^n\setminus B_r : u(y)<M\}$. Since $(u(y)-M)_+ \geq u_+(y)-M$, it follows from (c) of Lemma 2.4 that

$$(u(y)-M)_+^{p-1} \geq b_p u_+^{p-1}(y) - M^{p-1}$$

where $b_p = \mathbf{1}_{(1,2]}(p) + 2^{-(p-1)}\mathbf{1}_{(2,\infty)}(p)$. Thus the lower estimate on $J_{2,1}$ can be obtained as

$$J_{2,1} \geq d_2 M r^{-ps}|B_r|\left[\mathcal{T}_r(u_+;0)\right]^{p-1} - d_3 M^p r^{-ps}|B_r| \tag{4.16}$$

with universal constants $d_2, d_3 > 0$. If $x\in B_r$ and $y \in E_M$, then we observe that

$$\begin{aligned} (u(x)-u(y))_+^{p-1} &\leq a_p\big(|u(x)-M|^{p-1}+|M-u(y)|^{p-1}\big) \\ &\leq a_p M^{p-1} + a_p(M+u_-(y)-u_+(y))^{p-1} \\ &\leq a_p M^{p-1} + a_p(M+u_-(y))^{p-1} \\ &\leq a_p(1+a_p)M^{p-1} + a_p^2\,[u_-(y)]^{p-1} \end{aligned}$$

where $a_p = \mathbf{1}_{(1,2]}(p) + 2^{p-1}\mathbf{1}_{(2,\infty)}(p)$, because $u_+(y) < M + u_-(y)$ for any $y\in E_M$. Since $u_-(y)=0$ for all $y \in B_R$, the upper estimate on $J_{2,2}$ can thus be achieved by

$$\begin{aligned} J_{2,2} &\leq 4a_p(1+a_p)M^p\int_{\mathbb{R}^n\setminus B_r}\int_{B_r}\zeta^p(x)\,d_K(x,y) \\ &\quad + 4a_p M\int_{\mathbb{R}^n\setminus B_R}\int_{B_r}[u_-(y)]^{p-1}\zeta^2(x)\,d_K(x,y) \\ &\leq d_4 M^p r^{-ps}|B_r| + d_5 M R^{-ps}|B_r|\left[\mathcal{T}_R(u_-;0)\right]^{p-1} \end{aligned} \tag{4.17}$$

with universal constants $d_4, d_5 > 0$. Thus, by (4.16) and (4.17), we have

$$J_2 \geq -dM^p r^{-ps}|B_r| - dMR^{-ps}|B_r|\left[\mathcal{T}_R(u_-;0)\right]^{p-1} + eMr^{-ps}|B_r|\left[\mathcal{T}_r(u_+;0)\right]^{p-1} \tag{4.18}$$

where $d, e > 0$ are some universal constants depending only on n, s, λ and Λ.

Finally, it follows from (4.14), Hölder's inequality and the fractional Sobolev inequality that

$$\begin{aligned} J_3 &\geq -2M^p\int_{\mathbb{R}^n} V_+(x)\zeta^p(x)\,dx \\ &\geq -2M^p\|V_+\|_{L^q(\Omega)}\Big(\int_\Omega \zeta^{pq'}(x)\,dx\Big)^{1/q'} \\ &\geq -2M^p\|V_+\|_{L^q(\Omega)}\Big(\int_\Omega \zeta^{\frac{pn}{n-ps}}(x)\,dx\Big)^{\frac{n-ps}{n}}|\Omega|^{\frac{1}{q'}-\frac{n-ps}{n}} \\ &\geq -2M^p\|V_+\|_{L^q(\Omega)}|\Omega|^{\frac{1}{q'}-\frac{n-ps}{n}}\big(r^{-ps}\|\zeta\|^p_{L^p(B_r)} + [\zeta]^p_{W^{s,p}(B_r)}\big) \\ &\gtrsim -\|V_+\|_{L^q(\Omega)}M^p r^{-ps}|B_r| \end{aligned} \tag{4.19}$$

where $q > \frac{n}{ps} > 1$ and $1 < q' < \frac{n}{n-ps}$ with $\frac{1}{q}+\frac{1}{q'}=1$. Hence the estimates (4.13), (4.15), (4.18), and (4.19) give the required estimate. □

Next we shall obtain the local boundedness for nonnegative weak solutions of the nonlocal equation (1.3) by employing Theorem 1.1 and Lemma 4.3. It is interesting that this estimate no longer depends on the nonlocal tail term, whose proof is pretty simple.

Theorem 4.4. *Let $V \in \mathcal{P}^{s,p}_q(\mathbb{R}^n)$ and $g \in W^{s,p}(\mathbb{R}^n)$ for $q > \frac{n}{ps} > 1$ ($p > 1$, $s \in (0,1)$). If $u \in Y^{s,p}_g(\Omega)$ is a nonnegative weak solution of the nonlocal p-Laplace type Schrödinger equation* (1.3)*, then we have the estimate*

$$\sup_{B^0_r} u \le C \Big(\fint_{B^0_{2r}} u^p(x)\,dx \Big)^{1/p}$$

for any $r > 0$ with $B^0_{2r} \subset \Omega$.

Proof. We choose some $\delta \in (0,1]$ so that $1 - \delta d_0 > 0$ and take any $r > 0$ with $B^0_{2r} \subset \Omega$ where

$$d_0 = c_0(1 + \|V_+\|_{L^q(\Omega)}) > 0$$

for the universal constant $c_0 > 0$ given in Lemma 4.3. Then it follows from Theorem 1.1 and Lemma 4.3 that

$$\sup_{B^0_r} u \le \delta\, d_0 \big[\sup_{B^0_r} u + \mathcal{T}_{2r}(u^-; x_0)) \big] + C_0\, \delta^{-\frac{(p-1)n}{sp^2}} \Big(\fint_{B^0_{2r}} u^p(x)\,dx \Big)^{1/p}$$

Since $\mathcal{T}_{2r}(u^-; x_0) = 0$, we can easily derive the required result by taking

$$C = \frac{C_0\, \delta^{-\frac{(p-1)n}{sp^2}}}{1 - \delta d_0}.$$

Hence we complete the proof. □

5. Logarithm of a weak solution is a locally bounded mean oscillation function

In this section, we prove that the logarithm of a weak supersolution to the nonlocal p-Laplacian type Schrödinger equation (1.3) is a function with locally bounded mean oscillation. To do this, the following tool which is called the *fractional Poincaré inequality* is very useful.

Let $n \ge 1$, $p \ge 1$, $s \in (0,1)$ and $sp < n$. For a ball $B \subset \mathbb{R}^n$, let u_B denote the average of $u \in W^{s,p}(B)$ over B, i.e.

$$u_B = \fint_B u(y)\,dy.$$

Then it was shown in [2, 31] that

$$\|u - u_B\|^p_{L^p(B)} \le \frac{c_{n,p}(1-s)|B|^{\frac{sp}{n}}}{(n-sp)^{p-1}} [u]^p_{W^{s,p}(B)} \tag{5.1}$$

with a universal constant $c_{n,p} > 0$ depending only on n and p, which is usually very useful in getting the logarithmic estimate of weak supersolutions. Of course, the logarithmic estimate could be obtained as in [10], but we will not apply their approach to achieve it. Our method to realize the logarithmic estimate is easier and more simple than their method.

Proof of Theorem 1.2. For simplicity, we set $x_0 = 0$. So, in what follows, we write $B_r := B^0_r$ for $r > 0$. Take any $r > 0$ so that $B_{2r} \subset B_R$ where $B_R \subset \Omega$. Consider a radial function $\zeta \in C^\infty_c(B_{3r/2})$ with values in $[0,1]$ such that $\zeta|_{B_r} \equiv 1$, $\zeta|_{\mathbb{R}^n \setminus B_{2r}} \equiv 0$ and

$$|\nabla \zeta| \lesssim \frac{1}{r} \quad \text{in } \mathbb{R}^n.$$

We use the function

$$\varphi(x) = \frac{\zeta^p(x)}{u_b^{p-1}(x)}$$

as a testing function to the nonlocal p-Laplacian type Schrödinger equation (1.3), where $u_b(x) = u(x) + b$. Then we have

$$
\begin{aligned}
0 &\le \iint_{\mathbb{R}^n\times\mathbb{R}^n} H_p(u_b(x)-u_b(y))(\varphi(x)-\varphi(y))\,d_K(x,y) + \int_{\mathbb{R}^n} V(x)H_p(u(x))\varphi(x)\,dx \\
&= \iint_{B_{2r}\times B_{2r}} H_p(u_b(x)-u_b(y))(\varphi(x)-\varphi(y))\,d_K(x,y) \\
&\quad + 2\int_{\mathbb{R}^n\setminus B_{2r}}\int_{B_{2r}} H_p(u_b(x)-u_b(y))\,\varphi(x)\,d_K(x,y) + \int_{\mathbb{R}^n} V(x)|u(x)|^{p-2}u(x)\varphi(x)\,dx \\
&:= H(u,\varphi) + I(u,\varphi) + J(u,\varphi).
\end{aligned} \tag{5.2}
$$

Without loss of generality, we may assume that $u_b(x) \ge u_b(y)$ for the estimate $H(u,\varphi)$; for, by symmetry, the other case $u_b(x) < u_b(y)$ can be treated in the exactly same way. Then we have two possible cases: (a) $u_b(x) \le 2u_b(y)$ and (b) $u_b(x) > 2u_b(y)$.

Case (a): $u_b(y) \le u_b(x) \le 2u_b(y)$**.** By the mean value theorem, we note that

$$
\begin{aligned}
&\zeta(x) \ge \zeta(y) \Rightarrow \zeta^p(x) - \zeta^p(y) = p\int_{\zeta(y)}^{\zeta(x)} \tau^{p-1}d\tau \le p\zeta^{p-1}(x)(\zeta(x)-\zeta(y)), \\
&\zeta(x) < \zeta(y) \Rightarrow \zeta^p(x) - \zeta^p(y) = p\int_{\zeta(x)}^{\zeta(y)} (-\tau^{p-1})d\tau \le p\zeta^{p-1}(x)(\zeta(x)-\zeta(y)).
\end{aligned} \tag{5.3}
$$

Then it follows that

$$
\begin{aligned}
\varphi(x)-\varphi(y) &= \frac{\zeta^p(x)-\zeta^p(y)}{u_b^{p-1}(y)} + \zeta^p(x)\Big(\frac{1}{u_b^{p-1}(x)} - \frac{1}{u_b^{p-1}(y)}\Big) \\
&\le \frac{p\zeta^{p-1}(x)(\zeta(x)-\zeta(y))}{u_b^{p-1}(y)} + \zeta^p(x)\int_0^1 \frac{d}{d\tau}\Big(\frac{1}{[\tau(u_b(x)-u_b(y))+u_b(y)]^{p-1}}\Big)\,d\tau \\
&\le \frac{p\zeta^{p-1}(x)(\zeta(x)-\zeta(y))}{u_b^{p-1}(y)} - (p-1)\frac{\zeta^p(x)(u_b(x)-u_b(y))}{u_b^p(x)} \\
&\le \frac{p\zeta^{p-1}(x)\,|\zeta(x)-\zeta(y)|\,u_b(y)}{u_b^p(y)} - \frac{(p-1)}{2^p}\frac{\zeta^p(x)(u_b(x)-u_b(y))}{u_b^p(y)}.
\end{aligned} \tag{5.4}
$$

Applying Young's inequality with indices $p' = \frac{p}{p-1}, p, \varepsilon$, it follows from (5.2) that

$$
\begin{aligned}
H(u,\varphi) &\le c_{n,p,s}\Lambda p \iint_{B_{2r}\times B_{2r}} \frac{\varepsilon(u_b(x)-u_b(y))^p\zeta^p(x) + c_\varepsilon|\zeta(x)-\zeta(y)|^p u_b^p(y)}{u_b^p(y)}\frac{dx\,dy}{|x-y|^{n+ps}} \\
&\quad - \frac{c_{n,p,s}\lambda(p-1)}{2^p}\iint_{B_{2r}\times B_{2r}} \frac{(u_b(x)-u_b(y))^p\zeta^p(x)}{u_b^p(y)}\frac{dx\,dy}{|x-y|^{n+ps}}.
\end{aligned} \tag{5.5}
$$

If we choose $\varepsilon = \frac{\lambda(p-1)}{2^{p+1}p\Lambda}$ in (5.5), then we have

$$
\begin{aligned}
H(u,\varphi) &\lesssim -\iint_{B_{2r}\times B_{2r}} \zeta^p(x)\frac{(u_b(x)-u_b(y))^p}{u_b^p(y)}\frac{dx\,dy}{|x-y|^{n+ps}} + \iint_{B_{2r}\times B_{2r}} \frac{|\zeta(x)-\zeta(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\
&\lesssim -\iint_{B_{2r}\times B_{2r}} \zeta^p(x)\frac{(u_b(x)-u_b(y))^p}{u_b^p(y)}\frac{dx\,dy}{|x-y|^{n+ps}} + r^{n-ps}.
\end{aligned} \tag{5.6}
$$

because $x, y \in B_{2r}$. Since $0 \le u_b(x) - u_b(y) \le u_b(y)$, we have

$$
\big|\ln u_b(x) - \ln u_b(y)\big|^p = \Big(\int_0^1 \frac{u_b(x)-u_b(y)}{\tau(u_b(x)-u_b(y))+u_b(y)}\,d\tau\Big)^p \le \frac{(u_b(x)-u_b(y))^p}{u_b^p(y)}. \tag{5.7}
$$

Thus by (5.6) and (5.7) we have

$$\begin{aligned} H(u,\varphi) &\lesssim -\iint_{B_{2r}\times B_{2r}} \zeta^p(x)\Big|\ln\Big(\frac{u_b(x)}{u_b(y)}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} + r^{n-ps} \\ &\lesssim -\iint_{B_r\times B_r} \Big|\ln\Big(\frac{u_b(x)}{u_b(y)}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} + r^{n-ps}. \end{aligned} \tag{5.8}$$

Case (b): $u_b(x) > 2u_b(y)$. It follows from the inequality in Lemma 2.5 with $\varepsilon = (2^{p-1}-1)/2$ that

$$\begin{aligned} \varphi(x)-\varphi(y) &= \frac{\zeta^p(x)-\zeta^p(y)}{u_b^{p-1}(x)} + \zeta^p(y)\Big(\frac{1}{u_b^{p-1}(x)} - \frac{1}{u_b^{p-1}(y)}\Big) \\ &\le \frac{\zeta^p(x)-\zeta^p(y)}{u_b^{p-1}(x)} + \zeta^p(y)\Big(\frac{1}{2^{p-1}u_b^{p-1}(y)} - \frac{1}{u_b^{p-1}(y)}\Big) \\ &\le \frac{\varepsilon\zeta^p(y) + c_\varepsilon|\zeta(x)-\zeta(y)|^p}{u_b^{p-1}(x)} - (1-2^{-p+1})\frac{\zeta^p(y)}{u_b^{p-1}(y)} \\ &\le \frac{c\,|\zeta(x)-\zeta(y)|^p}{u_b^{p-1}(x)} - \Big(\frac{1}{2}-\frac{1}{2^p}\Big)\frac{\zeta^p(y)}{u_b^{p-1}(y)}. \end{aligned} \tag{5.9}$$

Since $u_b(x) \ge u_b(x)-u_b(y) \ge u_b(y)$, by (5.2) and (5.9) we have

$$\begin{aligned} \frac{H(u,\varphi)}{c_{n,p,s}} &\le c\Lambda \iint_{B_{2r}\times B_{2r}} \frac{|\zeta(x)-\zeta(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\ &\quad - c\lambda\Big(\frac{1}{2}-\frac{1}{2^p}\Big)\iint_{B_{2r}\times B_{2r}} \zeta^p(y)\frac{(u_b(x)-u_b(y))^{p-1}}{u_b^{p-1}(y)}\frac{dx\,dy}{|x-y|^{n+ps}}. \end{aligned} \tag{5.10}$$

Since

$$(\ln t)^p \le c\,(t-1)^{p-1} \quad \text{for } t>2,$$

we have

$$\big|\ln u_b(x) - \ln u_b(y)\big|^p \le c\Big(\frac{u_b(x)-u_b(y)}{u_b(y)}\Big)^{p-1} = c\frac{(u_b(x)-u_b(y))^{p-1}}{u_b^{p-1}(y)}. \tag{5.11}$$

Combining (5.10) with (5.11), it follows that

$$\begin{aligned} H(u,\varphi) &\lesssim -\iint_{B_{2r}\times B_{2r}} \zeta^p(y)\Big|\ln\Big(\frac{u_b(x)}{u_b(y)}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} + \iint_{B_{2r}\times B_{2r}} \frac{|\zeta(x)-\zeta(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\ &\lesssim -\iint_{B_r\times B_r} \Big|\ln\Big(\frac{u_b(x)}{u_b(y)}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} + r^{n-ps}. \end{aligned} \tag{5.12}$$

Hence, by (5.8) and (5.12), we conclude that

$$H(u,\varphi) \lesssim -\iint_{B_r\times B_r} \Big|\ln\Big(\frac{u_b(x)}{u_b(y)}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} + r^{n-ps}. \tag{5.13}$$

For the estimate of $I(u,\varphi)$, we note that (i) $u(y) \ge 0$ and $u(x)-u(y) \le u(x)$ for $(x,y) \in B_{2r}\times(B_R\backslash B_{2r})$ and (ii) $(u(x)-u(y))_+ \le u(x) + u_-(y)$ for $(x,y)\in B_{2r}\times(\mathbb{R}^n\backslash B_R)$. Since ζ is supported in $B_{3r/2}$, the above observations (i) and (ii) yield that

$$\begin{aligned} I(u,\varphi) &\le 2\,c_{n,p,s}\Lambda \int_{B_{3r/2}}\int_{\mathbb{R}^n\backslash B_{2r}} \frac{1}{|y-x_0|^{n+ps}}\,dy\,dx \\ &\quad + 2\,c_{n,p,s}\Lambda \int_{B_{3r/2}}\int_{\mathbb{R}^n\backslash B_R} \frac{u_-^{p-1}(y)}{u_b^{p-1}(x)}\frac{dy\,dx}{|y-x_0|^{n+ps}} \\ &\lesssim r^{n-ps} + \frac{r^{n-ps}}{b^{p-1}}\Big(\frac{r}{R}\Big)^{ps}[\mathcal{T}_R(u_-;x_0)]^{p-1} \end{aligned} \tag{5.14}$$

because $|y-x| \ge |y-x_0| - |x-x_0| \ge |y-x_0|/4$ for all $(x,y) \in B_{3r/2} \times (\mathbb{R}^n \backslash B_{2r})$. Also, it follows from Hölder's inequality and Proposition 2.1 that

$$\begin{aligned} J(u,\varphi) &\le \int_{\mathbb{R}^n} V_+(x)\zeta^p(x)\,dx \\ &\le \|V_+\|_{L^\tau(\Omega)} \Big(\int_\Omega \zeta^{p\tau'}(x)\,dx\Big)^{\frac{1}{\tau'}} \\ &\le \|V_+\|_{L^q(\Omega)} \Big(\int_\Omega \zeta^{\frac{pn}{n-ps}}(x)\,dx\Big)^{\frac{n-ps}{n}} |\Omega|^{\frac{1}{q'}-\frac{n-ps}{n}} \\ &\le \|V_+\|_{L^q(\Omega)} |\Omega|^{\frac{1}{q'}-\frac{n-ps}{n}} \big(r^{-ps}\|\zeta\|^p_{L^p(B_r)} + [\zeta]^p_{W^{s,p}(B_r)}\big) \\ &\le \|V_+\|_{L^q(\Omega)} |\Omega|^{\frac{1}{q'}-\frac{n-ps}{n}} r^{n-ps} \lesssim \|V_+\|_{L^q(\Omega)}\, r^{n-ps} \end{aligned} \tag{5.15}$$

where $q > \frac{n}{ps} > 1$, $p > 1$ and $1 < q' < \frac{n}{n-ps}$ with

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

By (5.13), (5.14) and (5.15), we obtain that

$$\iint_{B_r\times B_r} \Big|\ln\Big(\frac{u(x)+b}{u(y)+b}\Big)\Big|^p \frac{dx\,dy}{|x-y|^{n+ps}} \lesssim \frac{r^{n-ps}}{b^{p-1}}\Big(\frac{r}{R}\Big)^{ps} [\mathcal{T}_R(u_-;x_0)]^{p-1} + r^{n-ps}\big(1 + \|V_+\|_{L^q(\Omega)}\big)$$

for all $b \in (0,1)$ and $r \in (0, R/2)$, since $x, y \in B_{2r}$. Hence we complete the proof by applying (1.1). □

We now introduce a sort of local BMO spaces on $B_R^0 \subset \Omega$, i.e. $\mathrm{BMO}^p(B_R^0)$ for $p > 0$. The norm $\|\cdot\|_{\mathrm{BMO}^p(B_R^0)}$ is defined by

$$\|f\|_{\mathrm{BMO}^p(B_R^0)} = \sup_{r\in(0,R/2)} \Big(\fint_{B_r^0} \big|f(y) - f_{B_r^0}\big|^p\,dy\Big)^{1/p}$$

and the space is given by

$$\mathrm{BMO}^p(B_R^0) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}^p(B_R^0)} < \infty\}.$$

If $p = 1$, we write $\mathrm{BMO}^p(B_R^0) = \mathrm{BMO}(B_R^0)$. Then we easily see that

$$\|\,|f|\,\|_{\mathrm{BMO}(B_R^0)} \le \|f\|_{\mathrm{BMO}(B_R^0)} \tag{5.16}$$

because $\big||f| - |f|_{B_r^0}\big| \le |f - f_{B_r^0}|$ and

$$\|f \pm g\|_{\mathrm{BMO}(B_R^0)} \le \|f\|_{\mathrm{BMO}(B_R^0)} + \|g\|_{\mathrm{BMO}(B_R^0)}. \tag{5.17}$$

We observe that

$$a \wedge b = \frac{a+b-|a-b|}{2} \quad \text{and} \quad a \vee b = \frac{a+b+|a-b|}{2}$$

for any $a, b \in \mathbb{R}$. This implies that

$$\begin{aligned} \|f \vee g\|_{\mathrm{BMO}(B_R^0)} &\le \|f\|_{\mathrm{BMO}(B_R^0)} + \|g\|_{\mathrm{BMO}(B_R^0)}, \\ \|f \wedge g\|_{\mathrm{BMO}(B_R^0)} &\le \|f\|_{\mathrm{BMO}(B_R^0)} + \|g\|_{\mathrm{BMO}(B_R^0)}. \end{aligned} \tag{5.18}$$

In addition, we can obtain the following *John-Nirenberg inequality* (as in [16]) by using the Calderón-Zygmund decomposition in harmonic analysis as follows; there exists some constants $b_1, b_2 > 0$ depending only on the dimension n such that

$$\big|\{x \in B_r^0 : |f(x) - f_{B_r^0}| > \lambda\}\big| \le b_1 e^{-(b_2/\|f\|_{\mathrm{BMO}(B_R^0)})\lambda} |B_r^0|$$

for any $f \in \mathrm{BMO}(B_r^0)$, every $r > 0$ with $B_{2r}^0 \subset B_R^0$ and $B_R^0 \subset \Omega$, and every $\lambda > 0$. By standard analysis, this inequality makes it possible to easily show that

$$\begin{aligned} &\text{If } f \in \mathrm{BMO}(B_R^0) \text{ for } B_R^0 \subset \Omega \text{ and } 1 < p < \infty, \\ &\text{then } \|\cdot\|_{\mathrm{BMO}(B_R^0)} \text{ is norm-equivalent to } \|\cdot\|_{\mathrm{BMO}^p(B_r^0)}. \end{aligned} \tag{5.19}$$

Lemma 5.1. *If we set*

$$v(x) = \ln\Big(\frac{a+b}{u(x)+b}\Big) \quad \text{for } a, b \in (0,1),$$

where the function u satisfies the same assumption as Theorem 1.2, then we have

$$\fint_{B_r^0} |v(x) - v_{B_r^0}|^p \, dx \lesssim \mathfrak{R}_{b,r,R}(u_-; x_0)$$

for any $r \in (0, R/2)$, where

$$\mathfrak{R}_{b,r,R}(u_-; x_0) = \frac{1}{b^{p-1}}\Big(\frac{r}{R}\Big)^{ps} [\mathcal{T}_R(u_-; x_0)]^{p-1} + \big(1 + \|V_+\|_{L^q(\Omega)}\big).$$

It follows from this that $v \in \mathrm{BMO}(B_R^0)$, and moreover

$$\|v\|_{\mathrm{BMO}(B_R^0)} \le [\mathfrak{R}_{b,r,R}(u_-; x_0)]^{1/p} < \infty.$$

Proof. The first part easily follows from the fractional Poincaré inequality (5.1) and Theorem 1.2. Also the second part can be shown by applying the Remark of Theorem 1.1 and Hölder's inequality because $u \in W^{s,p}(\mathbb{R}^n)$. □

Corollary 5.2. *If we set $\bar{v} = (v \vee 0) \wedge d$ for $d > 0$ with the same v as in Lemma 5.1, then*

$$\fint_{B_r^0} |\bar{v}(x) - \bar{v}_{B_r^0}|^p \, dx \lesssim \mathfrak{R}_{b,r,R}(u_-; x_0) \quad \forall r \in (0, R/2),$$

where

$$\mathfrak{R}_{b,r,R}(u_-; x_0) = \frac{1}{b^{p-1}}\Big(\frac{r}{R}\Big)^{ps} [\mathcal{T}_R(u_-; x_0)]^{p-1} + \big(1 + \|V_+\|_{L^q(\Omega)}\big).$$

It follows from this that $\bar{v} \in \mathrm{BMO}(B_R^0)$, and moreover

$$\|\bar{v}\|_{\mathrm{BMO}(B_R^0)} \le [\mathfrak{R}_{b,r,R}(u_-; x_0)]^{1/p} < \infty.$$

Proof. Without loss of generality, assume that $x_0 = 0$. By Lemma 5.1, we have

$$\fint_{B_r} \big||v(x)| - |v|_{B_r}\big|^p \, dx \lesssim \mathfrak{R}_{b,r,R}(u_-; 0),$$

because $\big||v(x)| - |v|_{B_r}\big| \le |v(x) - v_{B_r}|$. Then we can easily derive from (5.17) and (5.19) that

$$\fint_{B_r} |\bar{v}(x) - \bar{v}_{B_r}| \, dx \lesssim [\mathfrak{R}_{b,r,R}(u_-; 0)]^{1/p}.$$

Finally, the second part can be done as in Lemma 5.1. □

6. Interior Hölder regularity

In this section, we establish an interior Hölder regularity of weak solutions to the nonlocal p-Laplacian type Schrödinger equation (1.3) by applying the previous results obtained in Sections 4 and 5.

Proof of Theorem 1.3. Fix any $p > 1$ and $0 < s < 1$ and take any $R > 0$ with $B_R(x_0) \subset \Omega$. For simplicity, without loss of generality, we assume that $x_0 = 0$. For any $k \in \mathbb{N} \cup \{0\}$ and $r \in (0, R/2)$, we set

$$r_k = \frac{\delta^k r}{2} \text{ for } \delta \in \big(0, \big(\frac{1}{4}\big)^{\frac{p-1}{ps}}\big), \quad B_k = B_{r_k}, \quad B_k^* = B_{2r_k}.$$

Let us set

$$\Xi(r_0) = 2\,\mathcal{T}_{r/2}(u; 0) + 2C_0\Big(\fint_{B_r} |u|^p \, dx\Big)^{1/p}$$

where $C_0 > 1$ is the constant given in Theorem 1.1. For $k \in \mathbb{N} \cup \{0\}$, we set

$$\Xi(r_k) = \Big(\frac{r_k}{r_0}\Big)^{\eta} \Xi(r_0)$$

where $\eta \in (0, \frac{ps}{p-1})$ is some constant to be determined later. For our proof, if we set $v = u/\Xi(r_0)$, then we have only to prove that

$$\operatorname{osc}_{B_k} v \le \Theta(r_k) := \frac{\Xi(r_k)}{\Xi(r_0)} \tag{6.1}$$

for any $k \in \mathbb{N} \cup \{0\}$.

We proceed by using the mathematical induction. By the remark of Theorem 1.1, we see that

$$\operatorname{osc}_{B_0} v \le \Theta(r_0).$$

Assume that (6.1) holds for all $k \in \{0, 1, \dots, m\}$. Then we will show that (6.1) is still true for $m+1$. For this proof, we consider two possible cases; either

$$\frac{\big|B^*_{k+1} \cap \big\{v \ge \inf_{B_k} v + \Theta(r_k)/2\big\}\big|}{|B^*_{k+1}|} \ge \frac{1}{2} \tag{6.2}$$

or

$$\frac{\big|B^*_{k+1} \cap \big\{v \le \inf_{B_k} v + \Theta(r_k)/2\big\}\big|}{|B^*_{k+1}|} \ge \frac{1}{2}\,. \tag{6.3}$$

If (6.2) holds, then we set $v_k = v - \inf_{B_k} v$, and if (6.3) holds, then we set

$$v_k = \Theta(r_k) - \big(v - \inf_{B_k} v\big).$$

In these two cases, we see that $v_k \ge 0$ in B_k and

$$\frac{\big|B^*_{k+1} \cap \{v_k \ge \Theta(r_k)/2\}\big|}{|B^*_{k+1}|} \ge \frac{1}{2}\,. \tag{6.4}$$

Furthermore, v_k is a weak solution satisfying

$$\sup_{B_k} |v_m| \le 2\Theta(r_k) \tag{6.5}$$

for all $k \in \{0, 1, \dots, m\}$. Under the induction hypothesis, if $m \ge 1$, then we now claim that

$$[\mathcal{T}_{r_k}(v_k; 0)]^{p-1} \le c\,\delta^{-(p-1)\eta}\,[\Theta(r_k)]^{p-1} \tag{6.6}$$

for $k \in \{0, 1, \dots, m\}$. Indeed, by (6.5) and that

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_0} \frac{|v_m(x)|^{p-1}}{|x|^{n+ps}}\,dx &\lesssim r_0^{-ps} \sup_{B_0} |v|^{p-1} + r_0^{-ps}\,[\Theta(r_0)]^{p-1} + \int_{\mathbb{R}^n \setminus B_0} \frac{|v(x)|^{p-1}}{|x|^{n+ps}}\,dx \\ &\lesssim r_1^{-ps}\,[\Theta(r_0)]^{p-1},\end{aligned}$$

we have the estimate

$$\begin{aligned}[\mathcal{T}_{r_m}(v_m; 0)]^{p-1} &= c\,r_m^{ps} \sum_{k=1}^{m} \int_{B_{k-1} \setminus B_k} \frac{|v_m(x)|^{p-1}}{|x|^{n+ps}}\,dx + c\,r_m^{ps} \int_{\mathbb{R}^n \setminus B_0} \frac{|v_m(x)|^{p-1}}{|x|^{n+ps}}\,dx \\ &\lesssim r_m^{ps} \sum_{k=1}^{m} \Big[\sup_{B_{k-1}} |v_m|\Big]^{p-1} \int_{\mathbb{R}^n \setminus B_k} \frac{1}{|x|^{n+ps}}\,dx + r_m^{ps} \int_{\mathbb{R}^n \setminus B_0} \frac{|v_m(x)|^{p-1}}{|x|^{n+ps}}\,dx \\ &\le \sum_{k=1}^{m} \Big(\frac{r_m}{r_k}\Big)^{ps} [\Theta(r_{k-1})]^{p-1} \\ &= \sum_{k=1}^{m} \Big(\frac{r_m}{r_k}\Big)^{ps} \Big(\frac{r_{k-1}}{r_0}\Big)^{(p-1)\eta} \\ &= \Big(\frac{r_m}{r_0}\Big)^{(p-1)\eta} \sum_{k=1}^{m} \Big(\frac{r_m}{r_k}\Big)^{ps-(p-1)\eta} \Big(\frac{r_{k-1}}{r_k}\Big)^{(p-1)\eta} \\ &= [\Theta(r_m)]^{p-1} \delta^{-(p-1)\eta} \sum_{k=1}^{m} \delta^{(m-k)[ps-(p-1)\eta]}\end{aligned}$$

$$\leq \delta^{-(p-1)\eta}\frac{\delta^{(m-1)[ps-(p-1)\eta]}}{1-\delta^{-[ps-(p-1)\eta]}}[\Theta(r_m)]^{p-1} \lesssim \delta^{-(p-1)\eta}[\Theta(r_m)]^{p-1}.$$

For k and $d>0$, we set

$$\bar{v}_k = \Big[\ln\Big(\frac{\Theta(r_k)/2+b}{v_k+b}\Big)\vee 0\Big]\wedge d.$$

Applying Corollary 5.2 with $a=\Theta(r_k)/2$, $b\in(0,1)$ and $d>0$, we have

$$\fint_{B^*_{k+1}} |\bar{v}_k(x)-(\bar{v}_k)_{B^*_{k+1}}|^p\,dx \lesssim \mathfrak{R}_{b,r_{k+1},r_k}((v_k)_-;0) \tag{6.7}$$

where

$$\mathfrak{R}_{b,r,R}(u_-;x_0)=\frac{1}{b^{p-1}}\Big(\frac{r}{R}\Big)^{ps}[\mathcal{T}_R(u_-;x_0)]^{p-1}+\big(1+\|V_+\|_{L^q(\Omega)}\big).$$

If we set $b=\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)$ in (6.7), then by (5.19) and (6.6) we obtain that

$$\begin{aligned}\fint_{B^*_{k+1}} |\bar{v}_k(x)-(\bar{v}_k)_{B^*_{k+1}}|\,dx &\leq \Big(\fint_{B^*_{k+1}} |\bar{v}_k(x)-(\bar{v}_k)_{B^*_{k+1}}|^p\,dx\Big)^{1/p}\\ &\leq c\big(1+\|V_+\|_{L^q(\Omega)}\big)^{1/p}\end{aligned} \tag{6.8}$$

where $c>0$ is a constant depending only on n,s,p,η,λ and Λ. From (6.4), we can derive the estimate

$$\begin{aligned} d &= \frac{1}{|B^*_{k+1}\cap\{v_k\geq\Theta(r_k)/2\}|}\int_{B^*_{k+1}\cap\{v_k\geq\Theta(r_k)/2\}} d\,dx\\ &= \frac{1}{|B^*_{k+1}\cap\{v_k\geq\Theta(r_k)/2\}|}\int_{B^*_{k+1}\cap\{\bar{v}_k=0\}} d\,dx\\ &\leq \frac{2}{|B^*_{k+1}|}\int_{B^*_{k+1}}(d-\bar{v}_k)\,dx = 2\big(d-(\bar{v}_k)_{B^*_{k+1}}\big).\end{aligned} \tag{6.9}$$

The estimates (6.8) and (6.9) make it possible to obtain the estimate

$$\begin{aligned}\frac{|B^*_{k+1}\cap\{\bar{v}_k=d\}|}{|B^*_{k+1}|}\,d &\leq \frac{2}{|B^*_{k+1}|}\int_{B^*_{k+1}\cap\{\bar{v}_k=d\}}\big(d-(\bar{v}_k)_{B^*_{k+1}}\big)\,dx\\ &\leq \frac{2}{|B^*_{k+1}|}\int_{B^*_{k+1}\cap\{\bar{v}_k=d\}}\big(\bar{v}_k-(\bar{v}_k)_{B^*_{k+1}}\big)\,dx\\ &\lesssim \big(1+\|V_+\|_{L^q(\Omega)}\big)^{1/p}.\end{aligned} \tag{6.10}$$

We now set

$$d=d_*:=\ln\Big(\frac{\Theta(r_k)/2+\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)}{3\,\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)}\Big).$$

Then we see that $d_*\sim\ln(1/\delta)$. By (6.10), we have

$$\frac{|B^*_{k+1}\cap\{v_k\leq 2\,\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)\}|}{|B^*_{k+1}|}\leq\frac{c\,\big(1+\|V_+\|_{L^q(\Omega)}\big)^{1/p}}{d_*}\leq\frac{c_0\,\big(1+\|V_+\|_{L^q(\Omega)}\big)^{1/p}}{\ln(1/\delta)}. \tag{6.11}$$

Now we proceed the next step with a well-known iteration process as follows. For $i\in\mathbb{N}\cup\{0\}$, we set

$$\rho_i=(1+2^{-i})r_{k+1},\quad \bar{\rho}_i=\frac{\rho_i+\rho_{i+1}}{2},\quad B_i=B_{\rho_i},\quad \bar{B}_i=B_{\bar{\rho}_i}.$$

For $i\in\mathbb{N}\cup\{0\}$, we consider a function $\zeta_i\in C^\infty_c(B_{\bar{\rho}_i})$ with $\zeta_i|_{B_{\rho_{i+1}}}\equiv 1$ such that $0\leq\zeta_i\leq 1$ and $|\nabla\zeta_i|\leq c\rho_i^{-1}$ in $\mathbb{R}^n$. Moreover we set $d_i=(1+2^{-i})\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)$ and $w_i=(d_i-v_k)_+$ and

$$N_i=\frac{|B_i\cap\{v_k\leq d_i\}|}{|B_i|}=\frac{|B_i\cap\{w_i\geq 0\}|}{|B_i|}.$$

From (6.11), we see that

$$N_0 \le \frac{c_0\big(1+\|V_+\|_{L^q(\Omega)}\big)^{1/p}}{\ln(1/\delta)}. \tag{6.12}$$

By Theorem 1.5, we have

$$\begin{aligned}
[w_i\zeta_i]^p_{W^{s,p}(B_{\rho_i})} &\lesssim \iint_{B_{\rho_i}\times B_{\rho_i}} [w_i(x)\vee w_i(y)]^p|\zeta_i(x)-\zeta_i(y)|^p\, d_K(x,y)\\
&\quad + \Big(\sup_{x\in \operatorname{supp}(\zeta_i)} \int_{\mathbb{R}^n\setminus B_{\rho_i}} [w_i(y)]^{p-1}\, K(x-y)\, dy\Big)\|w_i\zeta_i^p\|_{L^1(B_i)}\\
&:= A(\rho_i,w_i,\zeta_i)+B(\rho_i,w_i,\zeta_i).
\end{aligned}$$

Then we have the estimate

$$\begin{aligned}
A(\rho_i,w_i,\zeta_i) &\lesssim d_i^p \int_{B_{\rho_i}}\int_{B_{\rho_i}\cap\{v_k\le d_i\}} \frac{\sup_{\mathbb{R}^n}|\nabla\zeta_i|^p}{|x-y|^{n+ps-p}}\, dx\, dy\\
&\lesssim d_i^p\Big(\frac{1}{\rho_i}\Big)^p \int_{B_{\rho_i}\cap\{v_k\le d_i\}}\int_{B_{2\rho_i}} \frac{1}{|y|^{n+ps-p}}\, dy\, dx\\
&\lesssim d_i^p\, \rho_i^{-ps}\, |B_i\cap\{v_k\le d_i\}|.
\end{aligned} \tag{6.13}$$

By the fact that

$$|y-x|\ge |y|-|x|\ge \Big(1-\frac{\bar\rho_i}{\rho_i}\Big)|y|\ge 2^{-i-2}|y|$$

for all $y\in\mathbb{R}^n\backslash B_{\rho_i}$ and $x\in B_{\bar\rho_i}$, we obtain that

$$\begin{aligned}
B(\rho_i,w_i,\zeta_i) &\lesssim d_i 2^{i(n+ps)}|B_i\cap\{v_k\le d_i\}| \int_{\mathbb{R}^n\setminus B_{\rho_i}} \frac{|w_i(y)|^{p-1}}{|y|^{n+ps}}\, dy\\
&\lesssim 2^{i(n+ps)} d_i\rho_i^{-ps}|B_i\cap\{v_k\le d_i\}|\,[\mathcal{T}_{r_{k+1}}(w_i;0)]^{p-1}.
\end{aligned} \tag{6.14}$$

Thus it follows from (6.13) and (6.14) that

$$[w_i\zeta_i]^p_{W^{s,p}(B_{\rho_i})} \lesssim \big(d_i^p + 2^{i(n+ps)}\, d_i[\mathcal{T}_{r_{k+1}}(w_i;0)]^{p-1}\big)\rho_i^{-ps}|B_i\cap\{v_k\le d_i\}|. \tag{6.15}$$

From (6.6) and that $w_i\le 2\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)$ in B_k and $w_i\le |v_k|+2\delta^{\frac{ps}{p-1}-\eta}\Theta(r_k)$ in $\mathbb{R}^n$, we can derive that

$$\begin{aligned}
[\mathcal{T}_{r_{k+1}}(w_i;0)]^{p-1} &\lesssim r_{k+1}^{ps}\int_{B_k\setminus B_{k+1}} \frac{|w_i(y)|^{p-1}}{|y|^{n+ps}}\, dy + \Big(\frac{r_{k+1}}{r_k}\Big)^{ps}[\mathcal{T}_{r_k}(w_i;0)]^{p-1}\\
&\lesssim \delta^{ps-(p-1)\eta}\,[\Theta(r_k)]^{p-1} + \delta^{ps}[\mathcal{T}_{r_k}(v_k;0)]^{p-1} \lesssim d_i^{p-1}.
\end{aligned} \tag{6.16}$$

Thus by (6.15) and (6.16), we have

$$[w_i\zeta_i]^p_{W^{s,p}(B_{\rho_i})} \lesssim 2^{i(n+ps)}\, d_i^p\rho_i^{-ps}|B_i\cap\{v_k\le d_i\}|. \tag{6.17}$$

By applying (6.17) and the fractional Sobolev's inequality with exponent

$$\gamma = \frac{n}{n-ps},$$

we can deduce the inequalities

$$\begin{aligned}
\Big(\int_{B_{i+1}} |w_i|^{p\gamma}\, dx\Big)^{1/\gamma} &\le \Big(\int_{B_i} |w_i\zeta_i|^{\frac{pn}{n-ps}}\, dx\Big)^{\frac{n-ps}{n}}\\
&\lesssim [w_i\zeta_i]^p_{W^{s,p}(B_{\rho_i})} + \rho_i^{-ps}\|w_i\zeta_i\|^p_{L^p(B_{\rho_i})}\\
&\lesssim 2^{i(n+ps)}\, d_i^p\, \rho_i^{-ps}\, |B_i\cap\{v_k\le d_i\}|.
\end{aligned} \tag{6.18}$$

Since $|B_{i+1}|\sim \rho_i^n\sim |B_i|$ and

$$w_i = (d_i-v_k)_+ \ge (d_i-d_{i+1})\,\mathbf{1}_{\{v_k\le d_{i+1}\}} \ge 2^{-i-2}d_i\,\mathbf{1}_{\{v_k\le d_{i+1}\}},$$

estimate (6.18) yields that

$$(d_i - d_{i+1})^p \Big(\frac{|B_{i+1} \cap \{v_k \le d_{i+1}|}{|B_{i+1}|} \Big)^{1/\gamma} \le \frac{\rho_i^{ps}}{|B_i|} \Big(\int_{B_{i+1}} |w_i|^{p\gamma} \, dx \, dt \Big)^{1/\gamma} \lesssim 2^{i(n+ps)} \, d_i^p \frac{|B_i \cap \{v_k \le d_i\}|}{|B_i|},$$

which gives

$$N_{i+1}^{1/\gamma} \le c \frac{2^{i(n+ps)} \, d_i^p}{(d_i - d_{i+1})^p} N_i \le c 2^{i(n+ps+p)} N_i.$$

This leads us to

$$N_{i+1} \le c_1 2^{i\gamma(n+ps+p)} N_i^{1+\frac{ps}{n-ps}}.$$

If we could show that

$$N_0 = \frac{|B_0 \cap \{v_k \le 2\delta^{\frac{ps}{p-1}-\eta} \Theta(r_k)\}|}{|B_0|} \le c_1^{-\frac{n-ps}{ps}} 2^{-\frac{\gamma(n-ps)^2(n+ps+p)}{p^2 s^2}} := c_*, \tag{6.19}$$

then by Lemma 2.3 we conclude that $\lim_{i\to\infty} N_i = 0$, i.e.

$$\inf_{B_{k+1}} v_k > \delta^{\frac{ps}{p-1}-\eta} \Theta(r_k).$$

To guarantee (6.19), by (6.12) we observe that

$$c_* \ge \frac{c_0 \big(1 + \|V_+\|_{L^q(\Omega)}\big)^{1/p}}{\ln(1/\delta)} \;\Leftrightarrow\; \delta \le e^{-(c_0/c_*)(1+\|V_+\|_{L^q(\Omega)})^{1/p}},$$

and so we choose $\delta > 0$ as

$$\delta = e^{-(c_0/c_*)(1+\|V_+\|_{L^q(\Omega)})^{1/p}} \wedge \Big(\frac{1}{4}\Big)^{\frac{p-1}{ps}}.$$

If $v_k = v - \inf_{B_k} v$, then by (6.1) we have

$$\operatorname{osc}_{B_{k+1}} v = \operatorname{osc}_{B_{k+1}} v_k \le \operatorname{osc}_{B_k} v - \inf_{B_{k+1}} v_k \le (1 - \delta^{\frac{ps}{p-1}-\eta}) \Theta(r_k). \tag{6.20}$$

If $v_k = \Theta(r_k) - (v - \inf_{B_k} v)$, then we have

$$\operatorname{osc}_{B_{k+1}} v = \operatorname{osc}_{B_{k+1}} v_k = \Theta(r_k) - \inf_{B_{k+1}} v + \inf_{B_k} v - \inf_{B_{k+1}} v_k \le (1 - \delta^{\frac{ps}{p-1}-\eta}) \Theta(r_k). \tag{6.21}$$

From (6.20) and (6.21), we obtain that

$$\operatorname{osc}_{B_{k+1}} v \le (1-\delta^{\frac{ps}{p-1}-\eta}) \Theta(r_k) = (1-\delta^{\frac{ps}{p-1}-\eta}) \Big(\frac{r_k}{r_{k+1}}\Big)^\eta \Theta(r_{k+1}) = (1-\delta^{\frac{ps}{p-1}-\eta}) \delta^{-\eta} \Theta(r_{k+1}). \tag{6.22}$$

To find η so that $(1 - \delta^{\frac{ps}{p-1}-\eta}) \delta^{-\eta} \le 1$, we consider the function

$$\xi(\eta) = \delta^\eta + \delta^{\frac{ps}{p-1}-\eta}.$$

We note that

$$\xi'(\eta) = \delta^\eta \ln \delta \, \big(1 - \delta^{\frac{ps}{p-1}-2\eta}\big) = 0 \;\Leftrightarrow\; \eta = \frac{ps}{2(p-1)} := \eta_{p,s}\,,$$

$\xi'(\eta) < 0$ for $\eta < \eta_{p,s}$ and $\xi'(\eta) > 0$ for $\eta > \eta_{p,s}$. These facts imply that the graph of ξ is going down from the point $(0, 1 + \delta^{2\eta_{p,s}})$ to the point $(\eta_{p,s}, 1 + 2\delta^{\eta_{p,s}})$ and is going up the point $(2\eta_{p,s}, 1 + \delta^{2\eta_{p,s}})$ right after that. Since we see that

$$2\delta^{\eta_{p,s}} < 1 + \delta^{2\eta_{p,s}} \quad \text{and} \quad 2\delta^{\eta_{p,s}} < 1,$$

we can find exactly two η's inside $(0, \frac{ps}{p-1})$ so that

$$\delta^\eta + \delta^{\frac{ps}{p-1}-\eta} = 1. \tag{6.23}$$

If we set $Y = \delta^\eta$, then the above equation (6.23) will be transformed into

$$\frac{Y^2 - Y + \delta^{\frac{ps}{p-1}}}{Y} = 0$$

and its solutions are

$$\delta^\eta = Y = \frac{1 \pm \sqrt{1 - 4\delta^{\frac{ps}{p-1}}}}{2} \in (0, 1),$$

because $\delta < (1/4)^{\frac{p-1}{ps}}$. Hence it turns out that the solutions of (6.23) are

$$\eta_0^\pm = \frac{\ln\left(\frac{1\pm\sqrt{1-4\delta^{\frac{ps}{p-1}}}}{2}\right)}{\ln\delta}.$$

Then we see that $\xi(\eta) \geq 1$, i.e. $(1-\delta^{\frac{ps}{p-1}-\eta})\delta^{-\eta} \leq 1$ for all $\eta \in (0,\eta_0^-] \cup [\eta_0^+, 2\eta_{p,s})$. Thus, if $\eta \in (0,\eta_0^-] \cup [\eta_0^+, 2\eta_{p,s})$, then by (6.22) we conclude that

$$\operatorname{osc}_{B_{k+1}} v \leq \Theta(r_{k+1}).$$

Therefore we complete the proof. □

Acknowledgements. Yong-Cheol Kim was supported by School of Education, Korea University Grant in 2024.

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