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Reimagining Fixed Points: Exploring the Role of Occasionally Weakly Compatible Mappings in Fuzzy Metrics

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Abstract: In this work, we revisit the concept of fixed points by exploring the role of occasionally weakly compatible (OWC) mappings within the structure of fuzzy metric spaces. Building on the classical fixed point theory, we investigate new conditions under which tripled fixed points exist and are unique. By employing the framework of fuzzy metrics and leveraging the flexibility of OWC mappings, we establish a generalized tripled fixed point theorem that extends several known results. We also explore the concept of tripled fixed points for occasionally weakly compatible mappings within the framework of fuzzy metric spaces. We establish several novel tripled fixed-point theorems that extend existing results in this area. Additionally, to validate the applicability of our theorems, we provide detailed illustrative examples that demonstrate the effectiveness and relevance of the established results in fuzzy metric settings. These findings contribute to the broader understanding of fixed-point theory in fuzzy environments and open new avenues for future research in generalized metric spaces and their applications.

Keywords: Occasionally weakly compatible mappings; tripled fixed point; fuzzy metric space.

2000 Mathematics Subject Classification: 47H10; 54H25.

1 Introduction

Zadeh [14] defined Fuzzy sets. Kramosil and Michalek [7] introduced Fuzzy metric space, George and Veermani [3] modified the notion and gave a new notion with the help of continuous t-norms of fuzzy metric spaces. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions.

Fixed point theorems, involving four self-maps, began with the assumption that they are commuted. Sessa [10] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [4] and then pairwise weakly compatible maps [5]. Jungck and Rhoades [6] introduced the concept of occasionally weakly compatible maps (owc). Some of the work cited in references [1], [2], [8], [9], [11], [12] and [13] is also significant.

In this work, we introduce tripled fixed point for occasionally weakly compatible mappings in fuzzy metric space and also proved some tripled fixed-point theorem for occasionally weakly compatible mappings in fuzzy metric space. Our results extend and some recent results in literature. Some illustrative examples are offered to support our theorems. We have also given the diagram to demonstrate the viability and applicability of the result.

2 Preliminary Notes

Definition 2.1 A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2 A binary operation $*: [0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if $*$ is satisfying conditions:

(i) $*$ is an commutative and associative;

(ii) $*$is continuous;

(iii) $a * 1 = a$ for all a$\in$ [0,1];

(iv) $a * b \le c * d$ whenever $a \le c\, and\ b \le d$ and $a, b, c, d \in [0,1]$.

Definition 2.3 A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set ,$*$ is a continuous $t - norm$ and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$,

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1\ if\ and\ only\ if\ x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(v) $M(x, y, \cdot): (0,\infty) \rightarrow (0,1]$ is continuous.

Then M is called a $fuzzy\ metric$ on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.4 Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space.

Lemma 2.5 Let (X, M,$*$) be a fuzzy metric space. If there exists $q \in (0,1)$ such that M(x, y, qt) $\geq$ M(x ,y ,t) for all x, y $\in$ X and t>0, then x = y.

Definition 2.6 Let X be a non-empty set. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a given mapping $f: X \times X \times X \rightarrow X$ if $x = f(x, y, z), y = f(y, z, x), z = f(z, x, y)$.

Example 2.6.1 Let $X = R$ and $S: X \times X \times X \rightarrow X$is defined as $S(x, y, z) = x + xy + xz$ then

$$S(1,0,0) = 1, S(0,1,0) = 0, S(0,0,1) = 0$$

Then (1,0,0),(0,1,0) and (0,0,1) are tripled fixed point.

Definition 2.7 An element $x \in X$ is called a common tripled fixed point of the mappings $f: X \times X \times X \to X$ and $g: X \to X$ if

$x = f(x,x,x) = g(x)$.

Definition 2.8 An element $(x,y,z) \in X \times X \times X$ is called a tripled coincidence point of a mapping $f: X \times X \times X \to X$ and $g: X \to X$ if $gx = f(x,y,z), gy = f(y,z,x), gz = f(z,x,y)$ in this case (gx, gy, gz) is called a tripled point of coincidence.

Definition 2.9 let $f : X \times X \times X \to X$ and $g : X \to X$ be two mappings. f *and* g are said to be weakly compatible if they commute at their a tripled coincidence point, i.e., if (x,y,z) is a tripled coincidence point of g *and* f, then $gf(x,y,z) = f(g(x), g(y), g(z))$.

Example 2.9.1 Let $S: X \times X \times X \to X$& $T: X \to X$ be defined by

$$S(x,y,z) = x + xy + xz$$

$$T(x) = \begin{cases} 0, & \text{if } x \neq 1; \\ 1, & \text{if} x = 1. \end{cases}$$

Here, (1,0,0), (0,1,0) and (0,0,1) are triple coincidence points of S and T at which (S,T) commute. So, S and T are weakly compatible.

Example 2.9.2 Let $S: X \times X \times X \to X$& $T: X \to X$ be defined by

$$S(x,y,z) = xyz$$

$$T(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1; \\ 1, & \text{if} x \geq 1. \end{cases}$$

So, S and T are weakly compatible at (0, 0, 0) and (1, 1, 1).

Definition 2.10 The mappings$f: X \times X \times X \to X$ and $g: X \to X$ of a set X are occasionally weakly compatible(owc)iff there is a point $(x,y,z) \in X \times X \times X$ which is a coincidence point of f and g at which f and g commute i.e. (f,g) are occasionally weakly compatible maps iff $f(x,y,z) = g(x), f(y,z,x) = g(y), f(z,x,y) = gz$

implies $gf(x,y,z) = f(gx,gy,gz), gf(y,z,x) = f(gy,gz,gx), gf(z,x,y) = f(gz,gx,gy)$ for $(x,y,z) \in X \times X \times X$.

Example 2.10.1 Let $(X, \mathcal{F}, *)$ be a fuzzy metric space, where $X = [0,1]$ with $a * b = \min\{a,b\}$and

$$M(x,y,t) = \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Let $f: X \times X \times X \to X$& $g: X \to X$ be defined by

$$f(x,y,z) = \frac{2x + 2y + z}{2}$$

$$g(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ \frac{5}{2}, & \text{if} x \ge 1. \end{cases}$$

Here, (0,0,0) and (1,1,1) are two coincidence points of f and g. That is$f(0,0,0) = 0 = g(0), f(1,1,1) = 1 = g(1)butgf(0,0,0) = 0 = f(g0, g0, g0),\ gf(1,1,1) \neq f(g1, g1, g1)$. Thus f and g are owc but not weakly compatible. The Mesh Diagram for the given example is shown in Fig [2.1].

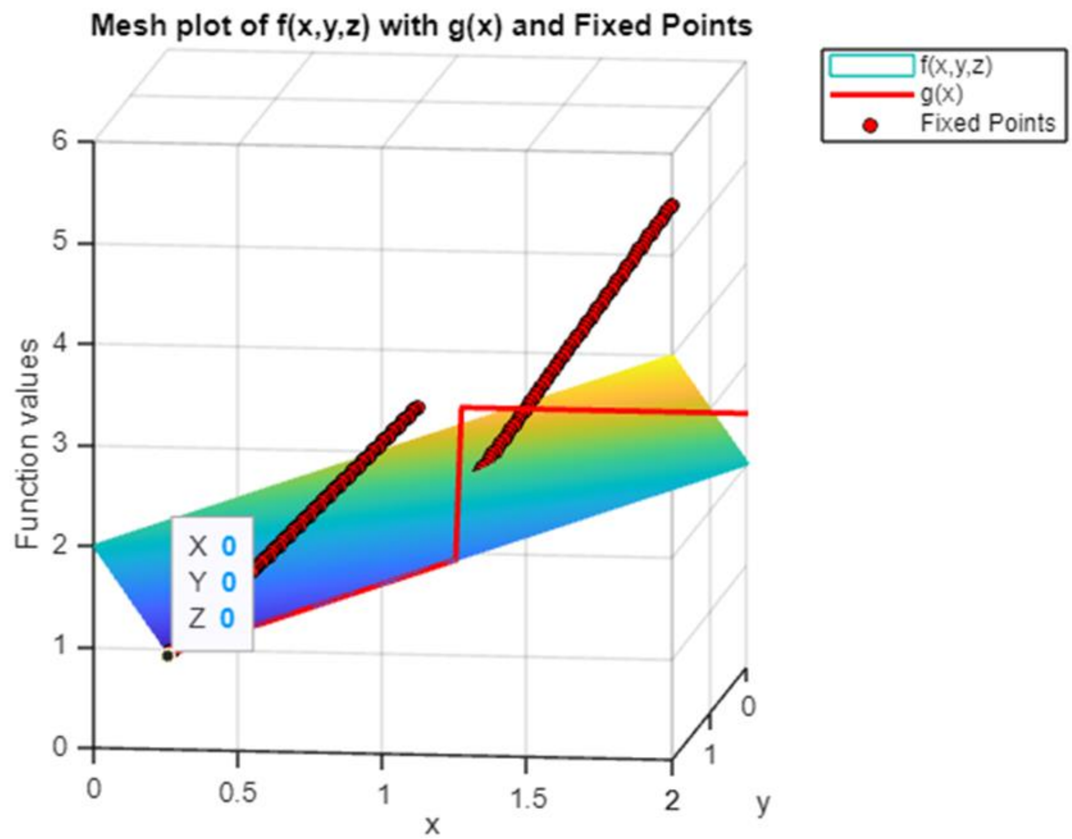

Fig [2.1]

3 Main Results

Theorem: 3.1 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M^P(A(x,y,z), B(u,v,w), qt) \geq$

$$\varphi\left[\begin{array}{c} a\, M^p(Sx, Tu, t) + (1-a) \min\{M^p(A(x,y,z), Sx, t), \\ M^p(B(u,v,w), Tu, t), M^{\frac{p}{2}}(A(x,y,z), Tu, t). M^{\frac{p}{2}}(B(u,v,w), Sx, t) \\ , \frac{1}{2}[M^p(A(x,y,z), Sx, t) + M^p(B(u,v,w), Tu, t)]\} \end{array}\right]$$

for all $x, y, z, u, v, w \in X$, $0 \leq a \leq 1, p \geq 1$ and $\varphi: R^+ \to R^+$ such that φ is upper semi continuous, non- increasing and $\varphi(t) > t$ for any t>0.

(ii) $y = B(x, y, z)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that $A(x,x,x) = T(x) = B(x,x,x) = S(x) = x$.

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a,b,c,a',b',c' in X such that

$A(a,b,c) = Sa, A(b,c,a) = Sb, A(c,a,b) = Sc$ and

$B(a',b',c') = Ta', B(b',c',a') = Tb', B(c',a',b') = Tc'$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$$M^p(A(a,b,c),B(a',b',c'),qt)$$
$$\geq \varphi\begin{bmatrix} a\, M^p(Sa,Ta',t) + (1-a)\min\{M^p(A(a,b,c),Sa,t), \\ M^p(B(a',b',c'),Ta',t), M^{\frac{p}{2}}(A(a,b,c),Ta',t).M^{\frac{p}{2}}(B(a',b',c'),Sa,t) \\ ,\frac{1}{2}[M^p(A(a,b,c),Sa,t) + M^p(B(a',b',c'),Ta',t)]\} \end{bmatrix}$$

$$= \varphi(a\, M^p(Sa,Ta',t) + (1-a)\min\{1,1,\ M^p(Sa,Ta',t),1\}$$
$$= \varphi(a\, M^p(Sa,Ta',t) + (1-a)\min M^p(Sa,Ta',t)$$
$$> M(Sa,Ta',t)$$
$$\Rightarrow Sa = Ta'$$

Therefore $A(a,b,c) = Ta' = Sa = B(a',b',c')$

Similarly $A(b,c,a) = Tb' = Sb = B(b',c',a')$

$A(c,a,b) = Tc' = Sc = B(c',a',b')$

Thus the pairs (A,S) and (B,T) have common coincidence points.

Let $A(a,b,c) = Ta' = Sa = B(a',b',c') = x$

and $A(b,c,a) = Tb' = Sb = B(b',c',a') = y$

$A(c,a,b) = Tc' = Sc = B(c',a',b') = z$

Since (A,S) and (B,T) are owc

So $Sx = SA(a,b,c) = A(Sa,Sb,Sc) = A(x,y,z)$

and $Sy = SA(b,c,a) = A(Sb,Sc,Sa) = A(y,z,x)$

$Sz = SA(c,a,b) = A(Sc,Sa,Sb) = A(z,x,y)$

Also $Tx = TB(a',b',c') = B(Ta',Tb',Tc') = B(x,y,z)$

$$Tz = TB(c',a',b') = B(Tc',Ta',Tb') = B(z,x,y)$$

Next we show that $x = y = z$, for this

putting $x = a$, $y = b$, $z = c$, $u = b'$, $v = c'$, $w = a'$ in (i),

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$$M^P(A(a,b,c),B(b',c',a'),qt) \geq \varphi\left[\begin{array}{c} a\,M^p(Sa,Tb',t)+(1-a)\min\{M^p(A(a,b,c),Sa,t), \\ M^p(B(b',c',a'),Tb',t),M^{\frac{p}{2}}(A(a,b,c),Tb',t).M^{\frac{p}{2}}(B(b',c',a'),Sa,t) \\ ,\frac{1}{2}[M^p(A(a,b,c),Sa,t)+M^p(B(b',c',a'),Tb',t)]\} \end{array}\right]$$

$$M^p(x,y,qt) \geq \varphi(a\ \ M^p(x,y,t)+(1-a)min\{1,1,M^p(x,y,t),1\}$$
$$= \varphi M^p(x,y,t)$$
$$> M(x,y,t)$$
$$\Longrightarrow x=y$$

Again putting $x=a$, $y=b$, $z=c$, $u=c'$, $v=a'$,$w=b'$ in (i),

$$M^P(A(a,b,c),B(c',a',b'),qt) \geq \varphi\left[\begin{array}{c} a\,M^p(Sa,Tc',t)+(1-a)\min\{M^p(A(a,b,c),Sa,t), \\ M^p(B(c',a',b'),Tc',t),M^{\frac{p}{2}}(A(a,b,c),Tc',t).M^{\frac{p}{2}}(B(c',a',b'),Sa,t) \\ ,\frac{1}{2}[M^p(A(a,b,c),Sa,t)+M^p(B(c',a',b'),Tc',t)]\} \end{array}\right]$$

$$M^p(x,z,qt) \geq \varphi(a\ \ M^p(x,z,t)+(1-a)min\{1,1,M^p(x,z,t),1\}$$
$$= \varphi M^p(x,z,t)$$
$$> M^p(x,z,t)$$
$$\Longrightarrow x=z$$
$$\Rightarrow x=y=z$$

Now we prove that $Sx=Tx$

$$M^P(A(x,y,z),B(y,z,x),qt) \geq \varphi\left[\begin{array}{c} a\,M^p(Sx,Ty,t)+(1-a)\min\{M^p(A(x,y,z),Sx,t), \\ M^p(B(y,z,x),Ty,t),M^{\frac{p}{2}}(A(x,y,z),Ty,t).M^{\frac{p}{2}}(B(y,z,x),Sx,t) \\ ,\frac{1}{2}[M^p(A(x,y,z),Sx,t)+M^p(B(y,z,x),Ty,t)]\} \end{array}\right]$$

$$M^p(Sx,Ty,qt) \geq \varphi(a\ \ M^p(Sx,Ty,t)+(1-a)min\{1,1,M^p(Sx,Ty,t),1\}$$
$$= \varphi M^p(Sx,Ty,t)$$
$$> M^p(Sx,Ty,t)$$
$$\Longrightarrow Sx=Tx$$
$$Sx=Tx=B(x,y,z)=A(x,y,z)$$

or $Sx=Tx=B(x,x,x)=A(x,x,x)$

Also by condition (ii) we have, $x = B(x,x,x)$

Thus $A(x,x,x) = T(x) = B(x,x,x) = S(x) = x$ ■

Example 3.1.1 Let $X = [0,1]$ with the metric d defined by $d(x,y) = |x - y|$ and for each $t \in [0,1]$, define

$$M(x,y,t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, *)$ be a fuzzy metric space, with $a * b = \min\{a, b\}$. Let $S, T: X \to X$ and $A, B: X \times X \times X \to X$ defined by

$$A(x,y,z) = \frac{2x + y + z}{2} \quad S(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ \dfrac{5}{2}, & \text{if } x \ge 1. \end{cases}$$

$$B(x,y,z) = y \quad T(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ 5, & \text{if } x \ge 1. \end{cases}$$

Also the pairs (A, S) and (B, T) are owc.

Clearly all the conditions of the above theorem are satisfied. Also

$$SA(0,0,0) = A(S0, S0, S0) \text{ and } TB(0,0,0) = B(T0, T0, T0)$$

So, (A, S) and (B, T) are owc maps and (0, 0, 0) is the common tripled fixed point of A, B, S and T. The Mesh Diagram for the given example is shown in Fig [3.1].

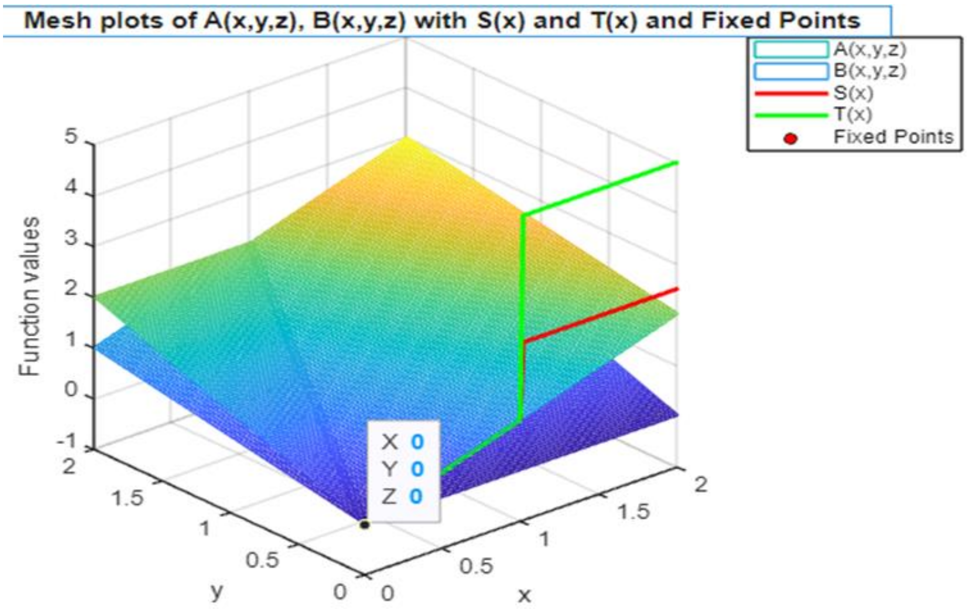

Fig [3.1]

Theorem: 3.2 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M^P(A(x,y,z), B(u,v,w), qt) \geq$
$\min\{ M^p(Sx,Tu,t), M(A(x,y,z),Tu,t). M^{p-1}(B(u,v,w),Sx,t)\}$
for all $x,y,z,u,v,w \in X$, $0 \leq a \leq 1, p \geq 1$.
(ii) $y = B(x,y,z)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that$A(x,x,x) = T(x) = B(x,x,x) = S(x) = x$.

Theorem: 3.3 Let $(X,M,*)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A,B: X \times X \times X \to X$ and $S,T: X \to X$ be four self-mappings satisfying the following conditions:

$$M(A(x,y,z), B(u,v,w), qt) \geq min\left\{M(Sx,Tu,t) + \frac{1}{2}\left(1 + \frac{M(A(x,y,z),Sx,t)}{M(B(u,v,w),Tu,t)}\right)\right\}$$

for all $x,y,u,v \in X$
(i) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that $A(x,x,x) = T(x) = B(x,x,x) = S(x) = x$.

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