The No Alternatives Argument

Richard Dawid   Stephan Hartmann   Jan Sprenger

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Abstract

Scientific theories are hard to find, and once scientists have found a theory H, they often believe that there are not many distinct alternatives to H. But is this belief justified? What should scientists believe about the number of alternatives to H, and how should they change these beliefs in the light of new evidence? These are some of the questions that we will address in this paper. We also ask under which conditions failure to find an alternative to H confirms the theory in question. This kind of reasoning (which we call the *No Alternatives Argument*) is frequently used in science and therefore deserves a careful philosophical analysis.

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1 Introduction

We typically confirm or disconfirm a scientific hypothesis with a piece of empirical evidence. For example, the observation of a black raven confirms the hypothesis that all ravens are black, and certain clicks in a particle detector confirm the existence of the top quark. However, there are situations where empirical evidence is unattainable over long periods of time. Such situations arise with particular force in contemporary high energy physics, where the characteristic empirical signatures of theories like Grand Unified Theories or string theory must be expected to lie many orders of magnitude beyond the reach of present day experimental technology. They are entirely common also in scientific fields such as palaeontology or anthropology, where scientists must rely on the scarce and haphazard empirical evidence they happen to find in the ground. Interestingly, scientists are at times quite confident

regarding the adequacy of their theories even when empirical evidence is largely or entirely absent.

Trust in a theory H in such cases must be based on what we call *non-empirical evidence.* Such evidence may be gathered by observation but it does not fall into the intended domain of H, and it cannot be related to H by another scientific theory. In particular, we cannot directly calculate the probability of the evidence given theory H. Rather than pertaining to any phenomenon described by H, non-empirical evidence for H can, for example, consist in observations about the research process leading up to the construction of H.

From an empiricist point of view, arguments relying on non-empirical evidence may be regarded as mere speculation: they neither contribute to actual theory confirmation nor do they have objective scientific weight. We challenge this claim by exploring the following case: scientists develop a considerable degree of trust in a theory H because, despite considerable efforts, no alternatives to H have been found that meet a number of theoretical and empirical constraints. We call this argument the *No Alternatives Argument* (NAA).[1] If valid, it would demonstrate the possibility of non-empirical theory confirmation.

In order to formalise this argument, we introduce the concept of the *number of alternative theories to* H. Then, we relate it to the empirical adequacy of H and the scientists' success at finding an alternative (Sect. 2).

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[1]The name of the argument stems from its crucial *premise* (scientists have not yet found a suitable alternative to H); it does not draw the *conclusion* that there are no alternatives to H.

On that basis, we construct a probabilistic model of NAA and prove the possibility of non-empirical theory confirmation (Sect. 3). Next, we show that the significance of NAA in scientific reasoning depends on the scientists' subjective judgements (Sect. 4). We provide a couple of results that show how these judgements are affected by evidence, and tentatively explore a *Meta-Inductive Argument* (MIA) for intersubjective agreement on the number of alternatives to H (Sect. 5). Finally, we put our findings into a broader context and briefly look at applications in epistemology and philosophy of science (Sect. 6). Throughout the paper, we operate in the framework of Bayesian epistemology.[2]

2 The conceptual framework

In order to understand the problem of non-empirical theory confirmation, we contrast it with its empirical counterpart. We call some evidence E empirical evidence for H if and only if (i) E falls into the (broadly construed) intended domain of H; (ii) E is logically or probabilistically related to H; (iii) E is observed.

Consider now the proposition T that hypothesis H is empirically adequate, which we understand as the property of being consistent with past *and* future observations. If one measures confirmation in terms of increase

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[2]Recent surveys of Bayesian epistemology are Hájek and Hartmann ([2010]) and Hartmann and Sprenger ([2010]). Applications of Bayesian epistemology to scientific reasoning are given in Bovens and Hartmann ([2003]) and Howson and Urbach ([2006]). Throughout this paper, we follow the convention that propositional variables are printed in italic script, and that the instantiations of these variables are printed in roman script.

in degrees of belief, as Bayesians typically do, T is confirmed by E whenever $P(\text{T}|\text{E}) > P(\text{T})$. Throughout the paper, we use this inequality as a criterion for when a piece of evidence confirms a hypothesis. The evidence can be observed perceptually or by means of measurement instruments, as common in modern science.

We would like to investigate whether the observation F—the lack of alternatives to H—constitutes evidence for T in this Bayesian sense. But how is it possible that F raises the subjective degree of belief in T? After all, F is neither deductively nor probabilistically implied by H. It does not even fall into the intended domain of H. Does F then qualify as (non-empirical) evidence in an *argument from ignorance* (Walton [1995]; Hahn and Oaksford [2007]; Sober [2009]), such as: if H were not empirically adequate, then we would have disproved it before?

The most plausible way to solve this problem is to deploy a two step process. First, we find a statement that does predict evidence of the type F. Then, we show that this statement is probabilistically relevant to the empirical adequacy of H. In the case of NAA, our non-empirical evidence F_A consists in the fact that scientists have not found any alternatives to a specific solution of a research problem, despite looking for them with considerable energy and for a long time. Then it is straightforward to identify a natural candidate for a statement that predicts F_A, namely that there are *few or no alternative theories to* H. Apparently, this would render F_A more likely than a scenario where a large (not necessarily infinite) number of possible alternative theories can be constructed: in the latter case, one might expect that scientists would have already found an alternative.

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The number k of possible scientific theories which can account for a certain set of data is in turn relevant for the degree of belief in the empirical adequacy of H. We assume that scientists who develop a theory in accordance with available data do not have a perfectly reliable method to select the true theory if false theories can be constructed that are also consistent with the available data. Under this condition, a lower number of possible scientific theories that can account for a certain set of empirical data increases the degree of belief that the theory developed by scientists is adequate. The stated assumption seems to be fairly plausible in science: scientists often come up with an incorrect, but fruitful theory when they begin to investigate a new field. Bohr's model of the atom is perhaps a good example for this claim.

Based on this reasoning, we introduce a random variable Y measuring the number of alternatives to H and taking values in the natural numbers. $Y_k :=$ $\{Y = k\}$ expresses the proposition that there are k adequate and distinct alternatives which satisfy a set of theoretical constraints $\mathcal{C}$, are consistent with the existing data $\mathcal{D}$, and give distinguishable predictions for the outcome of some set $\mathcal{E}$ of future experiments. We will later show that, via its effect on the Y_k, the non-empirical evidence F_A confirms H under plausible conditions.

Inferences about the number of alternatives to a theory H naturally depend on what counts as a genuine alternative. This is, in turn, sensitive to the specific scientific context. Therefore we leave the individuation problem to the scientific community which typically has the best grip on what should count as a distinct theory. Moreover, for the No Alternatives Argument, we only require the premise that the number of alternatives to H be *possibly*

finite, that is,

$$P(Y = \infty) < 1. \tag{1}$$

In order to motivate this assumption, we put two constraints on the individuation of scientific theories which, in our opinion, duly reflect accepted scientific practice.

First, different theories make different predictions. If two theories make exactly the same predictions, then we consider them to be identical. For example, we consider the De Broglie-Bohm version and the Copenhagen version of quantum mechanics as representing the same theory as long as they do not give different empirical predictions (Cushing [1994]). As a consequence, we are only interested in arriving at empirically adequate theories, and not in the more ambitious goal of finding theories that are true under a given interpretation (cf. van Fraassen [1980], and footnote 5 later on).

Second, different theories provide different solutions to a given scientific problem. That is, theories which only differ in a detail, such as the precise value of a parameter or the existence of a physically meaningless dummy variable, do not count as different theories. For example, the simple Higgs model in particle physics is treated as *one* theory, although the hypothesised and perhaps finally discovered Higgs particle could have different mass values. Generally, if it were enough to slightly modify the value of a certain parameter in order to arrive at a new theory, then coming up with new theories would be an easy and not very creative task. Inventing a novel mechanism or telling a new story of why a certain phenomenon came about is much harder.

Beyond these general guidelines, the specifics of theory individuation must

depend on the context in which a no alternatives argument is formulated by scientists. Scientists often formulate no alternatives arguments at the level of general conceptual principles while allowing for a large spectrum of specific realizations of those principles.[3] In such cases, they have a fairly inclusive conception of what counts as one theory. But apart from satisfying (1), our argument does not depend on how these distinctions are made. We are now ready to proceed to a formal analysis of the No Alternatives Argument.

3   The No Alternatives Argument

In the No Alternatives Argument or NAA, the non-empirical evidence consists in the observation that scientists have not yet found an alternative to H. This observation is taken to indicate that there are actually not too many alternatives to H, and thus indirectly as an argument for H. Focusing on the case of string theory, Dawid ([2006], [2009]) calls this the *argument of no choice.*

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[3]For example, since the 1980s particle physicists strongly supported a no alternatives argument with respect to the Higgs mechanism. That is, they believed that no alternatives to a gauge theory that was spontaneously broken by a Higgs sector of scalar fields could account for the available empirical data. They did not claim, however, that only one way of realizing the Higgs sector was possible. They knew that the Higgs sector could consist of one or several complex scalar fields, of elementary or constituent scalars, etc. Therefore, physicists strongly believed based on NAA that the Higgs sector would be observed at the LHC experiment but did not have particular trust in any of the specific models of the Higgs sector. Their NAA clearly was placed at the level of physical principles rather than specific models.
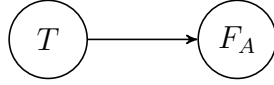
Figure 1: The Bayesian Network representation of the two-propositions scenario.

Following this line of reasoning, we reconstruct NAA based on the notion that there exists a specific but unknown number k of possible scientific theories. As stated above, these theories have to satisfy constraints $\mathcal{C}$, explain data $\mathcal{D}$ and predict the outcomes of the experiments $\mathcal{E}$. We will then show that failure to find an alternative to H raises the probability of H being empirically adequate and thus confirms H.

To do so, we introduce the binary propositional variables T and F_A, already briefly encountered in Sect. 2. T takes the values

    T The hypothesis H is empirically adequate.

  ¬T The hypothesis H is not empirically adequate.

and F_A takes the values

   F_A The scientific community has not yet found an alternative to H that fulfills $\mathcal{C}$, explains $\mathcal{D}$ and predicts the outcomes of $\mathcal{E}$.

¬F_A The scientific community has found an alternative to H that fulfills $\mathcal{C}$, explains $\mathcal{D}$ and predicts the outcomes of $\mathcal{E}$.

We would now like to explore under which conditions F_A confirms H, that is, when

$$P(\text{T}|\text{F}_\text{A}) > P(\text{T}).\qquad(2)$$

This equation suggests a direct influence of T on F_A. See Figure 1 for a Bayesian Network representation of this scenario. But since a direct influence is blocked by the non-empirical nature of F_A, we introduce a third variable Y which mediates the connection between T and F_A. Like in the previous section, Y has values in the natural numbers, and Y_k corresponds to the proposition that there are exactly k hypotheses that fulfil $\mathcal{C}$, explain $\mathcal{D}$ and predict the outcomes of $\mathcal{E}$.

We should also note that the value of F_A—that scientists find/do not find an alternative to H—does not only depend on the number of available alternatives, but also on the difficulty of the problem, the cleverness of the scientists, or the available computational, experimental, and mathematical resources. Call the variable that captures these complementary factors D, and let it take values in the natural numbers, with $D_j := \{D = j\}$ and $d_j := P(D_j)$. The higher the values of D, the more difficult the problem.[4] It is clear that D has no direct influence on Y and T (or vice versa), but that it matters for F_A and that this influence has to be represented in our Bayesian Network.

We now list five plausible assumptions that we need for showing the validity of the No Alternatives Argument.

A1. The variable T is conditionally independent of F_A given Y:

$$T \perp\!\!\!\perp F_A | Y \tag{3}$$

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[4]For the purpose of our argument, it is not necessary to assign a precise operational meaning to the various levels of D. It is sufficient that they satisfy a natural monotonicity assumption with regard to their impact on F_A—see condition **A3** in the main text.
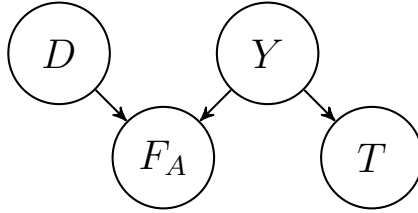
Figure 2: The Bayesian Network representation of the four-propositions scenario.

Hence, learning that the scientific community has not yet found an alternative to H does not alter our belief in the empirical adequacy of H if we already know that there are exactly k viable alternatives to H.

A2. The variable D is (unconditionally) independent of Y:

$$D \perp\!\!\!\perp Y \tag{4}$$

Recall that D represents the aggregate of those context-sensitive factors that affect whether scientists find an alternative to H, but that are not related to the number of suitable alternatives. In other words, D and Y are orthogonal to each other by construction.

These are our most important assumptions, and we consider them to be eminently sensible. Figure 2 shows the corresponding Bayesian Network. To complete it, we have to specify the prior distribution over D and Y and the conditional distributions over F_A and T, given the values of their parents. This is done in the following three assumptions.

A3. The conditional probabilities

$$f_{kj} := P(\mathrm{F_A}|\mathrm{Y_k}, \mathrm{D_j}) \tag{5}$$

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are non-increasing in k for all $j \in \mathbb{N}$ and non-decreasing in j for all $k \in \mathbb{N}$.

The (weak) monotonicity in the first argument reflects the intuition that for fixed difficulty of a problem, a higher number of alternatives does not decrease the likelihood of finding an alternative to H. The (weak) monotonicity in the second argument reflects the intuition that increasing difficulty of a problem does not increase the likelihood of finding an alternative to H, provided that the number of alternatives to H is fixed.

A4. The conditional probabilities

$$t_k := P(\mathrm{T}|\mathrm{Y_k}) \tag{6}$$

are non-increasing in k.

This assumption reflects the intuition that an increase in the number of alternative theories does not make it more likely that scientists have already identified an empirically adequate theory.

A5. There is at least one pair (i, k) with $i < k$ for which (i) $y_i\, y_k > 0$ where $y_k := P(\mathrm{Y_k})$, (ii) $f_{ij} > f_{kj}$ for some $j \in \mathbb{N}$, and (iii) $t_i > t_k$.

In particular, this assumption implies that $y_k < 1$ for all $k \in \mathbb{N}$ because otherwise, a pair satisfying (i) could not be found.

With these five assumptions, we can show that (proof in appendix A):

Theorem 1. *If Y takes values in the natural numbers $\mathbb{N}$ and assumptions* **A1** *to* **A5** *hold, then* $\mathrm{F_A}$ *confirms* T, *that is,* $P(\mathrm{T}|\mathrm{F_A}) > P(\mathrm{T})$.

We have therefore shown that F_A confirms the empirical adequacy of H under rather weak and plausible assumptions.

In line with the introduction of Y in section 2, we have assumed that Y only takes values in the natural numbers. This might be seen as evading the skeptical argument that there may be infinitely many (theoretically adequate, empirically successful, ...) alternatives to H. Therefore we now extend the theorem by explicitly allowing for the possibility $Y_\infty := \{Y = \infty\}$, and we modify our assumptions accordingly. In particular, we observe that **A5** entails $P(Y_\infty) < 1$, define $f_{\infty j} := P(F_A | Y_\infty, D_j)$, $t_\infty := P(T | Y_\infty)$ and demand that

$$f_{ij} \geq f_{\infty j} \; \forall i, j \in \mathbb{N} \qquad\qquad f_{\infty i} \leq f_{\infty j} \; \forall i, j \in \mathbb{N} \text{ with } i < j \qquad (7)$$

$$t_i \geq t_\infty \; \forall i \in \mathbb{N}. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (8)$$

These requirements naturally extend assumptions **A3** and **A4** to the case of infinitely many alternatives. Then, we obtain the following generalization of the NAA:

Theorem 2. *If Y takes values in $\mathbb{N} \cup \{\infty\}$ and assumptions **A1** to **A5** hold together with their extensions (7) and (8), then F_A confirms T, that is, $P(T | F_A) > P(T)$.*

That is, even if we concede to the skeptic that there may be infinitely many alternatives to H, she must still acknowledge the validity of NAA as long as her degrees of belief satisfy $P(Y_\infty) < 1$.[5] This is, in our mind, a quite substantial and surprising result. Philosophy of science has focused on logical

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[5] At this point, the reader is able to acknowledge why we make inferences about the empirical adequacy of H, instead of its *truth*. In that case, we could always construct

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and probabilistic relations between theory and evidence, but has neglected this particular evidential support and failed to acknowledge its validity.

Note that only a dogmatic skeptic who insists on $P(Y_\infty) = 1$ can deny the validity of NAA. But Theorem 2 convinces anyone whose attitude is *genuinely skeptical*, that is, someone who does not want to commit herself with respect to the probability of (in)finitely many alternatives to H. Convincing such a fair and non-committal skeptic is, to our mind, much more important than convincing dogmatists who just deny our premises.

The following two sections discuss the *amount* of confirmation that NAA confers on H, and provide more mathematical results regarding the problem of assessing the number of alternatives to H.

4   Discussion I: A quantitative analysis of NAA

We have seen that NAA can be used in support of a proposed theory. The question remains, however, whether the resulting support is of significant strength and whether using NAA in a specific situation is justified. To facilitate the reading, we conduct this analysis for the finite case (Theorem 1) only; the infinite case is analogous.

The Bayesian Network representation of NAA in Figure 2 suggests that such significance is difficult to attain by NAA on its own without further supportive reasoning. According to Figure 2, F_A may confirm an instance of

infinitely many empirically indistinguishable alternatives to H (e.g., by introducing meaningless dummy variables), jeopardizing the validity of NAA.

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D (limitations to the scientists' abilities to solve difficult problems) as well as an instance of Y (limitations to the number of possible theories). It is then easy to see that for all $l \in \mathbb{N}$,

$$P(\mathrm{D_l}|\mathrm{F_A}) = \frac{P(\mathrm{D_l}, \mathrm{F_A})}{P(\mathrm{F_A})} = \frac{d_l \cdot \sum_k y_k \, f_{kl}}{\sum_{j,k} d_j \, y_k \, f_{kj}} \, . \tag{9}$$

Hence the ratio measure of confirmation[6] can be computed as

$$r(\mathrm{D_l}, \mathrm{F_A}) := \frac{P(\mathrm{D_l}|\mathrm{F_A})}{P(\mathrm{D_l})} = \frac{\sum_k y_k \, f_{kl}}{\sum_{j,k} d_j \, y_k \, f_{kj}} \, . \tag{10}$$

We cannot provide fully general conditions for when this expression is greater than 1. However, we observe that the expression on the right hand side of equation (10) is non-decreasing in l since the f_{kl} are non-decreasing in l for fixed k (see assumption **A3**). That is, the degree of confirmation that $\mathrm{F_A}$ lends to D_l, as expressed by the ratio measure, typically increases with l. Thus, $\mathrm{F_A}$ confirms the claim that the problem at hand is rather complicated (i.e., that it has a high rank l) and disconfirms the claim that it is not particularly complicated (i.e., that it has a low rank l). The turning point l^* depends on the precise values of the parameters in question.

To accentuate the resulting problem, note that the situation could be such that $\mathrm{D}^* := \{D \geq l^*\}$—the proposition that the problem has difficulty rank l^* or higher—receives more confirmation than T. While failure to find an alternative confirms the empirical adequacy of H, this failure would also confirm, and to a larger degree, the hypothesis that the problem is too complicated for our current science. This alternative explanation of $\mathrm{F_A}$ weakens

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[6]The choice of this particular confirmation measure is guided by simplicity and transparency; nothing significant changes if we move to a different confirmation measure.

the significance of NAA. To successfully apply NAA, one has to amend the *qualitative* claim shown in section 3 with a *comparative* claim, namely that F_A confirms T more than D^*. But such a statement is sensitive to the specific parameter assignments as well as to the chosen confirmation measure—and therefore hard to prove in general.

So far we have left the parameters d_k, f_{kj}, t_k and y_k largely unrestricted and assumed that they reflect the subjective degrees of belief of a scientist. Hence, different scientists may assign different values to these parameters, which implies that the assessment of the significance of NAA will differ from scientist to scientist in the absence of further rational constraints. Given that science strives for objectivity or at least for intersubjective agreement, this is an unfortunate situation. So we have to examine what empirical evidence can tell us about the (probable) number of alternatives to a given theory H. This is the subject of the next section.

5 Discussion II: The number of alternatives and the problem of underdetermination

In this section, we present some results on how evidence changes our beliefs about the number of alternatives to a theory H. In other words, we try to develop a dynamic perspective on the underdetermination problem, as opposed to the more static NAA.

Some of our results here only apply to an arbitrarily large, but finite number of alternatives. Notably, this does not trivialize the problem. As

the following result shows, the fundamental problem of theoretical underdetermination persists even in that case (proof of all results in this section in appendix B):

Proposition 1. *For any $N \in \mathbb{N}$ and any $1 \geq \varepsilon > 0$, an agent's belief function P may jointly satisfy (i) $P(Y = \infty) = 0$, (ii) $P(Y \leq N) \geq 1 - \varepsilon$, and (iii) $\langle Y \rangle := \sum_{k=0}^{\infty} k\, P(Y_k) = \infty$.*

In this notation, $\langle Y \rangle$ denotes the *expectation value* of Y. In other words, an agent might rule out an infinite number of alternatives to H, be strongly convinced that there are few alternatives to H, and yet retain the belief that our best guess regarding the number of alternatives to H is "indefinitely large" or "greater than any number that we can imagine".[7]

In other words, Proposition 1 points out the possibility of a strong epistemic tension within a single agent regarding the number of alternatives to a theory H. This tension gives an interesting twist to the problem of theoretical underdetermination: the agent might believe that H is fundamentally underdetermined by evidence (because our best guess for the number of alternatives is indefinitely large), but still rule out that there are infintely many alternatives.

Let us now study whether such a belief structure is responsive to evidence E, be it empirical or non-empirical. First, we ask the following question: Can an agent who believes that $\langle Y \rangle = \infty$, and perhaps even that $P(Y = \infty) > 0$, rationally revise her belief in the light of evidence E such that $\langle Y \rangle_{\mathrm{E}}$—the

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[7]This phenomenon is well-known from paradoxes of decision theory, such as the valuation of the St. Petersburg Game, but to our knowledge, this epistemic counterpart has not been explored before.

expectation value of Y under the posterior distribution $P(\cdot|\mathrm{E})$—is finite? In other words, is it possible that the (strong) underdetermination problem vanishes? The answer is yes. The following theorem lists four different sufficient conditions for such a belief change.

Theorem 3. *Assume that $\langle Y \rangle = \infty$ and that $P(Y = \infty) < 1$. Then any of the following conditions on evidence E with $P(\mathrm{E}) \neq 0$ is sufficient for $\langle Y \rangle_{\mathrm{E}} < \infty$.*

1. *The sequence $(k \cdot P(\mathrm{E}|Y_k))_{k \in \mathbb{N} \cup \infty}$ is bounded.*

2. *There are $\alpha, \beta > 0$ be such that $\alpha + \beta > 2$, and that $(k^\alpha P(\mathrm{E}|Y_k))_{k \in \mathbb{N} \cup \infty}$ and $(k^\beta P(Y_k))_{k \in \mathbb{N}}$ are bounded.*

3. *$\sum_{k=0}^{\infty} P(\mathrm{E}|Y_k) < \infty$, $P(E|Y_\infty) = 0$ and there is a $N_0 \in \mathbb{N}$ such that $(P(Y_k))_{k \in \mathbb{N}}$ is, for all $k \geq N_0$, monotonically decreasing.*

4. *$P(\mathrm{E}|Y_k) \to 0$ and there is an $\alpha > 0$ such that*

$$\limsup_{k \to \infty} k^{2+\alpha} \, |P(\mathrm{E}|Y_k) - P(\mathrm{E}|Y_{k-1})| < \infty. \tag{11}$$

These four conditions have different rationales, but all of them constrain the rate of decline of $P(\mathrm{E}|Y_k)$ as k increases. That is, the more alternatives there are, the less likely is E. The first and second condition basically amount to $P(\mathrm{E}|Y_k) \in \mathcal{O}(1/k^\alpha)$ for a suitable exponent $\alpha > 0$. The third condition makes a similar constraint by demanding that $\sum_{k=0}^{\infty} P(\mathrm{E}|Y_k)$ converges, and the fourth condition controls the differences between the values of $P(\mathrm{E}|Y_k)$ for neighboring values of k. In particular, the intuitive condition $P(\mathrm{E}|Y_k) = 1/k$ is sufficient for the theorem to hold.

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Note that only the second condition makes an assumption about the rate of decline of $P(\mathrm{Y_k})$. This is in line with the idea that we have little grip on the rational beliefs about the number of empirically adequate alternatives, whereas we are in a better position to assess how our evidence E is affected by the number of alternatives.

As already stated, the punch line of all four conditions is that $P(\mathrm{E|Y_k})$ converges fast enough to zero. For evidence E that is related to an empirical test of H, this assumption is reasonable: if there are more and more alternatives, why should H, instead of an unconceived alternative (Stanford [2006]), survive empirical tests? Thus, if large values of Y make little difference regarding our trust in the predictions of H, then we will abandon the belief that the expected number of alternatives is infinite. This is exactly what we would expect intuitively.

Second, we ask under which circumstances evidence E *lowers* the expected number of alternatives if that value is already finite. In answer to this question, we can demonstrate the following theorem:

Theorem 4. *Assume $P(Y = \infty) = 0$. Let $\mathrm{Y_k^+}$ denote the proposition that there are at least k alternatives to theory H, and let $\mathrm{Y_k^-}$ denote the proposition that there are at most $k-1$ alternatives to H. Then, if $P(\mathrm{E|Y_k^+}) \leq P(\mathrm{E|Y_k^-})$ for all $k \in \mathbb{N}$ and $P(\mathrm{E|Y_k^+}) < P(\mathrm{E|Y_k^-})$ for at least one $k > 0$, it will also be the case that $\langle Y \rangle > \langle Y \rangle_{\mathrm{E}}$.*

In other words, if the likelihood of evidence E decreases with the available number of alternatives to H, then the expected number of alternatives will be smaller *a posteriori* than it was *a priori*.

The condition of the theorem can be satisfied by empirical as well as non-empirical evidence. For *contrastive* empirical evidence, that is, data which confirm a particular set of theories and disconfirm others, this is straightforward. And this covers most of everyday data in scientific experiments. But even non-empirical evidence such as $F_A :=$ "the scientists have not yet found an alternative to H" may satisfy the conditions of Theorem 4 since such an observation supports H only via an inference about the number of suitable alternatives.

Finally, we also ask the question of whether the belief dynamics of under-determination are unidirectional or bidirectional. That is, can an agent who believes that Y takes finite values only (i.e., that $\langle Y \rangle < \infty$) come to the belief that $\langle Y \rangle_E = \infty$ for some evidence E? Interestingly, this is impossible. No empirical evidence is able to overturn the verdict that the expected number of alternatives to H is finite:

Proposition 2. *If $\langle Y \rangle < \infty$, then for any evidence E (empirical or non-empirical) with $P(E) \neq 0$, $\langle Y \rangle_E < \infty$.*

This means that the belief that the expected number of alternatives is finite is *not* responsive to empirical evidence: once you believe it, you will always believe it, independently of which evidence you receive. This points to an interesting asymmetry: evidence can change the belief that there are infinitely many alternatives, but it cannot reverse the belief that there are finitely many alternatives. The asymmetry between Theorem 3 and Proposition 2 confirms the suspicion that empirical evidence usually lowers the expected number of alternatives. This finding agrees with the observation that

convergence in scientific research occurs more frequently than divergence.

That said, it is still unclear what determines our beliefs on the number of alternatives in the first place. What should our priors for the Y_k look like? This question is hard to answer in general, but we would like to sketch a reasoning procedure, called the *Meta-Inductive Argument* (MIA) that might help us to get a grip on the distribution of the number of alternative theories.

The gist of MIA is best illustrated by a special case. It is notoriously difficult to find a theory that makes the correct *predictions*, rather than just to accommodate existing data (Kahn et al. [1992]; Hitchcock and Sober [1994]). But remarkably, scientists have often succeeded at identifying such a theory. Now, if there are a lot of alternative solutions to a given problem, then there is no reason to assume that the scientists identified the one theory which will prevail in the future. Thus, repeated predictive success within a particular scientific research program suggests a particular explanation, namely that there are few suitable alternative theories in the given theoretical context. This argument resembles the no miracles argument in the debate about scientific realism.

Now, assume that a novel theory H shows similarities to theories H_1, H_2, etc., in the same scientific research program. The joint feature of these theories may be a certain theoretical approach, a shared assumption, or any other relevant characteristic. Let us assume that a substantial share of the theories to which H is similar have been empirically confirmed. Assume further that for those theories, we have empirically grounded posterior beliefs about the number of alternatives. Then, it seems reasonable to use these posteriors as priors for the number of alternatives to H. After all, H is quite

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similar to H_1, H_2, etc. In statistics, such a way of grounding "objective" prior beliefs in past experience is referred to as the *empirical Bayes method* (Carlin and Louis [2000]).

If this move is accepted, then one is in a much better position to evaluate the significance of NAA, due to agreement on the prior probabilities of the Y_k. Admittedly, our account of MIA remains informal and provides at best a partial justification for the practical significance of NAA. On the other hand, formalizing MIA and strengthening the link between both arguments strikes us as a promising route for further research. Be this as it may, we would like to stress that even without MIA, the *validity* of the NAA is a surprising and substantial philosophical result and that even the degree of confirmation that this argument provides can in principle be large.

6   Conclusions

In this paper, we have completed three tasks: (i) we have formalised the No Alternatives Argument and explored under which conditions non-empirical evidence confirms a scientific theory H; (ii) we have studied the problem of theoretical underdetermination from the angle of how beliefs about the number of alternatives to H change in the light of evidence; and (iii) we have sketched the Meta-Inductive Argument for assessing the number of alternatives to H.

We conclude this paper by sketching future research projects. From a normative point of view, a rigorous formalization of MIA would be desirable. Also, we plan to relate the formal argument of this paper more closely to

case studies from scientific practice. Here we are particularly interested in the case of string theory and reasoning strategies employed in fields such as palaeontology and anthropology where contingent evolutionary details have to be reconstructed based on scarce and highly incomplete evidence. We will explore what role NAA plays in these fields, and how good the argument actually is.

Finally, the scope of NAA needs to be determined more exactly. In particular, we dare an outlook on whether NAA can be applied in philosophy, too. Two potential applications come to mind. First, *Inference to the Best Explanation* (Lipton [2004]; Douven [2011]) can, to a certain extent, be explicated in terms of NAA. In as much as the notion "best explanation" is understood as "the only genuinely satisfactory explanation", the fact that no other genuinely satisfactory explanation has been found can play the role of the claim of no alternatives in our argument, supporting the empirical adequacy of the currently best explanation. The structure of the argument and the formal result would be similar; only the interpretation would change from "empirically adequate" to "best explanation". We conjecture that the validity of IBE can sometimes be analyzed in terms of a NAA.

Second, one may ask whether NAA could also play a role in confirming general philosophical theories. The reputation of a philosophical theory is often based on the understanding that no other consistent answer has been found or is perhaps not even conceivable. Can reasoning of this kind be supported by NAA? In principle, the answer to this question is yes, but there is a problem: philosophical theories do not have a record of empirical testing. We will be unable to quantify the significance of NAA with empirical

data. Philosophy thus provides us with a neat example of the promises and limits of non-empirical theory confirmation beyond scientific contexts.

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A   Proof of the No Alternative Theorems

Proof of Theorem 1: F_A confirms T if and only if $P(T|F_A) - P(T) > 0$, that is, if and only if

$$\Delta := P(T, F_A) - P(T)P(F_A) > 0.$$

We now apply the theory of Bayesian Networks to the structure depicted in Figure 2, using assumptions **A1** $(T \perp\!\!\!\perp F_A | Y)$ and **A2** $(D \perp\!\!\!\perp Y)$:

$$
\begin{aligned}
P(\mathrm{F_A}) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(\mathrm{F_A | Y_i, D_j}) P(\mathrm{Y_i, D_j}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, f_{ij} \\
P(\mathrm{T}) &= \sum_{k=0}^{\infty} P(\mathrm{T | Y_k}) P(\mathrm{Y_k}) = \sum_{k=0}^{\infty} t_k \, y_k \\
P(\mathrm{T, F_A}) &= \sum_{i=0}^{\infty} P(\mathrm{F_A, T | Y_i}) P(\mathrm{Y_i}) = \sum_{i=0}^{\infty} y_i \, P(\mathrm{F_A | Y_i}) \, P(\mathrm{T | Y_i}) \\
&= \sum_{i=0}^{\infty} y_i \, t_i \left(\sum_{j=0}^{\infty} P(\mathrm{F_A | Y_i, D_j}) \, P(\mathrm{D_j | Y_i}) \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, t_i \, f_{ij}
\end{aligned}
$$

Hence, we obtain, using $\sum_{k \in \mathbb{N}} y_k = 1$,

$$
\begin{aligned}
\Delta &= \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, t_i \, f_{ij} \right) - \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, f_{ij} \right) \left(\sum_{k=0}^{\infty} y_k \, t_k \right) \\
&= \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, t_i \, f_{ij} \right) \left(\sum_{k=0}^{\infty} y_k \right) - \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j \, y_i \, f_{ij} \right) \left(\sum_{k=0}^{\infty} t_k \, y_k \right) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(d_j \, y_i \, y_k \, t_i \, f_{ij} - d_j \, y_i \, y_k \, t_k \, f_{ij} \right) \\
&= \sum_{j=0}^{\infty} d_j \sum_{i=0}^{\infty} \sum_{k \neq i=0}^{\infty} y_i \, y_k \, f_{ij} \, (t_i - t_k) \\
&= \sum_{j=0}^{\infty} d_j \sum_{i=0}^{\infty} \sum_{k>i} \left(y_i \, y_k \, f_{ij} \, (t_i - t_k) + y_k \, y_i \, f_{kj} \, (t_k - t_i) \right) \\
&= \sum_{j=0}^{\infty} d_j \sum_{i=0}^{\infty} \frac{1}{2} \sum_{k \neq i=0}^{\infty} y_i \, y_k \, \left(f_{ij} \, (t_i - t_k) + f_{kj} \, (t_k - t_i) \right) \\
&= \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k \neq i=0}^{\infty} d_j \, y_i \, y_k \, (t_i - t_k) \, (f_{ij} - f_{kj}) \\
&> 0
\end{aligned}
$$

because of **A3**-**A5** taken together: **A3** entails that the difference $(f_{ij} - f_{kj})$ is non-negative, **A4** does the same for the $(t_i - t_k)$, and **A5** entails that these differences are strictly positive for at least one pair (i, k). Hence, the entire double sum is strictly positive. $\qquad\qquad\square$

Proof of Theorem 2: We perform essentially the same calculations as in the proof of Theorem 1 and additionally include the possibility $\{Y = \infty\}$.[8] Defining $f_{\infty j} := P(F_A|D_j Y_\infty)$ leads us to the equalities

$$
\begin{aligned}
P(F_A) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_j\, y_i\, f_{ij} + \sum_{j=0}^{\infty} d_j\, y_\infty\, f_{\infty j} \\
P(T) &= \sum_{k=0}^{\infty} t_k\, y_k + t_\infty\, y_\infty \\
P(F_A, T) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_j\, y_i\, t_i\, f_{ij} + \sum_{j=0}^{\infty} d_j\, t_\infty\, y_\infty\, f_{\infty j}
\end{aligned}
$$

from which it follows, using $\lim_{K\to\infty}\sum_{k=1}^{K} y_k = 1 - y_\infty$, that

$$
\begin{aligned}
P(F_A)P(T) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j\, t_k\, y_i\, y_k\, f_{ij} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_j\, t_\infty\, y_i\, y_\infty\, f_{ij} \\
&\quad + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j\, t_k\, y_k\, y_\infty\, f_{\infty j} + \sum_{j=0}^{\infty} d_j\, t_\infty\, y_\infty^2\, f_{\infty j} \\
P(F_A, T) &= \frac{1}{1 - y_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j\, t_i\, y_i\, y_k f_{ij} + \sum_{j=0}^{\infty} d_j\, t_\infty\, y_\infty\, f_{\infty j}
\end{aligned}
$$

With the definition

$$
\Delta := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j\, t_i\, y_i\, y_k f_{ij} - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_j\, t_k\, y_i\, y_k\, f_{ij}
$$

---

[8]The notation suggests that ∞ is already included in the summation index, but the infinity sign on top of the sum is just the shortcut for the limit of the sequence of all natural numbers. Thus, the case $Y = \infty$ has to be treated separately.

we observe that $\Delta > 0$, as shown above in the proof of Theorem 1 (the parameter values satisfy the relevant conditions **A3-A5**). It then follows that

$$
\begin{aligned}
& P(\mathrm{F_A}, \mathrm{T}) - P(\mathrm{T})\, P(\mathrm{F_A}) \\
=\;& \Delta + \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, t_i\, y_i\, y_k f_{ij} + \sum_{j=0}^{\infty} d_j\, t_\infty\, y_\infty\, (1 - y_\infty)\, f_{\infty j} \\
& - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j\, t_\infty y_i\, y_\infty\, f_{ij} - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_j\, t_i\, y_i\, y_\infty\, f_{\infty j} \\
=\;& \Delta + \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, t_i\, y_i\, y_k f_{ij} + \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, t_\infty\, y_i\, y_k\, f_{\infty j} \\
& - \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, t_\infty y_i\, y_k\, f_{ij} - \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, t_i\, y_i\, y_k\, f_{\infty j} \\
=\;& \Delta + \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, y_i\, y_k (t_i f_{ij} + t_\infty f_{\infty j} - t_\infty f_{ij} - t_i f_{\infty j}) \\
=\;& \Delta + \frac{y_\infty}{1 - y_\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_j\, y_i\, y_k (t_i - t_\infty)(f_{ij} - f_{\infty j}) \\
>\;& 0
\end{aligned}
$$

since the extensions of **A3** and **A4** imply $f_{ij} \geq f_{\infty j}$ and $t_i \geq t_\infty$ (equations (7) and (8)), independent of the values of i and j. $\qquad\square$

B Proof of the results in Section 5

Proof of Proposition 1: The proof proceeds by construction. For instance, let $P(Y \leq N) = 1 - \varepsilon$, let $P(Y_k) = C/k^2 \;\forall k > N$, and choose C such that $\sum_{k>N} P(Y_k) = \varepsilon$ is satisfied. (The series $\sum_k 1/k^2$ converges.) Then, it is

easy to check that

$$\langle Y \rangle \geq \sum_{k=N+1}^{\infty} k\, P(Y_k) \geq C \sum_{k=N+1}^{\infty} \frac{1}{k} = \infty. \quad \square$$

Proof of Theorem 3: Note first that, since $P(\mathrm{E}) \neq 0$,

$$
\begin{aligned}
\langle Y \rangle_{\mathrm{E}} &= \sum_{k \in \mathbb{N} \cup \infty} k\, P(Y_k | \mathrm{E}) \\
&= \frac{1}{P(\mathrm{E})} \sum_{k \in \mathbb{N} \cup \infty} k\, P(Y_k)\, P(\mathrm{E}|Y_k) \\
&= \frac{1}{P(\mathrm{E})} \left(\sum_{k=1}^{\infty} k\, P(Y_k)\, P(\mathrm{E}|Y_k) + \lim_{K \to \infty} K\, P(Y_\infty)\, P(\mathrm{E}|Y_\infty) \right).
\end{aligned}
$$

All four conditions of the theorem require, either explicitly or implicitly, that $P(\mathrm{E}|Y_\infty)$ be equal to zero. Therefore, the second summand above vanishes and we can focus our analysis of $\langle Y \rangle_{\mathrm{E}}$ on $k < \infty$.

We begin by proving the sufficiency of the first condition. Assume that the expression $(k \cdot P(\mathrm{E}|Y_k))_{k \in \mathbb{N}}$ is bounded, that is, there is a $B > 0$ such that $k \cdot P(\mathrm{E}|Y_k) < B$. Then it will be the case that

$$
\begin{aligned}
\langle Y \rangle_{\mathrm{E}} &= \frac{1}{P(\mathrm{E})} \sum_{k=1}^{\infty} k\, P(Y_k)\, P(\mathrm{E}|Y_k) \\
&\leq B \cdot \frac{1}{P(\mathrm{E})} \sum_{k=1}^{\infty} P(Y_k) \\
&< \infty,
\end{aligned}
$$

proving the sufficiency of the first condition.

Related to this is the case that $k^\alpha \cdot P(\mathrm{E}|Y_k) \leq A_\alpha$ and $k^\beta \cdot P(Y_k) \leq A_\beta$ for all $k \in \mathbb{N}$ and some constants $A_\alpha, A_\beta > 0$, with the additional constraints

$\alpha, \beta > 0$ and $\alpha + \beta > 2$. Then we have

$$
\begin{aligned}
\langle Y \rangle_{\mathrm{E}} &= \frac{1}{P(\mathrm{E})} \sum_{k=1}^{\infty} k^{1-\alpha-\beta} \left(k^{\alpha} P(\mathrm{E}|Y_k) \right) \left(k^{\beta} P(Y_k) \right) \\
&\leq \frac{1}{P(\mathrm{E})} A_{\alpha} A_{\beta} \sum_{k=1}^{\infty} k^{1-(\alpha+\beta)} \\
&< \infty
\end{aligned}
$$

because by assumption, $1 - (\alpha + \beta) < -1$, ensuring the convergence of the series. In the remainder of the proof we will focus on the properties of the series

$$
\sum_{k=1}^{\infty} k\, P(Y_k) P(\mathrm{E}|Y_k) \tag{12}
$$

which is sufficient for examining the convergence properties of $\langle Y \rangle_{\mathrm{E}}$.

We now proceed to proving the sufficiency of the third condition. We assume that $\sum_{k=1}^{\infty} P(\mathrm{E}|Y_k) < \infty$ and that there is a $N_0 \in \mathbb{N}$ such that $P(Y_k) \geq P(Y_{k+1})$ for all $k \geq N_0$. By Dirichlet's criterion (Knopp [1964], p. 324), $\sum_{k=1}^{\infty} k\, P(Y_k) P(\mathrm{E}|Y_k)$ converges if (i) $\sum_{k=1}^{\infty} P(\mathrm{E}|Y_k) < \infty$ and (ii) $k\, P(Y_k) \to 0$ monotonically. The first condition is fulfilled by assumption. The second clause of the criterion can, without loss of generality, be replaced by demanding that for $N_0 \in \mathbb{R}$, $(k\, P(Y_k))_{k \in \mathbb{N}}$ be monotonically decreasing for all $k \geq N_0$.

Assume that the second clause of the criterion is not satisfied, and that there is a sequence of natural numbers n_k such that

$$
n_k P(Y_{n_k}) < n_{k+1} P(Y_{n_{k+1}}). \tag{13}
$$

Then the (sub)sequence $(n_k\, P(Y_{n_k}))_k$ would not converge to zero, and consequently, $(k\, P(Y_k))_k$ would not converge to zero. However, for some $k \geq N_0$,

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$P(Y_k)$ is by assumption a monotonically decreasing sequence. Furthermore, for such sequences, if $\sum_k P(Y_k)$ exists (which is the case here), then also $k\,P(Y_k) \to 0$ (Knopp [1964], p. 125). Hence, a subsequence $(n_k\,P(Y_{n_k}))_k$ with property (13) cannot exist and the second part of the Dirichlet criterion is satisfied. Thus, the third condition of Theorem 3 is indeed sufficient.

Finally, we demonstrate the joint sufficiency of (i) $P(E|Y_k) \to 0$ and (ii) there is an $\alpha > 0$ such that

$$\limsup_{k\to\infty} k^{2+\alpha}\,|P(E|Y_k) - P(E|Y_{k-1})| < \infty.$$

In particular, there exists a $C > 0$ such that $k^{2+\alpha}\,|P(E|Y_k) - P(E|Y_{k-1})| \leq C$, $\forall k \in \mathbb{N}$. Moreover, let $C' := 2C \sum_{k=1}^{\infty} 1/k^{1+\alpha}$.

By Abel's formula (Knopp [1964], p. 322), we can rewrite the partial sums of the series $\sum_{k=1}^{\infty} k\,P(Y_k)\,P(E|Y_k)$ in the following way:

$$\sum_{k=1}^{N} k\,P(Y_k)\,P(E|Y_k) \;=\; \sum_{k=1}^{N}\left(\sum_{j=1}^{k} j\,P(Y_j)\right)(P(E|Y_k) - P(E|Y_{k+1}))$$
$$+ \left(\sum_{j=1}^{N} j\,P(Y_j)\right)P(E|Y_{N+1}).$$

Note that the re-ordering of the terms does not affect the convergence properties since (12) has only positive members. It is now sufficient to show that both summands on the right side are uniformly bounded in N since this would mean that (12) has bounded partial sums and is thus convergent.

We begin by showing that the first summand is uniformly bounded:

$$
\left| \sum_{k=1}^{N} \left(\sum_{j=1}^{k} j\, P(Y_j) \right) \left(P(E|Y_k) - P(E|Y_{k+1}) \right) \right|
$$

$$
\leq \; \sum_{k=1}^{N} \left(\sum_{j=1}^{k} \frac{j}{k}\, P(Y_j) \right) \frac{1}{k^{1+\alpha}} k^{2+\alpha} |P(E|Y_k) - P(E|Y_{k+1})|
$$

$$
\leq \; C \sum_{k=1}^{N} \left(\sum_{j=1}^{k} P(Y_j) \right) \frac{1}{k^{1+\alpha}}
$$

$$
\leq \; C \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha}}
$$

$$
\leq \; C',
$$

and the resulting bound is independent of N.

For the second term, because of $P(E|Y_k) \to 0$, there is, for any $k \in \mathbb{N}$, a $N_0(k)$ such that

$$
\left(\sum_{j=1}^{k} j\, P(Y_j) \right) P(E|Y_{N_0(k)}) \leq C'/2. \tag{14}
$$

Then we can calculate

$$\left(\sum_{j=1}^{k} j\, P(Y_j)\right) P(E|Y_{k+1})$$

$$\leq \ \left(\sum_{j=1}^{k} j\, P(Y_j)\right) |P(E|Y_k) - P(E|Y_{k+1})| + \left(\sum_{j=1}^{k} j\, P(Y_j)\right) P(E|Y_{k+1})$$

$$\leq \ \ldots$$

$$\leq \ \left(\sum_{j=1}^{k} j\, P(Y_j)\right) \left(\sum_{l=k}^{N_0(k)-1} |P(E|Y_l) - P(E|Y_{l+1})|\right) + \left(\sum_{j=1}^{k} j\, P(Y_j)\right) P(E|Y_{N_0(k)})$$

$$\leq \ \left(\sum_{j=1}^{k} \frac{j}{k}\, P(Y_j)\right) \left(\sum_{l=k}^{N_0(k)-1} \frac{k}{l^{2+\alpha}} l^{2+\alpha} |P(E|Y_l) - P(E|Y_{l+1})|\right) + C'/2$$

$$\leq \ \left(\sum_{j=1}^{k} P(Y_j)\right) \left(\sum_{l=k}^{N_0(k)-1} \frac{C}{l^{1+\alpha}}\right) + C'/2$$

$$\leq \ C \left(\sum_{l=1}^{\infty} \frac{1}{l^{1+\alpha}}\right) + C'/2$$

$$\leq \ C',$$

proving the uniform boundedness of the second summand and thereby the sufficiency of the fourth and last condition for $\langle Y \rangle_E < \infty$. □

Proof of Theorem 4: Let us define

$$Y_k^+ := \{Y \geq k\} \qquad\qquad Y_k^- := \{Y < k\}$$

We have assumed that $P(E|Y_k^+) \leq P(E|Y_k^-)\, \forall k \in \mathbb{N}$, with inequality for at least one $k > 0$. Since Y_k^+ and Y_k^- are an exhaustive partition of the probability space, this entails that Y_k^+ and E are negatively relevant to each other, and that

$$P(Y_k^+|E) \leq P(Y_k^+)\ \forall k \in \mathbb{N}, \tag{15}$$

with inequality for at least one $k > 0$. Since $P(Y_k) = P(Y_k^+) - P(Y_{k+1}^+)$, we obtain by a simple diagonalization trick

$$
\begin{aligned}
\langle Y \rangle &= \sum_{k=0}^{\infty} k \, P(Y_k) \\
&= \sum_{k=0}^{\infty} \left(k \, P(Y_k^+) - k \, P(Y_{k+1}^+) \right) \\
&= 0 \cdot P(Y_0^+) + \sum_{k=1}^{\infty} \left(k P(Y_k^+) - (k-1) P(Y_k^+) \right) \\
&= \sum_{k=1}^{\infty} P(Y_k^+),
\end{aligned}
\tag{16}
$$

and similarly

$$
\langle Y \rangle_{\mathrm{E}} = \sum_{k=1}^{\infty} P(Y_k^+ | \mathrm{E}).
\tag{17}
$$

Combining (16) and (17), we conclude

$$
\langle Y \rangle_{\mathrm{E}} = \sum_{k=1}^{\infty} P(Y_k^+ | \mathrm{E}) < \sum_{k=1}^{\infty} P(Y_k^+) = \langle Y \rangle
$$

because of $P(Y_k^+ | \mathrm{E}) \leq P(Y_k^+) \, \forall k \in \mathbb{N}$ (see (15)), and because we have assumed inequality for at least one $k > 0$. $\qquad\square$

Proof of Proposition 2: By a straightforward application of Bayes' Theorem:

$$
\begin{aligned}
\langle Y \rangle_{\mathrm{E}} &= \sum_{k=1}^{\infty} k \, P(Y_k | \mathrm{E}) = \frac{1}{P(\mathrm{E})} \sum_{k=0}^{\infty} k \, P(Y_k) P(\mathrm{E} | Y_k) \\
&\leq \frac{1}{P(\mathrm{E})} \sum_{k=0}^{\infty} k \, P(Y_k) = \frac{1}{P(\mathrm{E})} \langle Y \rangle \\
&< \infty. \quad \square
\end{aligned}
$$

Richard Dawid

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Department of Philosophy and Institute Vienna Circle

University of Vienna

Universitätsstr. 7

1010 Vienna

Austria

http://homepage.univie.ac.at/richard.dawid/

richard.dawid@univie.ac.at

Stephan Hartmann

Munich Center for Mathematical Philosophy

Ludwig Maximilians-Universität München

Ludwigstr. 31

80539 München

Germany

http://www.stephanhartmann.org

Stephan.Hartmann@lrz.uni-muenchen.de

Jan Sprenger

Tilburg Center for Logic and Philosophy of Science

Tilburg University

PO Box 90153

5000 LE Tilburg

The Netherlands

http://www.laeuferpaar.de

j.sprenger@uvt.nl

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