Quantum Field Theory: Underdetermination, Inconsistency, and Idealization

Doreen Fraser
Department of Philosophy
University of Waterloo
200 University Ave. W.
Waterloo, ON, Canada N2L 3G1
dlfraser@uwaterloo.ca

ABSTRACT: Quantum field theory (QFT) presents a genuine example of the underdetermination of theory by empirical evidence. There are variants of QFT which are empirically indistinguishable yet support different interpretations. This case is of particular interest to philosophers of physics because, before the philosophical work of interpreting QFT can proceed, the question of which variant should be subject to interpretation must be settled. At one end of the spectrum of variants of QFT is the version which is found in introductory textbooks and employed by most working physicists; this is the variant of QFT which introduces renormalization procedures to facilitate the calculation of scattering matrix elements. At the other end of the spectrum are axiomatic presentations of QFT, which are rigorous but remote from practical applications. New arguments are offered for basing the interpretation of QFT on a rigorous axiomatic variant of the theory. The pivotal considerations are the roles that consistency and idealization play in this case.

KEYWORDS: quantum field theory, algebraic quantum field theory, Lagrangian quantum field theory, underdetermination, inconsistency, idealization

1 Introduction

Quantum field theory (QFT) is an example of a mature field within a mature science in which there are parallel research programs. Different communities of researchers (including experimentalists, theoreticians, and applied mathematicians) employ different variants of QFT. At one end of the spectrum of variants of QFT is the version of the theory which is found in the first part of most introductory textbooks and employed by most working physicists, viz. the version of QFT which introduces renormalization procedures to facilitate the calculation of scattering matrix elements. At the other end of the spectrum are axiomatic presentations of QFT, such as the Haag-Kastler algebraic formulation and the Wightman axiomatization, which are rigorous but remote from practical applications. There is no consensus on how philosophers should deal with this multitude of QFTs. Before the philosophical work of interpreting quantum field theory (QFT) can begin, a question must be addressed: which variant of QFT should be subject to interpretation? Arguments have been made in favour of interpreting variants of QFT on both ends of the spectrum. Teller treats the standard textbook variant of QFT in his book *An Interpretive Introduction to Quantum Field Theory*; Wallace's article "In defence of naiveté: The conceptual status of Lagrangian quantum field theory" is devoted to defending the use of textbook QFT for foundational purposes; and, most recently, MacKinnon has advanced arguments in favour of the same approach in this journal (MacKinnon 2008). In contrast, the recent work of Ruetsche, Earman, Halvorson, Clifton, and others examines the algebraic formulation of QFT.

This debate is an instance of a more general conflict of desiderata of philosophers of science, a conflict that seems particularly endemic to philosophy of physics. On the one hand, it is desirable to stick as close to actual scientific practice as possible. This means that philosophers should focus attention on the versions of theories that practicing scientists actually use and, in particular, how theories get applied. However, these theories are often messy. A theme of the work of Batterman and Cartwright (among others) is that the messy context of application is important for foundational and interpretive questions. A second desideratum of philosophers of science is to clarify the foundations of theories and to provide interpretations of theories, where necessary. These goals are often more easily achieved by focusing on cleaner versions of theories which are farther removed from actual applications. For instance, 'toy' models–models which represent idealized situations–have been used to investigate many foundational and interpretive questions.

The choice between these two desiderata is particularly stark in the QFT case because the desiderata are best fulfilled not by different aspects of a single theoretical framework, but by different theoretical frameworks. The textbook variant of QFT has a range of applications and its predictions have been borne out to an impressive degree of accuracy, but it is not mathematically rigorous. In contrast, rigorous axiomatizations of QFT have been proposed, but to date no physically realistic models of any set of axioms has been found. More specifically, there is no known model for any interaction in four spacetime dimensions.[1] The arguments that have been offered for focusing on a particular variant of QFT track the general debate. For instance, Teller explains that he does not discuss "formal and rigorous work in axiomatic field theory" in his book because "[a]lthough [axiomatic field theory] is a useful enterprise in the study of formal properties of quantum field theories, axiomatic quantum field theory as it exists today does not appear usefully to describe real physical phenomena" (Teller 1995, p. 146, fn. 22). In his survey article on the philosophical significance of algebraic QFT, Halvorson concedes that the algebraic approach is "indeed idiosyncratic in the sense of demographics," but argues that philosophers of physics should study algebraic QFT because "[t]here remains an implicit working assumption among many philosophers that studying the foundations of a theory requires that the theory have a mathematical description" and, moreover, "whether or not having a mathematical description is mandatory, having such a description greatly facilitates our ability to draw inferences securely and efficiently" (Halvorson and Müger 2007).

I contend that an interpretation of QFT should be based on a rigorous axiomatic variant of QFT rather than any of the other variants. I will argue that there is more to recommend a rigorous variant of QFT than ease of interpretation and transparency of the foundations of the theory. *A fortiori*, the choice among formulations of QFT does not come down to a subjective preference for a certain methodology in philosophy of science. The content of QFT is the point at issue.

[1]Recently, a method has been developed which can be used to construct models with non-trivial scattering matrix elements in any number of spacetime dimensions, but these models are not physically realistic insofar as the S-matrix breaks Lorentz symmetry (Buchholz and Summers 2008). A model takes the form of a set of wedge-localized operators which commute at spacelike distances, transform covariantly under the underlying representation of the Poincare group, and admit a scattering theory (Buchholz and Summers 2008, p. 1).

2 Set up: Three variants of QFT, Empirical Indistinguishability, and Underdetermination

To set up the discussion, I will categorize the variants of QFT on the basis of the types of renormalization procedures that are invoked. In the course of the discussion, it will become clear why this is a natural categorization. This set of distinctions will also help to clarify the foundational significance of renormalization.

For the sake of concreteness, consider the admittedly physically unrealistic case of a $(\phi^4)_2$ interaction (i.e., an interaction represented by a Lagrangian with a scalar ϕ^4 self-interaction term on two-dimensional spacetime). A ϕ^4 interaction is the simplest non-trivial, physically meaningful interaction because a quadratic self-interaction term represents a free system and a cubic self-interaction term is not physically meaningful because even the classical wave equation has energy that is unbounded from below and has singular solutions[2] (Jaffe 1999, p. 133; Keller 1957). The restriction to two spacetime dimensions allows the ultraviolet divergences to be completely removed by normal-ordering (Glimm and Jaffe 1968, p. 175). The Hamiltonian for the $(\phi^4)_2$ interaction is defined as follows (Glimm and Jaffe 1968, p. 1945):

$$H = H_F + \lambda \int : \phi^4(x, t_0) : dx \tag{1}$$

where : : denotes normal-ordering. In classic introductory textbook presentations of QFT (e.g., Schweber (1961)), this Hamiltonian acts on the Hilbert space of the interaction picture representation.[3] For the moment, the only relevant feature of the interaction picture representation is that, by assumption, there is some time t_0 at which the Hilbert space representation for the interaction coincides with the Fock space $\mathcal{F}$ associated with a neutral scalar field of the same mass. (For further details, see Schweber (1961, pp. 316-325) or Fraser (2006, §2.1).) In particular, at time t_0, the vacuum state for the interacting system coincides with the vacuum state for the associated free system, Ω_0. At time t_0, applying the full interaction Hamiltonian H to the vacuum state Ω_0 yields the following well-known result:

$$H\Omega_0 = \infty \tag{2}$$

From a theoretical point of view, this result is problematic because the vacuum state is supposed to be invariant under the time translation operator e^{-iHt}. From a practical point of view, this is problematic because the formula for calculating scattering matrix elements involves applying H to Ω_0, and the scattering matrix elements encode the experimental predictions. (Naturally, this traditional introductory textbook presentation lacks the sophistication of the modern approach to renormalization (i.e., renormalization group methods); however, the modern refinements do not affect the arguments of this paper. See Section 6 below.)

The variants of QFT adopt different solutions to this problem. In the classic textbook presentation of QFT, this result is interpreted as a sign that the vacuum self-energy must be renormalized. That is, the infinite vacuum self-energy counterterm E_0 is introduced to make the lowest

[2]Though recently this conventional widsom has been challenged by advocates of non-Hermitian Hamiltonians. See, for example, Bender (2007).

[3]The interaction picture representation is also known as the Dirac picture representation.

eigenvalue of the renormalized interaction Hamiltonian zero:

$$(H - E_0)\Omega_0 = 0 \tag{3}$$

$$H_{ren} = H_F + \lambda \int : \phi^4(x, t_0) : dx - E_0 \tag{4}$$

(As a result of the restriction to two spacetime dimensions, the vacuum self-energy E_0 is the only renormalization counterterm that is infinite in this case (Glimm and Jaffe 1970a, p. 205)). The classic textbook approach is to accept the infinite counterterm E_0 and to introduce the Hamiltonian H_{ren}. I will refer to the resulting formulation of QFT as the *infinitely renormalized* variant of QFT.

There are two other ways of responding to the result that application of the $(\phi^4)_2$-interaction Hamiltonian H to the vacuum Ω_0 yields infinity. One is to introduce a spatial cutoff function into the Hamiltonian which "cuts off" long-distance contributions:

$$H(g) = H_F + \lambda \int : \phi^4(x, t_0) : g(x) dx - E_0' \tag{5}$$

$$g(x) = 1 \text{ in some finite region } R \text{ and } g(x) = 0 \text{ outside of } R$$

The introduction of the long-distance spatial cutoff function renders the vacuum self-energy counterterm E_0' finite. This is the *cutoff* variant of QFT. In general (i.e., for interaction terms in general), short-distance cutoffs will also be needed. (Long-distance cutoffs alone are sufficient for the $(\phi^4)_2$ interaction because the ultraviolet divergences are rendered finite by normal-ordering.)[4]

The third variant of QFT–the *formal* variant–responds by taking a more formal approach. In the context of formal, rigorous mathematics, the fact that applying the interaction Hamiltonian H to the interaction picture vacuum Ω_0 yields infinity is an indication that the vector Ω_0 is not in the domain of H (see, for example, Glimm and Jaffe (1970a, p. 363)). Furthermore, it turns out that the only vector in $\mathcal{F}$ that is in the domain of H is the zero vector (Glimm 1969). From this more formal point of view, the proper response is thus to find another Hilbert space on which to represent the interaction Hamiltonian (i.e., a Hilbert space representation of the canonical commutation relations that is unitarily inequivalent to $\mathcal{F}$). For the $(\phi^4)_2$ interaction this has been achieved: Glimm and Jaffe's $(\phi^4)_2$ model supplies a Hamiltonian operator for the interaction without an infinite counterterm that is well-defined on a Hilbert space representation of the canonical commutation relations that is unitarily inequivalent to $\mathcal{F}$ at all times. The Glimm-Jaffe $(\phi^4)_2$ model satisfies all of the Wightman and the Haag-Kastler axioms (Glimm, Jaffe, and Spencer 1974; Glimm and Jaffe 1970a; Cannon and Jaffe 1970; Glimm and Jaffe 1970b, p. 208). This formal approach to QFT has been pursued by the mathematical physicists and mathematicians named in this paragraph. One strategy for pursuing this approach is the formulation of axioms for QFT. The complementary strategy is the construction of models for particular interactions, which can then be checked for their compatibility with different sets of axioms.

The infinitely renormalized, cutoff, and formal variants of QFT are all alternative formulations of the same theory. This raises the question: How should these variants of QFT be regarded?

[4] An alternative to introducing a long-distance spatial cutoff function is compactifying space. This is also known as "quantization in a box with periodic boundary conditions" (Glimm and Jaffe 1971, p. 5).

This is an important question for the purposes of interpretation, as mentioned above, but also for foundational purposes (e.g., for determining which variant should inform the development of future theories, perhaps including quantum gravity). There is an obvious line of response to the question: perhaps this matter can be settled empirically. Perhaps only one variant of QFT is empirically adequate. However, it turns out that this is not the case. The three variants are empirically indistinguishable. In QFT, the quantities which are subject to experimental test are scattering matrix elements. Scattering matrix elements encode the outcomes of scattering experiments. The three variants of QFT are empirically indistinguishable in the sense that the sets of scattering matrix elements generated by the three variants can be brought into arbitrarily close agreement. Consider the infinitely renormalized and cutoff variants. In the limit as the cutoffs are removed, the scattering matrix elements of the cutoff formulation approach the scattering matrix elements of the infinitely renormalized formulation. (See Fraser (2006, §3.3.2) for details). Thus, the sets of scattering matrix elements generated by these two variants can be brought into arbitrarily close agreement by choosing large (or small) enough cutoff functions.[5] It has been verified that the sets of scattering matrix elements yielded by the rigorous Glimm-Jaffe $(\phi^4)_2$ model also agree with those yielded by the infinitely renormalized variant, modulo the fact that in the infinitely renormalized framework scattering matrix elements can only be calculated approximately to some order in a perturbative expansion (Jaffe 1999, p. 140; Wightman 1986, p. 213). Since a model for a given interaction is required to calculate the scattering matrix elements for that interaction, one might worry that models that are constructed in the future will reveal disagreements between the predictions of the formal variant and those of the other variants. However, this worry is addressed by the process of model construction: one of the means of establishing that a constructed model is a model for a particular interaction is to verify that the scattering matrix elements calculated using the model agree with those calculated using other variants (Glimm 1969, p. 103). In general, then, one would expect agreement between the scattering matrix elements generated by the formal variant and the other variants.

Since experimental tests cannot settle the matter, one might wonder whether these three variants of QFT are genuinely distinct. Perhaps, for example, they are merely notational variants. However, it is clear that proponents of these approaches to QFT believe that they differ significantly. As Wightman described the situation in a 1962 lecture, "[t]he root-mean-square deviation from the mean of opinion on what is a sensible thing to try to do in elementary particle theory seems to be one of those unrenormalizable infinities one hears about" (Wightman 1963, p. 11). From the perspective of philosophy rather than sociology, there is also reason to regard the variants as genuinely distinct: they yield different metaphysical interpretations. The infinitely renormalized and cutoff representations for the $(\phi^4)_2$ interaction are genuinely distinct from the Glimm-Jaffe representation because they support different ontologies. For instance, they disagree about whether QFT describes quanta. ("Quanta" are entities that resemble classical particles insofar as they are countable and possess the same energies as classical, relativistic, non-interacting particles.) The complete argument for this claim that the variants of QFT disagree about whether QFT describes quanta is a paper in itself (see Fraser (2008)), but it can be

[5] In this analysis, the cutoff is not interpreted realistically. If the cutoff is interpreted realistically, then its value cannot be changed arbitrarily and the limiting procedure is illegitimate. However, as discussed in Section 4 below, nobody seems to actually hold the view that QFT dictates that space must be discrete and the universe spatially finite.

sketched in a few paragraphs.

Consider the cutoff and Glimm-Jaffe representations. The cutoff representation for an interaction admits an interpretation in terms of quanta, but the Glimm-Jaffe representation does not admit a quanta interpretation. In QFT, the quanta interpretation is derived from the Fock space representation for a system. A Fock space representation for a free system has a basis of state vectors–the n-particle state vectors–each of which can be interpreted as representing a state in which a definite number of quanta is present. In a cutoff representation for an interacting system, this quanta interpretation can be extended from free systems to interacting systems. Since the number of degrees of freedom is finite,[6] the Stone-von Neumann theorem applies and all standard representations[7] of the ETCCR's are unitarily equivalent. In particular, the representation for the interacting system is unitarily equivalent to the Fock representation for the corresponding free system. Therefore, any state vector describing the interacting system can be written in terms of the n-particle states for the corresponding free system. Thus, in the cutoff representation, the interacting system can also be given a quanta interpretation. In contrast, the Glimm-Jaffe representation does not admit a quanta representation. The Fock representation for the corresponding free system is not unitarily equivalent to the Glimm-Jaffe representation for the interacting system. (See Fraser (2008) for the argument.) Thus it is not possible to extend the quanta interpretation from the free system to the interacting system, and there are no other candidates for quanta states. As a result, the cutoff and the Glimm-Jaffe representations disagree about the existence of quanta.

It is less clear how to interpret the infinitely renormalized variant. The difficulties are a direct result of lack of mathematical rigor. The argument can be made that the infinitely renormalized representation can be given a quanta interpretation. By assumption, there is some time t_0 at which the Hilbert space representation for the interaction coincides with the Fock space $\mathcal{F}$ for the corresponding free system. Also by assumption, time evolution is unitary; therefore, the Hilbert space representation for the interacting system is unitarily equivalent to $\mathcal{F}$ at all times. That is, the state vector for the interacting system is always in Fock space, so it is always possible to write it in terms of the n-particle states for the free system. Thus, it seems that, in the context of the infinitely renormalized variant, the interacting system can be given a quanta interpretation. However, infinite renormalization complicates matters. The key assumption that time evolution is unitary is undermined by infinite renormalization: the renormalized Hamiltonian for the interaction H_{ren} is not a well-defined self-adjoint operator on the Hilbert space for the interacting system because it contains an infinite term; consequently, the time translation operator $exp(-iH_{ren}t)$ is not unitary.[8] Deciding which interpretation the infinitely renormalized variant actually supports will require a judgment about which principles this variant actually includes. The standard criticism leveled against unrigorous theories–that they are difficult to analyze and interpret–certainly applies in this case.

The disagreement on matters of metaphysics among the three variants of QFT can also be made out in more general terms. The variants disagree about what is possible. According to the cutoff variant, it is possible for the interacting system to be in the same state as any other system,

[6] At least in the typical cases in which both long and short distance cutoffs are introduced.

[7] i.e., All irreducible representations of the Weyl form of the equal time canonical commutation relations.

[8] Another possibility is that, according to the infinitely renormalized variant, there is an informal sense in which the state of the interacting system 'remains in' the Fock representation for the free system even though, formally, the two representations are not unitarily equivalent.

governed by any dynamics. The infinitely renormalized variant incorporates the assumption that it is possible for the interacting system to be in the same state as a free system. (Indeed, scattering theory in the interaction picture works on the basis of the assumption that the free and interacting systems are actually in the same state at $t = \pm\infty$.) This disagreement is the upshot of their respective assumptions about unitary equivalence. In contrast, according to the formal variant, it is not possible for the interacting system to be in the same state as a system governed by any other dynamics. In addition, Baker (2009) argues that the considerations which rule out a quanta interpretation of the formal variant also rule out a natural field interpretation of the formal variant. Thus, the disagreement between the variants is not limited to the existence of quanta. Even if we decide that QFT does not describe quanta we can expect that the representations will still disagree on matters of metaphysics. Other respects in which the interpretations differ are discussed below.

The three variants are empirically indistinguishable yet carry different metaphysical implications. In other words, this is a genuine case of the classic problem of underdetermination of theory by all possible evidence. To emphasize, this is not a contrived example of underdetermination, but a real case of underdetermination that actually arises in a mature physical theory. This addresses a criticism leveled by skeptics about underdetermination: that the only examples of the phenomenon are 'toy theories' invented by philosophers. However, the anti-realist should not be too quick to celebrate. I contend that this case of underdetermination does not provide support for scientific anti-realism because there are good arguments for paying heed to the metaphysical implications of one variant of QFT and disregarding the others.

3 Inconsistency

The starting point for resolving the question of which variant of QFT to adopt is a better understanding of the point of departure for the variant formulations of QFT. What went wrong with the construction of the interaction picture[9]? Theorists adopted a set of seemingly plausible principles for a relativistic quantum theory but ended up with nonsensical results (i.e., a vacuum state with infinite energy and infinite probabilities for the outcomes of scattering experiments). Historically, this is precisely the difficulty that Feynman et. al. had to overcome in order to formulate QED. Renormalization methods were the remedy, but they do not directly address the question of what went wrong in the first place.

Haag's theorem provides some insight into what went wrong with the interaction picture. Haag's theorem can be viewed as a 'no go' theorem for the interaction picture. The theorem establishes that if a theory adopts the specified set of assumptions $\{T\}$, then the theory necessarily describes a free system. The interaction picture goes wrong by adopting the complete set of assumptions $\{T\}$. The set $\{T\}$ contains the following assumptions:[10]

The interacting system and the corresponding free system are each described by a

[9]Here and in the following discussion "interaction picture" denotes the pre-renormalization representation of the system adopted in the introductory textbook approach to QFT.

[10]There are several versions of Haag's theorem. The following discussion is based on the version of the theorem presented in Hall and Wightman (1957) for the special case in which one of the fields is a free field. For further details see Earman and Fraser (2006) and Fraser (2006).

neutral scalar[11] field ϕ_j, $j = 1, 2$, and a conjugate momentum field π_j such that

(i) Each pair (ϕ_j, π_j) gives an irreducible representation of the equal time CCR (ETCCR)

$$[\phi_j(\mathbf{x}, t), \pi_j(\mathbf{x}', t)] = i\delta(\mathbf{x} - \mathbf{x}') \qquad j = 1, 2 \tag{6}$$
$$[\phi_j(\mathbf{x}, t), \phi_j(\mathbf{x}', t)] = [\ \pi_j(\mathbf{x}, t), \pi_j(\mathbf{x}', t)] = 0.$$

(ii) Poincaré transformations (a, Λ) (where a stands for a spacetime translation and Λ stands for a Lorentz transformation) are induced by unitary transformations $U_j(a, \Lambda)$.

(iii) The fields transform under $U_j(a, \Lambda)$ as follows:

$$U_j(a, \Lambda)\phi_j(x, t)U_j^{-1}(a, \Lambda) = \phi_j(\Lambda x + a, t) \tag{7}$$
$$U_j(a, \Lambda)\pi_j(x, t)U_j^{-1}(a, \Lambda) = \pi_j(\Lambda x + a, t)$$

(iv) There exist unique normalizable states $|0_j\rangle$ invariant under Poincaré transformations (i.e., vacuum states):[12]

$$U_j(a, \Lambda)|0_j\rangle = |0_j\rangle \tag{8}$$

(v) The fields are related at some time t by a unitary transformation V:

$$\phi_2(\mathbf{x}, t) = V\phi_1(\mathbf{x}, t)V^{-1}, \qquad \pi_2(\mathbf{x}, t) = V\pi_1(\mathbf{x}, t)V^{-1} \tag{9}$$

(vi) No states of negative energy exist.

These assumptions are, taken individually, each plausible. However, Haag's theorem establishes that, when combined, this package of assumptions has the unwanted consequence that the field that was intended to describe an interacting system actually describes a free system. More specifically, what is proven is that all of the vacuum expectation values (VEVs) of the two fields are equal:

$$\langle 0_2|\phi_2(x_1)...\phi_2(x_n)|0_2\rangle = \langle 0_1|\phi_1(x_1)...\phi_1(x_n)|0_1\rangle \text{ for all } n \tag{10}$$

This means that the 'interacting' representation makes exactly the same predictions as the representation for the free system. The set of all VEVs for a field contains the set of all its scattering matrix elements. Thus, the representation that was intended to describe an interacting system actually describes a system with trivial scattering matrix elements (i.e., the scattering matrix is the identity); that is, the representation actually describes a free system in which the initial state is identical to the final state. Assurance that theories that share a complete set of VEVs share the same dynamics in the sense of having the same Hamiltonian is provided by Wightman's reconstruction theorem (Streater and Wightman 2000, pp. 117-126).

[11] The case of neutral scalar fields is treated for notational convenience. Haag's theorem also holds in the more general case of fields with spin indices (Streater and Wightman 2000, p. 166). The commutation relations below are then CCR's or CAR's, as dictated by the Spin-Statistics Theorem.

[12] Actually, this is stronger than necessary. The weaker assumptions that suffice are that there exist unique normalizable states $|0_j\rangle$ which are invariant under Euclidean transformations and that these states $|0_j\rangle$ are also invariant under Poincaré transformations.

Put in slightly different terms, Haag's theorem demonstrates that the interaction picture adopts an inconsistent set of principles. Inconsistency is a straightforward consequence of the theorem. Let F be the statement that the system described is free. By Haag's theorem, $\{T\} \Longrightarrow F$. But, the interaction picture was introduced for the purpose of treating interacting systems; thus, by assumption, the system described by the interaction picture is not free. This sets up a *reductio ad absurdum*: $\{T\}\&\neg F \Longrightarrow F\&\neg F$. Thus, Haag's theorem informs us that the source of the problem with the interaction picture is that it is inconsistent. Furthermore, Haag's theorem establishes that this is an entirely generic problem; the theorem does not hinge on any assumptions about the specific form taken by the interaction.[13]

The three variants of QFT espouse different responses to the *reductio* of the interaction picture. The infinitely renormalized variant modifies the principles of the interaction picture by inserting infinite renormalization counterterms into the Hamiltonian. The practical consequence is that the scattering matrix elements become nontrivial and finite; the theoretical consequence is that Haag's theorem is inapplicable to the interaction picture after the renormalization procedures have been carried out. Strictly speaking, the assumptions of Haag's theorem no longer hold. For example, the renormalized interaction Hamiltonian $H_{ren} = H_F + \lambda \int : \phi^4(x, t_0) : dx - E_0$ is not a formally well-defined self-adjoint operator[14] in virtue of the infinity of E_0. By Stone's theorem, since H_{ren} is not self-adjoint, the operator $e^{-iH_{ren}t}$ is not unitary. Thus, time translations are represented by operators that are not unitary, contrary to assumption (ii) of Haag's theorem. Put another way, infinite renormalization introduces informal mathematical reasoning in the form of infinite subtractions and informally defined mathematical expressions. Hall and Wightman's proof of Haag's theorem relies on heavy machinery from advanced mathematics (e.g., the theory of analytic functions). The infinitely renormalized variant of QFT dodges Haag's theorem by shifting to the context of informal, unrigorous mathematics, in which these techniques of formal, rigorous mathematics are inapplicable.

The cutoff variant of QFT responds to the *reductio* of the interaction picture by Haag's theorem by denying one of the common assumptions of the interaction picture and Haag's theorem: that the fields transform appropriately under Poincaré transformations (assumption (iii) above). The Hamiltonian $H(g)$ with the spatial cutoff function $g(x)$ ($g(x) = 1$ in region R and $g(x) = 0$ outside of R) is associated with the 'cutoff' field equation

$$\left(\Box + m_0^2\right) \phi_g(x) + 4g\lambda_0\phi_g^3(x) = 0 \tag{11}$$

The 'cutoff field' $\phi_g(x)$ that satisfies this field equation is clearly not invariant under spatial translations:[15]

$$e^{-iP\cdot x}\phi_g(x,t)e^{iP\cdot x} \neq \phi_g(x + x', t) \tag{12}$$
$$\text{for } x \text{ such that } g(x) = 1,\ x' \text{ such that } g(x') = 0$$

[13]Though interaction terms which take particular forms will violate some of the assumptions of Haag's theorem. For an example see the discussion below of Hamiltonians incorporating cutoff functions.

[14]On the Hilbert space $\mathcal{F}$, the Fock space for the corresponding free system. As noted above, strictly speaking, the domain of H_{ren} on $\mathcal{F}$ contains only the zero vector (Glimm and Jaffe 1970b, p. 363).

[15]Introduction of a short distance cutoff function would also be sufficient to violate spatial translation invariance and thus would render the version of Haag's theorem presented above inapplicable. However, there are other versions of Haag's theorem (e.g., the Streit-Emch version) which are applicable in the presence of short distance spatial cutoffs (i.e., to lattices) (See Emch (1972, pp. 247-253), Fraser (2006, pp. 63-64)).

In general, the cutoff variant of QFT introduces both long- and short-distance spatial cutoff functions, which serve to reduce the theory from an infinite number of degrees of freedom to a finite number. Haag's theorem is only applicable in the presence of an infinite number of degrees of freedom. The cutoff variant also invalidates assumption (iv) of Haag's theorem, that there exists a unique normalizable state $|0\rangle_j$ (i.e., the vacuum state) that is invariant under Poincaré transformations. For a finite number of degrees of freedom the assumption of uniqueness fails. (See Fraser (2006, Section 1.2.2) for further discussion.) Naturally, when the time comes to evaluate the cutoff variant of QFT, the physical motivations for rejecting these assumptions will be a central issue.

The formal variant of QFT also confronts the inconsistency of the interaction picture by seeking to modify or reject at least one of the assumptions of the interaction picture. The rigorous Glimm-Jaffe model for the $(\phi^4)_2$ interaction rejects assumption (v), that the fields are related at some time t by a unitary transformation V. This is the natural assumption to reject because, unlike the other assumptions, it does not have a strong physical motivation. While it seems reasonable to guess that the representation for the interaction will coincide with the representation for the corresponding free system in the limit of infinitely early and late times, when the interaction is negligible, this is not an essential feature of a relativistic quantum theory. Historically, the motivation for introducing this assumption was not foundational, but practical; it made it possible to extract predictions from QFT via renormalization. From a mathematical perspective, the assumption of unitary equivalence of the free and interacting representations is dubious. In the context of ordinary non-relativistic quantum mechanics with a finite number of degrees of freedom, the Stone-von Neumann theorem guarantees that all representations of the ETCCRs are unitarily equivalent. However, the Stone-von Neumann theorem fails in QFT due to the infinite number of degrees of freedom; in fact, there exist uncountably many unitarily inequivalent representations of the ETCCRs (Gårding and Wightman 1954).

4 $QFT = SR + QT$

With a view towards evaluating them, these three variants of QFT can be characterized as providing either a principled response to the inconsistency of the interaction picture or a pragmatic response. The formal variant is a clear example of the principled response: the inconsistency of the interaction picture must be addressed by fixing the principles of the interaction picture because the inconsistency of the principles reflects a problem with the foundations of the theory. In contrast, the infinitely renormalized variant is a clear example of the pragmatic response: the expedient of introducing infinite renormalization is adequate as a means of getting around the problem of inconsistency because it allows predictions to be derived from the theory. (Which category the cutoff variant falls into depends on how the cutoffs are justified, as discussed below.) The principled approach is to cure the disease; the pragmatic approach is to treat the symptoms.

For scientific theories in general, it is not clear that the principled strategy of revising the principles of the theory to make it consistent is always preferable. The argument has been made that, even for mature scientific theories, it can be appropriate to retain an inconsistent theory and to merely treat the symptoms.[16] However, in the case of QFT, there is a compelling reason to

[16] For example, see daCosta and French (2002), among other contributions to Meheus (2002).

demand a consistent formulation of the theory. Quantum field theory is *by definition* the theory that best unifies quantum theory (QT) and the special theory of relativity (SR).[17] Historically, it was clear by the mid-1920's, when physicists had obtained non-relativistic quantum mechanics, that physicists should work on formulating a relativistic version of quantum theory.[18] Since $QFT = QT+SR$, the project of formulating quantum field theory cannot be considered successful until either a consistent theory that incorporates both relativistic and quantum principles has been obtained or a convincing argument has been made that such a theory is not possible. The big foundational question lying in the background is, of course, whether the principles of quantum theory and special relativity are consistent. I do not presume that this question has a positive answer. However, I do maintain that the project of developing QFT cannot be considered complete until this central foundational question has been answered. The pragmatic response to the inconsistency of the interaction picture is inadequate because it leaves the question of whether QFT is possible unanswered. To put the point provocatively, it is not clear that any variant of QFT that adopts a pragmatic response to the infinities is a theory that satisfies the definition of QFT. Either a principled response or a demonstration that a principled response is not possible is required.

The infinitely renormalized variant of QFT is unsatisfactory because it neither furnishes a manifestly consistent set of principles for relativistic quantum theory nor provides a reason to believe that a consistent relativistic quantum theory is impossible. On the latter score, the inconsistency of the interaction picture does not constitute evidence that it is not possible to consistently formulate a relativistic quantum theory because we have no reason to believe that the interaction picture is the only possible way to realize such a theory. For example, the formal variant furnishes examples of alternatives (e.g., the Wighman axiomatization and the Haag-Kastler axiomatization). With respect to the former point, after infinite renormalization procedures have been carried out, the resources of formal, rigorous mathematics cannot be brought to bear to test the consistency of the theory. Consequently, it would be difficult to determine whether the infinitely renormalized variant of QFT is consistent. Note that these criticisms have nothing to do with the efficacy of infinite renormalization procedures for the purpose of deriving predictions. To the best of my knowledge, no set of contradictory predictions has been derived in the more than fifty year history of applications of the theory. I grant that it is extremely unlikely that physicists will ever unearth a contradiction.[19] However, this fact should not be taken as evidence in favor of the consistency of the theory. It is possible that physicists are just reasoning very carefully from an inconsistent set of theoretical principles. That is, they may be employing inferential restrictions. Many historical examples of this phenomenon have been brought to light by

[17]Two qualifications of this definition. (1) This is intended to be a definition of *relativistic* quantum field theory (Galilean QFTs, for example, are excluded). (2) It is not obvious a priori that the theory that best unifies SR and QT must be a field theory; I take it that non-field theories (e.g., S-matrix theory) will be ruled out either on the grounds of not providing the best unification or on the grounds of being failed programs.

[18]In fact, it was already clear before non-relativistic quantum mechanics had been worked out: Schrödinger experimented with a relativistic wave equation en route to his wave mechanics and the earliest version of the Klein-Gordon equation was published in 1925 (Schweber 1994, p. 57; Mehra and Rechenberg 2001, p. 445; Kragh 1990, pp. 49-50).

[19]Huggett (2002) offers an argument based on renormalization group theory that S-matrix elements follow as deductive consequences from the interaction picture framework. Note that renormalization group methods do not solve the problem with the interaction picture that is brought to light by Haag's theorem; however, they may fall into the category of inferential restrictions.

historians and philosophers of science, including the old quantum theory and Newtonian gravity (see Meheus (2002)).

In contrast, the formal variant of QFT offers a satisfactory line of response to the *reductio* of the interaction picture because the response is principled. The goal of the formal variant is to find a consistent reformulation of the principles of the interaction picture. Again, the consistency of quantum and relativistic principles is desired, but not presupposed. Ultimately, the success of this project will hinge on whether there exist realistic models of the resultant axioms. (I will return to this issue in Section 5).

The short argument that the cutoff variant is unsatisfactory is that it does not satisfy the definition of QFT. Setting aside the principled-pragmatic distinction, the cutoff variant cannot, strictly speaking, be considered a relativistic theory because the fields are not Poincaré covariant.

The long argument that the cutoff variant is unsatisfactory appeals to the principled-pragmatic distinction. Whether the cutoff variant of QFT falls into the category of a principled or a pragmatic response to the *reductio* depends on how the cutoffs are interpreted. If the cutoffs are regarded as a convenient device that is employed for the purpose of facilitating the derivation of predictions, then the cutoff variant offers a pragmatic response to the inconsistency of the interaction picture. However, the cutoff variant also contains the resources for a principled response. As explained above, the cutoff variant rejects common assumptions of the interaction picture and Haag's theorem; consequently, the principles of the cutoff variant are consistent. In general, consistency is achieved by reducing the theory to a finite number of degrees of freedom by defining the fields on a spatial lattice of finite extent. However, to provide a genuine principled response to the inconsistency of the interaction picture, the cutoffs must be regarded as essential elements of the theory and not merely dispensable add-ons to facilitate calculation. On the latter, instrumental view, the content (i.e., core theoretical principles) of the cutoff variant of QFT is identical to that of the infinitely renormalized variant; it is merely the approach to renormalization as a tool for deriving predictions that differs.[20] The way in which the cutoffs are viewed will have repercussions for their interpretation. If the cutoffs are taken seriously, then they must be interpreted realistically; that is, space is really discrete and of finite extent according to the cutoff variant of QFT.[21] Thus, the cutoffs must take particular fixed values (though these may not be presently known). If the cutoffs are not taken seriously, then they may be interpreted instrumentally; that is, space is really continuous and of infinite extent according to the cutoff variant of QFT. This is compatible with assigning the cutoffs arbitrary values. The upshot is that the cutoff variant of QFT can only be regarded as a candidate formulation of relativistic quantum theory if it makes sense to regard space as a lattice of finite extent.

Can a case be made for interpreting the cutoffs realistically? If QFT were true, would space be discrete and finite in extent? It is telling that–to the best of my knowledge–nobody defends the position that QFT provides evidence that space is discrete and the universe is finite. Of course, proponents of some quantum theories of gravity have claimed that space is discrete. However, even if these claims are borne out, the fact that quantum gravity indicates that space is discrete would not help settle the question of how to interpret the cutoff variant of QFT

[20]Wallace (2006) endorses a different version of the cutoff variant; this position is discussed in Section 6.

[21]An apparent alternative for taking the cutoffs seriously is to regard space as continuous and of infinite extent and QFT as breaking down at small and large distance scales. Since I take an interpretation of QFT to involve providing a description of a possible world in which QFT is true (see Section 6), I regard this apparent alternative as being equivalent to regarding space as discrete and finite in extent.

because gravitational considerations are external to QFT. The point at issue is whether QFT dictates that space is necessarily discrete and finite in extent; that is, whether the discreteness and finitude of space is a foundational principle of QFT.

One reason that a realistic interpretation of the cutoffs is not compelling is illuminated by considering, once again, the cutoffs as a response to the *reductio* of the interaction picture. As a response to the *reductio*, the introduction of the assumption that space is discrete and the universe is finite seems *ad hoc*. It does circumvent Haag's theorem and produces a consistent set of principles for QFT, but it does not have an independent motivation. In contrast, arguments from quantum gravity that space is discrete are supported by deep theoretical considerations about how gravity is to be quantized. Arguments from cosmology that the universe is spatially finite are based on a combination of theory and a variety of experimental results. The desire for a consistent formulation of QFT is not sufficient to justify the introduction of the assumption that space is discrete and the universe is finite because there is another strategy for obtaining a consistent formulation of QFT: the strategy of modifying other principles of the interaction picture, which is adopted by the formal variant. *If* we had good reason to believe that this strategy will fail, then perhaps we would have good reason to interpret the cutoffs realistically. However, even in that case, it would remain an option to concede that it is not possible to formulate a consistent relativistic quantum theory. (This issue will be taken up in Section 5.) In sum, the cutoff variant of QFT is unsatisfactory because the cutoffs are not interpreted realistically and when the cutoffs are interpreted instrumentally the cutoff variant is subject to the same criticisms as the infinitely renormalized variant.

The formal variant is the only one of the three variants to provide an adequate response to the *reductio* of the interaction picture. Viewed in these terms, the choice among the variants of QFT is not (merely) a matter of picking the variant that is easiest to subject to analysis and interpretation; the choice is a matter of determining the content of QFT. Each of the three variants of QFT adopts a different set of theoretical principles. In this sense, each of the three variants prescribes a different foundation for QFT. Since the content of the theory is what is at stake, it is not surprising that the matter is settled by foundational considerations rather than purely interpretive considerations. The formal variant is not singled out because it supports a particular ontology or even because the other candidates succumb to interpretive difficulties, but because QFT should be a relativistic quantum field theory (if such a thing is possible) and it is the only variant that satisfies this condition. In some respects, the situation in QFT parallels the situation in non-relativistic quantum mechanics. In non-relativistic quantum mechanics, alternative, physically distinct formulations of the theory (e.g., Bohmian mechanics) were also sought out in response to foundational problems with the standard formulation of the theory. The significant point of contrast is, of course, that the consistency of relativistic and quantum principles is a different kind of foundational problem than the measurement problem.

As an aside, the foregoing considerations about consistency may also pose a difficulty for the interpretation of infinitely renormalized QFT. A consistent set of theoretical principles may all be true simultaneously; an inconsistent set is certainly not.[22] If a theory contains an inconsistent set

[22] Unless, that is, one agrees with Priest (2002) that contradictions do occur in nature. Briefly, my response is that we would need to be driven to this by compelling considerations, and I am arguing that QFT does not supply them.

da Costa and French (2002) also argue that it is appropriate to regard inconsistent theories as true, but as partially true rather than wholly true, a notion that they explicate using model theory. This approach does not of theoretical principles, some members of the set must be false. A metaphysical interpretation of a theory should be based on the true theoretical principles, but not the false ones. The need to identify the true principles complicates the project of interpreting an inconsistent theory. This must be borne in mind when interpreting the infinitely renormalized variant of QFT because it is not known whether this variant is consistent.

5 But we possess no realistic models of any set of rigorous axioms for QFT!

The position that I have been advocating–that the formal variant of QFT should be treated as the official formulation of QFT–seems counterintuitive in an important respect: it is not clear that any of the proposed sets of rigorous, consistent principles for QFT holds for any realistic system! To date, no rigorous model for a realistic interaction has been obtained–for any axiomatization of QFT. This is why I have been referring to the Glimm-Jaffe model for the $(\phi^4)_2$ interaction. David Wallace worries that "pending the discovery of a realistic interacting [model for an axiomatization of QFT]...we have only limited reason to trust that our results apply to the actual world" (Wallace 2006, p. 34). This is a reasonable concern; if the formal variant of QFT is to be viable, it must be addressed.

This concern can be alleviated by recognizing that the formal variant of QFT is best viewed as a program which has yet to be completed. This axiomatic program cannot be considered complete until models of a rigorous set of axioms are found for realistic interactions. The starting point for the program is the interaction picture and the *reductio* of the interaction picture underwritten by Haag's theorem. As Haag himself recognized in Haag (1955), the paper which contains the first presentation of the eponymous theorem, the first step was to reject the assumption of the interaction picture that there is a time at which the representation for the interacting system is unitarily equivalent to the representation for the free system.[23] That this was the first step was confirmed by the first success of the program: the construction of the rigorous model for the $(\phi^4)_2$ interaction by Glimm and Jaffe in the early 1970's. This model satisfies all of the assumptions of the Hall-Wightman version of Haag's theorem with the sole exception of the assumption that there is a time at which the representation is unitarily equivalent to the Fock representation for a free system. However, even before the $(\phi^4)_2$ model had been obtained, it was clear that further refinements to the interaction picture would be necessary to obtain a consistent set of principles applicable to realistic interactions. Recall that Haag's theorem pertains only to the divergence in vacuum self-energy, which can be treated by introducing a long-distance cutoff. For realistic interactions, there are other types of divergences (e.g., ultraviolet divergences, which are treated using short-distance cutoffs). It would be very useful if it were possible to prove a sequence of Haag-type no-go theorems: e.g., a $(\text{Haag})^2$ theorem which establishes that a certain set of the principles embraced by the interaction picture imply that the system under consideration is a free system when the dimension of spacetime is greater than two. Such a sequence of theorems

undermine my argument because we would still be left with the problem of how to determine which parts of the theory are true. In their terms, I am arguing that the formal variant is to be preferred because it may be wholly true, while the infinitely renormalized variant may only be partially true.

[23] In the abstract, he writes "[i]t is shown that ... Dyson's matrix $U(t_1, t_2)$ for finite t_1 or t_2 cannot exist" (Haag 1955, p. 1).

might give an indication of which assumptions of the interaction picture would need to be revised or abandoned to treat particular interactions.

By the early 1960's, Wightman had proposed one set of axioms for QFT (Wightman 1959) and Haag and Kastler had proposed another set of axioms (Haag and Kastler 1964). These axiom systems were conjectures about how the principles of the interaction picture would need to be refined in order to consistently accommodate realistic interactions. The provisional nature of these proposed axiomatizations was recognized from the outset. In one of the first textbook presentations of the formal approach to QFT, Jost coined the term "general field theory" because he did not like the connotations of the term "axiomatic" (Jost 1965, p. xi). Haag gives the following explanation of why he also prefers to employ Jost's terminology:

> ...the word "axiom" suggests something fixed, unchangeable. This is certainly not intended here. Indeed, some of the assumptions are rather technical and should be replaced by more natural ones as deeper insight is gained. We are concerned with a developing area of physics which is far from closed and should keep an open mind for modifications of the assumptions, additional structural principles as well as information singling out a specific theory within a general frame. (Haag 1996, p. 58)

For the axiomatic program to achieve completion, realistic models of some set of principles for relativistic quantum theory must be found. The construction of realistic models for either (or both) the Wightman or Haag-Kastler axioms[24]–or any other set of axioms that has been proposed, for that matter–would complete the program, but this is not the only way in which the program could be completed. It is also possible that the sets of principles proposed by Wightman and Haag-Kastler must be further refined before they admit realistic models. For example, Bender (2007) advocates dropping the requirement that the that the Hamiltonian be Hermitian and Rivasseau (2007) makes a case for adopting non-commutative spacetime.

This approach to the formal variant gains further plausibility when QFT is viewed in its proper historical context. QFT is hardly unusual in requiring refinements over a period of time. There are many cases in which an inconsistent formulation of a theory was replaced by a consistent one in the course of its historical development. There is also historical precedent for an informally formulated theory later being given a formal reformulation. Arguably, the evolution of Newtonian mechanics from Newton's *Principia* to its modern-day textbook formulation is an example of this.

From this perspective, the fact that no rigorous model for a realistic interaction has been constructed is not an argument for disregarding the formal variant of QFT. In fact, precisely the opposite is the case: the Wightman and Haag-Kastler axiomatizations improve upon the interaction picture by refining its axioms. The rigorous models for idealized interactions which have been constructed–interactions which exhibit tamer instances of the same types of divergences that plague realistic interactions–provide evidence that the refinements that have been made are on the right track. In contrast, the infinitely renormalized and cutoff variants of QFT do not institute appropriate refinements of the principles of the interaction picture, as I've argued. The fact that no rigorous model for a realistic interaction has been obtained is not an argument

[24]Streater (1988) rehearses the considerations that suggest that QED in four dimensions and QCD cannot be fit into the framework of the Wightman axioms. He remarks that "[i]t should be possible to fit QCD into the framework of the Haag-Kastler algebraic axiomatization" (p. 147).

against the formal variants and for the alternatives, but this fact should not be overlooked by philosophers studying the foundations and interpretation of QFT. The provisional nature of the proposed axiomatizations of QFT carries the implication that foundational and interpretive conclusions based on these axiomatizations are also provisional.

What if the formal variant of QFT is a program which it is not possible to complete? That is, what if it turns out not to be possible to find a consistent set of relativistic and quantum principles that are applicable to realistic interacting systems? This is a hypothetical question because mathematical physicists are still working towards constructing models for realistic interactions. The results that have been obtained so far do not give any reason to believe that this an unattainable goal. However, it is interesting to speculate on what the failure of the program to formulate an applicable yet consistent relativistic quantum theory would mean. One interpretation of the result is that relativistic and quantum principles are inconsistent. This seems to be what Streater and Wightman had in mind when they wrote in the introduction to their text book *PCT, Spin, and Statistics, and All That* that "the Main Problem of quantum field theory turned out to be to kill it or cure it: either to show that the idealizations involved in the fundamental notions of the theory (relativistic invariance, quantum mechanics, local fields, etc.) are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics" (Streater and Wightman 2000, p. 1). Another way of interpreting the failure of the formal program would be as a victory for the cutoff variant: as discussed in Section 4, failure to obtain a consistent set of principles on infinite, continuous space could be cited as a justification for treating space as finite and discrete. A third alternative is that the no-go result for the formal program could be ambiguous.[25] The requirement is that relativistic quantum theory include recognizably relativistic principles and recognizably quantum principles; however, there is a certain amount of latitude in deciding what counts as a relativistic principle and what counts as a quantum principle. In any case, it is worth pursuing the formal program because either success or failure would be a significant result for the foundations and interpretation of QFT.

6 In Defence of Sophistication

I have been arguing against basing the interpretation of QFT on the cutoff variant of the theory. In a paper entitled "In defence of naiveté: The conceptual status of Lagrangian quantum field theory," David Wallace defends the cutoff variant of QFT. By "interpretation" I mean the activity of giving an answer to the following hypothetical question: "if QFT were true, what would reality be like?" In contrast, the interpretive question that Wallace focuses on is "given that QFT is approximately true, what is reality (approximately) like?" The fact that QFT does not furnish a true description of the actual world makes this a substantial point of disagreement. QFT marries special relativity and quantum theory, but does not incorporate general relativity. Following Cao

[25]Perhaps the Coleman-Mandula result is an example of this. Supersymmetric theories were introduced in response to the Coleman-Mandula 'no go' theorem for QFT's in four spacetime dimensions (Coleman and Mandula 1967). Supersymmetric theories generalize the framework of QFT by allowing the algebra of generators of symmetries to contain both communtation and anticommutation relations (Wess and Bagger 1992, p. 4). The resultant algebra is not a Lie algebra, but a pseudo-Lie algebra (also known as a superalgebra or a graded Lie algebra) (Haag, Lopuszański, and Sohnius 1975, pp. 257-8; Wess and Bagger 1992, p. 2).

(1997), Wallace frames the debate in terms of which one of two attitudes should be adopted to the foundational status of QFT:[26]

> (i) "The current situation is genuinely unsatisfactory: we should reject the cutoff theories ....[27], and continue to look for nontrivial theories defined at all lengthscales," or
>
> (ii) "QFT's as a whole are to be regarded as only approximate descriptions of some as-yet-unkown deeper theory [theory X], which gives a mathematically self-contained description of the short-distance physics" (p. 45)

Wallace elaborates that

> the difference between [(i)] and [(ii)] is that the former rejects current QFT *in toto*, and looks for mathematically rigorous versions of *our current QFTs*: QED, $\lambda\phi^4$, the Standard Model, etc. By contrast [(ii)] accepts that these current theories are indeed best understood in terms of Lagrangian QFT, and looks for a deeper-level theory in which Lagrangian QFT as a whole can be grounded. (p. 45)

In order to resolve the "attendant foundational problem" with option (ii)–that is, the problem of how we can "give a clean conceptual description of a theory which can be rigorously defined only as the low-energy limit of another theory which we do not yet have"[28] (p. 46)–Wallace adopts what I have been calling the cutoff variant of QFT. A a short-distance cutoff which is much shorter than the length scales at which QFT is applied and tested is imposed.

Recall that in Section 4 I argued that the cutoff variant is unsatisfactory because it offers only a pragmatic response to the *reductio* of textbook QFT. These arguments are not straightforwardly applicable to Wallace's position because the above arguments are predicated on the assumption that interpreting QFT involves describing what reality would be like if QFT were true. In accordance with attitude (ii), Wallace instead regards QFT as providing an approximate description of the actual world at suitably large distance scales. As he puts it, "[s]uccess in [algebraic QFT, string theory, or another theory-of-everything candidate] ... would, of course, revolutionise physics, but that success would scarcely change the current status of Lagrangian QFT: as an inherently approximate, but nonetheless extraordinarily powerful tool to analyse the deep structure of the world" (Wallace 2006, p. 75).This stance allows Wallace to argue that there is a principled physical justification for introducing the cutoffs without necessarily being forced to defend the view that space is discrete and finite in our world. Wallace lists three alternatives which each provide a "physical justification...for imposing a cutoff in relativistic QFT" (p. 43).

[26]Following Cao (1997), Wallace lists as a third option "the picture of 'an infinite tower of effective field theories'," but Wallace does not discuss this possibility.

[27]He adds "as not mathematically well-defined," but–as Wallace himself goes on to argue–the (finitely renormalized) cutoff variant does not suffer from this problem; only the (infinitely renormalized) variant *without* cutoffs is not mathematically well-defined.

[28]Indeed, Wallace summarizes the aim of his paper as follows: "[t]his paper is an investigation of whether Lagrangian QFT is sufficiently well-defined conceptually and mathematically that it too can be subject to foundational analysis" (p. 2). It should be clear that I agree that the cutoff variant–Lagrangian QFT with cutoffs–is sufficiently well-defined that its foundations *could* be analyzed; however, I maintain that the cutoff variant *should not* be subject to foundational analysis.

Two of these possible physical justifications involve the physics imposing an effective cutoff for QFT without space being discrete. For example, it is possible that the field-theoretic description breaks down at a very high energy scale and is replaced by a different kind of theory (e.g., string theory) (p. 44). Wallace's attempted defence of the cutoff variant is not successful. My first line of objection is that, even if it were granted that QFT should be regarded as an approximate guide to the ontology of relatively large distance scales, it does not follow that the cutoff variant of QFT succeeds in describing large-scale ontology. I take issue with the premise that the content and interpretation of the cutoff variant is approximately equivalent to that of algebraic QFT (the leading strain of the formal variant of QFT). The more important line of objection is the second one: that QFT should not be regarded as *merely* an approximate guide to the ontology of relatively large distance scales. Wallace's insistence that we respect the fact that QFT is not our final theory actually supports the pursuit of the formal variant of QFT.

An underlying assumption of Wallace's argument is that there is approximate agreement on matters of ontology between the cutoff variant of QFT and algebraic QFT (an instance of what I have been calling the formal variant). This assumption is incorrect. As Wallace represents matters,

> ... we can see that Lagrangian QFT (as I have defended it) is not really in conflict with A[lgebraic] QFT at all. Success in the A[lgebraic] QFT program would leave us with a field theory exactly defined on all scales, and such a theory would be a perfectly valid choice for 'theory X': furthermore, even if we found such an exact QFT it would not prevent us from defining low-energy, 'effective' QFTs–which would not be well defined without a cutoff... (p. 41; see also p. 35 and p. 75).

One of the main conclusions of the foregoing sections is that the variants of QFT differ with respect to both content and matters of interpretation. Changing the subject from precise agreement to approximate agreement does not affect matters. Wallace contends that cutoff QFT approximately satisfies the axioms of QFT.[29] For example, the axioms of Poincaré covariance and local commutativity are strictly false, but approximately satisfied over large distance scales (Wallace 2006, p. 50). However, the cutoff variant does not have even approximately the same content as algebraic QFT because the cutoff variant has a finite number of degrees of freedom and therefore does not admit unitarily inequivalent representations; in contrast, algebraic QFT has an infinite number of degrees of freedom and therefore admits unitarily inequivalent representations.[30] Spontaneous symmetry breaking is one case in which these unitarily inequivalent representations are put to use (see Earman (2004) for an exposition and preliminary analysis of this case). Cutoff and algebraic QFT admit different interpretations for the same reason. A theory according to which quanta exist is not approximately equivalent to a theory according to which quanta do not exist. The flaw in Wallace's reasoning is that the fact that we do not care about what QFT tells us about short distance scales does not license chopping short distance

[29]Note, however, Wallace's important qualification that "[o]bviously, this is not intended to be precise" (p. 49) Note also that, at this point, the discussion shifts from cutoff Lagrangian QFTs to algebraic QFTs with cutoffs. The relevant comparison is between cutoff Lagrangian QFTs and algebraic QFTs *without* cutoffs.

[30]Wallace advances an alternative interpretation of free bosonic cutoff QFT, according to which particles are emergent entitites (Wallace 2001). "Particles" in Wallace's sense differ from "quanta" in my sense in that Wallace's "particles" satisfy a locality condition.

scales out of the theory. Cutting off the theory at some short distance scale has the effect of changing the content of the theory as a whole, including its description of long-distance scales. The difference in physical content is also reflected in the fact that the variants include different sets of theoretical principles. A consequence of these differences in content is that the theories support different ontologies.

Wallace does address the issue of unitarily inequivalent representations. He summarizes his argument as follows:

> From a realist perspective the sting of the representation ambiguity has largely been drawn. Locally, any representation ambiguity is artificial, caused by the presence of unphysical degrees of freedom beyond the high-energy limit of the theory's validity. Globally, there may indeed be representation ambiguities – depending on cosmology, and the topology of the universe – but the inaccessible information which they encode is 'respectable', analogous to the classical inaccessibility of the long-distance structure of the universe. (p. 58)

In the present context, this argument is not compelling because the issue of whether the "degrees of freedom beyond the high-energy limit of the theory's validity" need to be taken into account is precisely the point of contention.

A source of motivation for Wallace's position is the thought that renormalization group methods provide a warrant for subjecting the cutoff variant of QFT to interpretation. As Wallace points out, renormalization group methods establish that the imposition of a short distance cutoff "has no practical consequences" for the predictions of the theory (p. 44). However, this contingent fact about some QFTs is not by itself sufficient warrant for the conclusion that the cutoff variant should be subject to interpretation. Renormalization group methods establish that the *predictions* of the cutoff variant are in approximate agreement with the predictions of the infinitely renormalized variant but they do not furnish any evidence about approximate agreement of either the *theoretical principles* or the *interpretations* supported by the cutoff and infinitely renormalized variants. *A fortiori*, renormalization group methods have no bearing on whether or not there is approximate agreement between either the theoretical principles or the interpretations of the cutoff and algebraic variants of QFT. This brings the discussion back to underdetermination.[31]

There is a more fundamental difficulty with Wallace's position. Wallace contends that the correct attitude to adopt towards QFT is to accept that our "current theories [i.e., the Standard Model, QED, etc.] are indeed best understood in terms of Lagrangian QFT, and looks for a deeper-level theory in which Lagrangian QFT as a whole can be grounded" (p. 45). The problem with this approach is that it gets things the wrong way around. QFT, as the predecessor theory, is valuable for the hints that it gives about the content of the successor theory, theory X. Theory X is the theory that furnishes an adequate description of short-distance physics and, thus, includes quantum gravity. But, as the name indicates, theory X is an as-yet-undiscovered theory. How are we to determine the content of theory X? One of the starting points for formulating theory

[31] There is certainly more to be said here. This topic will be taken up in a paper to be published with the proceedings of the April 2009 University of Western Ontario Philosophy of Quantum Field Theory Workshop.

X is QFT.[32] Wallace commits himself to the following brand of scientific realism (at least with respect to Lagrangian QFT): Lagrangian QFT is "an inherently approximate, but nonetheless extraordinarily powerful tool to analyse the deep structure of the world" (Wallace 2006, p. 75). If (Lagrangian) QFT is to play this role, then theory change from QFT to theory X must involve approximate continuity at the theoretical level and not merely approximate continuity at the empirical level. This makes the theoretical content of QFT particularly relevant for finding theory X. As I have stressed, the content of QFT is precisely what is at issue in the choice among variants of QFT. How the disagreements about the appropriate theoretical principles for QFT get settled is germane to the development of theory X. Resting content with Lagrangian QFT is thus a poor strategy for finding theory X; the better strategy is to continue to work on these questions about the content of QFT.

This is an entirely general point about theoretical change that does not hinge on the details of theory X. Of course, until we actually find theory X and can determine the respects in which there is approximate continuity between theory X and QFT, we have no way of knowing how helpful it would have been to work out the details of QFT. The fact remains that–given our limited knowledge at the present time–continuing to pursue the development of QFT is a good strategy. However, one can hypothesize about how pursuing the axiomatic program might illuminate the content of theory X. Entering into the realm of pure speculation, here are some examples. As Wallace himself points out, the fact that the cutoff variant is approximately (but not exactly) Poincaré covariant is of little help for finding theory X; approximate Poincaré covariance at large distance scales does not provide any indication about whether theory X is covariant or not ("understood as, say, the absence of a preferred spacetime foliation") (pp. 51-52). In contrast, either a formal variant of QFT which contains exact Poincaré covariance as a principle or the knowledge that it is not possible to consistently combine Poincaré covariance with other field-theoretic principles (e.g., knowledge that the axiomatic QFT program fails) would be useful for finding theory X. If the debate were to be settled in favour of the cutoff variant with realistically construed cutoffs, this might lend support to approaches to quantum gravity according to which space is discrete. More fundamentally, if it were to turn out that even special relativity and quantum theory are incompatible (if, say, the axiomatic program proves to be incompletable), then the strategy for combining general relativity and quantum theory could be affected. More generally, it is conceivable that the quest for a rigorous model of a realistic model will uncover a more general mathematical framework which could prove more suitable for formulating quantum gravity. (For hints in this direction see Bender (2007) and Rivasseau (2007)) I concede that these examples are pure speculation. I hope that this does not deflect attention from the argument that settling questions about the theoretical content of QFT might aid the search for theory X. Resting content with the cutoff variant of QFT because it is empirically adequate at large distance scales would be a strategic mistake because it would hinder the search for theory X.

[32] Of course, there is disagreement among theorists working on rival programs for quantum gravity about the extent to which QFT should be used as a starting point. However, theory X is a theory of everything, including non-gravitational forces; therefore, QFTs such as QED are relevant starting points.

7 Conclusion

This paper has explored the implications of a genuine case of underdetermination in contemporary physics. The infinitely renormalized, cutoff, and formal variants of QFT are empirically equivalent in the sense that they are empirically indistinguishable on the basis of past and future scattering experiments. The three variants differ with respect to physical content; each variant adopts a different set of theoretical principles. The three variants also admit different interpretations. For example, the cutoff variant supports a quanta ontology but the formal variant does not. I argued that the proper moral to draw from this case is not anti-realism, but that empirical considerations narrowly construed are not the only grounds on which to choose among rival theories. In this case, consistency is also a relevant criterion because quantum field theory is, by definition, the theory which integrates quantum theory and the special theory of relativity. Consistency is relevant to QFT for theoretical reasons–not for practical reasons (e.g., the derivation of predictions). As a result, it is necessary to either formulate a consistent theory or else show that this criterion cannot be satisfied (i.e., that there is no consistent theory with both quantum and special relativistic principles). The infinitely renormalized variant fails to satisfy this criterion because its theoretical principles are not manifestly consistent. The cutoff variant fails to satisfy the criterion because, while its theoretical principles are consistent, they are not well-motivated. The formal variant is the only variant that satisfies the criterion; its set of theoretical principles are both consistent and well-motivated. Neither the infinitely renormalized nor cutoff variant furnishes an argument that a consistent formulation of QFT is impossible; such an argument would require making the case that the axiomatic program cannot be completed.

This covers the terms "underdetermination" and "inconsistency" in the title. Idealization also plays an interesting role in this case. The standard criticism of the formal variant of QFT–that it has yet to be established that any set of rigorous axioms applies to the real world because no realistic models have been constructed–can be construed as a complaint that the formal variant of QFT is an idealization. In response to this criticism, I have argued that–if it does turn out that none of the sets of axioms that have been proposed admit realistic models–then the proposed axiomatizations could be modified. Thus, if it does turn out that the proposed axiomatizations are idealizations in this sense, the idealization is in principle dispensable; the idealization could be removed by appropriate modification of the axioms (modulo the possibility that there is no set of relativistic plus quantum axioms that admits realistic models).

The cutoff variant is an idealization in another sense: when the theoretical principles of the cutoff variant are interpreted literally, the cutoff variant describes a world in which space is discrete and of finite extent; this is an idealization in the sense that the possible worlds in which QFT is true are presumed to be worlds in which space is continuous and infinite. This idealization is indispensable insofar as it is not possible to remove the cutoffs entirely. (Removing the cutoffs by taking infinite limits would turn the cutoff variant into the infinitely renormalized variant, which would mean adopting a different set of theoretical principles and introducing the attendant set of problems.)[33]

[33] The distinction that I am drawing here between dispensable and indispensable idealizations is similar to Batterman's distinction between traditional and nontraditional views of mathematical modeling. One difference is that revising the axioms of a theory is not what he has in mind when he says that "the traditional view aims, ultimately, to 'de-idealize' by adding more details so as to bring about a convergence to a complete and accurate description" (Batterman 2009, p. 4). Another difference is pointed out in the discussion of infinite limits in the

The significance of idealization in quantum statistical mechanics (QSM) has recently been the subject of debate (see Callendar (2001), Batterman (2005) and Ruetsche (2003)). Since there are many similarities between QFT and QSM, one might expect this debate about idealization to map onto QFT; however, idealization plays opposing roles in the two cases. The root issues in the QSM case are the same as in the QFT case: an infinite number of degrees of freedom and unitarily inequivalent representations. In QSM, the thermodynamic limit is taken, which is an idealization insofar as the system under consideration (e.g., a steaming cup of coffee) is represented as containing an infinite number of particles. The thermodynamic limit, which makes available unitarily inequivalent representations, must be invoked to represent phase transitions (e.g., a transition from a liquid to a gas). Another point of similarity is that renormalization group methods are employed in both QFT and QSM. However, there is a crucial difference between QSM and QFT with an infinite number of degrees of freedom (i.e., either the infinitely renormalized variant or the formal variant): whereas the description of a system as containing an infinite number of particles furnished by QSM is taken to be false, the description of space as continuous and infinite that is furnished by QFT with an infinite number of degrees of freedom is taken to be true. As far as idealization is concerned, the parallel should not be drawn between QSM and QFT *with an infinite number of degrees of freedom*, but between QSM and QFT *with a finite number of degrees of freedom* (i.e., the cutoff variant). In the context of QFT, representing space as being discrete and finite is the same sort of idealization as, in the context of QSM, representing a system as containing an infinite number of particles. Both idealizations are also indispensable, in the above sense. However, the fact that there is no alternative to QSM which is both capable of handling phase transitions and does not employ the idealization means that there is an argument for adopting QSM that is not available for QFT with cutoffs.

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