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# A "FUNDAMENTAL" AXIOMATIZATION OF MULTIPLICATIVE POWER RELATIONS AMONG THREE VARIABLES*

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Suppose that entities composed of two independent components are qualitatively ordered by a relation that satisfies the axioms of conjoint measurement. Suppose, in addition, that each component has a concatenation operation that, together either with the ordering induced on the component by the conjoint ordering or with its converse, satisfies the axioms of extensive measurement. Without further assumptions, nothing can be said about the relation between the numerical scales constructed from the two measurement theories except that they are strictly monotonic. An axiom is stated that relates the two types of measurement theories, seems to cover all cases of interest in physics, and is sufficient to establish that (the multiplicative form of) the conjoint measurement scales are power functions of the extensive measurement scales.[2]

**1. Introduction.** To illustrate the problem considered in this paper, consider a specific physical relation such as $E = \frac{1}{2}mv^2$, where $E$ is the kinetic energy of an object, $m$ is its mass, and $v$ is its velocity. Classically, each of these quantities is measured separately and the numerical relation among them is empirically verifiable. In principle, at least, both $m$ and $v$ can be measured fundamentally by means of the theory of extensive measurement, which, in each case, rests upon having an empirical operation of concatenation: two masses can be concatenated (placed together) to form a third mass; two velocities in a given direction can be concatenated (added) to form a third velocity in that direction. Energy is not usually measured fundamentally, but rather in some derived fashion.

Applying the general results of Luce and Tukey [1], to this example, it can be shown that if an empirical method exists for determining which of two moving objects has the greater kinetic energy and if that qualitative relation satisfies certain axioms (see Sec. 2), which it would according to classical physics, then "energy" measures $\phi_1$ and $\phi_2$ can be assigned to the mass and velocity components in such a way that the object $(m, v)$ qualitatively has greater kinetic energy than $(m', v')$ if and only if

$$\phi_1(m) + \phi_2(v) > \phi_1(m') + \phi_2(v'),$$

or, setting $u = \exp \phi$, if and only if

$$u_1(m)u_2(v) > u_1(m')u_2(v').$$

Thus, this system of fundamental measurement, which is called conjoint measurement, yields in this case the simultaneous measurement of the energy contribution of all three variables, and it provides an alternative to the theory of extensive measurement.

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It should be noted that no concatenation operations are postulated in the axioms of conjoint measurement; rather, the construction is based upon a trade-off between the two independent components. It therefore offers a possible means of fundamental measurement when no operation of concatenation has been identified, and so it is of at least potential interest to the behavioral sciences.

A problem is created, however, in those cases, such as the one above, in which concatenation operations exist, because we now have two equally plausible methods of fundamental measurement. And there is no clear assurance that they yield the same measures. One would like to know that the two measures are, in fact, the same in physically interesting cases—in this example, that $u_1(m) = \beta_1 m$ and $u_2(v) = \beta_2 v^2$. Actually, the relations cannot possibly be quite this simple since the multiplicative conjoint measurement scales are unique only up to the choice of scale and a common exponent, i.e., if $u_1$ and $u_2$ satisfy the representation then so do $\beta_1 u_1{}^\alpha$ and $\beta_2 u_2{}^\alpha$, where $\alpha > 0$. Thus, the strongest result that we can hope to establish for this example is that $u_1(m) = \beta_1 m^\alpha$ and $u_2(v) = \beta_2 v^{2\alpha}$.

The purpose of this paper is to state an axiomatic structure sufficient to prove a result of which the above is a special case. In essence, I assume that the axioms of conjoint measurement hold for quantities having two independent components, that the axioms of extensive measurement hold for each of these components separately, and that a new axiom relates the two measurement systems. From these assumptions it is shown that the desired relation holds between each of the two pairs of numerical measures. Moreover, if such relations hold between the numerical scales, the axiom relating the measurement systems is necessary. This result justifies the "it can be shown" assertion about the relation between the conjoint measurement of momentum and the extensive measurement of mass and velocity in Sec. III of [1].

**2. Axioms and Preliminary Results.** Let $A_1$ and $A_2$ be sets, $R$ a binary relation over $A_1 \times A_2$, and $o_\rho$ a binary operation on $A_\rho$, $\rho = 1, 2$. Denoting the converse of $R$ by $R'$, then in the usual fashion let $I = R \cap R'$ and $P = R - I$. The first three of the four axioms of conjoint measurement are:

**Axiom C1.** *$R$ is a weak ordering of $A_1 \times A_2$.*

**Axiom C2.** *For each $a$ in $A_1$ and $p$, $q$ in $A_2$, the equation $(a, p)I(f, q)$ has a solution $f$ in $A_1$; and for each $a$, $b$ in $A_1$ and $p$ in $A_2$, the equation $(a, p)I(b, x)$ has a solution $x$ in $A_2$.*

**Axiom C3.** *For $a, b, f$ in $A_1$ and $p, q, x$ in $A_2$, if $(a, x)R(f, q)$ and $(f, p)R(b, x)$, then $(a, p)R(b, q)$.*

**Definition 1.** *Define the relations $R_\rho$ on $A_\rho$, $\rho = 1, 2$, by: $aR_1b$ if and only if $(a, x)R(b, x)$ for some $x$ in $A_2$; $pR_2q$ if and only if $(f, p)R(f, q)$ for some $f$ in $A_1$.*

**Theorem 1.** *If axioms C1–C3 hold, then $R_\rho$ is a weak ordering of $A_\rho$.*

*Proof.* Theorems V H and V K, [1].

Next we introduce all save one of Suppes's, [2], axioms for extensive measurement in $\langle A_\rho, R_\rho{}^*, o_\rho \rangle$, $\rho = 1, 2$, where $R_\rho{}^*$ denotes either $R_\rho$ or its converse, $R_\rho'$. This complication about which relation is involved is necessary in order to cope with laws of the form $F = mr^{-2}$, for which the conjoint ordering on the second component corresponds to $1/r$ whereas the extensive measurement theory is based on an ordering that corresponds to $r$. Observe that there is no need to distinguish between $I_\rho$ and $I_\rho{}^*$ since $I_\rho{}' = I_\rho$. There is no real need to state the first axiom of the Suppes system since the transitivity of $R_\rho{}^*$ is guaranteed by Theorem 1, but in order to retain his numbering I state it explicitly. In all axioms, $a$, $b$, $c$ are in $A_\rho$.

**Axiom E1.** *$R_\rho{}^*$ is transitive* (Theorem 1).

**Axiom E2.** *$ao_\rho b$ is in $A_\rho$.*

**Axiom E3.** $[(ao_\rho b)o_\rho c]R_\rho{}^*[ao_\rho(bo_\rho c)]$.

**Axiom E4.** *If $aR_\rho{}^*b$, then $(ao_\rho c)R_\rho{}^*(co_\rho b)$.*

**Axiom E5.** *If $aP_\rho{}^*b$, then there exists $c$ in $A_\rho$ such that $aI_\rho(bo_\rho c)$.*

**Axiom E6.** $(ao_\rho b)P_\rho{}^*a$.

Suppes's final (Archimedean) axiom is not listed since it will be derived from the Archimedean axiom of conjoint measurement and the other axioms. To state the final axiom of conjoint measurement, we need:

**Definition 2.** *A doubly infinite sequence of pairs $\{a_i, p_i\}$, $i = 0, \pm 1, \pm 2, \ldots$, where $a_i$ is in $A_1$ and $p_i$ is in $A_2$, is a* non-trivial dual standard sequence (dss) *provided that for all $i$*

i. *not* $(a_{i+1}I_1a_i)$ *and not* $(p_{i+1}I_2p_i)$,
ii. $(a_i, p_{i+1})I(a_{i+1}, p_i)$,
iii. $(a_{i+1}, p_{i-1})I(a_i, p_i)$.

The Archimedean axiom for conjoint measurement is:

**Axiom C4.** *If $\{a_i, p_i\}$ is a non-trivial dss, $b$ is in $A_1$, and $q$ is in $A_2$, then there exist integers $j$, $k$ such that*

$$(a_k, p_k)R(b, q)R(a_j, p_j).$$

**Definition 3.** *Let $o_\rho$ be a binary operation of $A_\rho$. For $a$ in $A_\rho$ and $i$ an integer, $ia$ is defined recursively by*: $1a = a$, $ia = (i-1)ao_\rho a$.

The next axiom guarantees that $A_\rho$ includes all "rational fractions" of elements. It is not needed in the usual statement of extensive measurement, but I have been unable to prove the desired result (Theorem 3) without it.

**Axiom F.** *For each $a$ in $A_\rho$ and for each positive integer $i$, there exists $b$ in $A_\rho$ such that $aI_\rho ib$.*

It follows immediately from this axiom that the following quantities exist.

**Definition 4.** *Let $r = i/j$ where $i$ and $j$ are positive integers and let $a$ be in $A_\rho$. Denote by $ra$ any element, which is unique up to $I_\rho$, such that $iaI_\rho j(ra)$.*

Throughout the rest of the paper, let $i, j, k, l$ denote positive integers and $r = i/j$ and $s = k/l$ rationals.

**Lemma 1.** *Suppose that Axioms* E1–E6 *and* F *hold. If $r > s$, then for all $a$ in $A_\rho$, $raP_\rho{}^*sa$.*

*Proof.* Since $r > s$, $il > jk$. By Def. 4 and Axiom F,

$$j(ra)I_\rho ia \quad \text{and} \quad l(sa)I_\rho ka.$$

By Theorems 11, 12, and 15 of Suppes [2]

$$lj(ra)I_\rho l(ia)I_\rho(li)aP_\rho{}^*(jk)aI_\rho j(ka)I_\rho jl(sa),$$

and the assertion follows from Theorem 16.

**Lemma 2.** *Suppose that Axioms* E1–E6 *and* F *hold. Then* $r(sa)I_\rho(rs)a$.

*Proof.* By Def. 4 and Axiom F,

$$j[r(sa)]I_\rho i(sa), \quad jl[(rs)a]I_\rho(ik)a, \quad l(sa)I_\rho ka.$$

By Theorems 11 and 15 of Suppes [2],

$$lj[r(sa)]I_\rho li(sa)I_\rho i(ka)I_\rho(ik)aI_\rho jl[(rs)a],$$

from which the assertion follows by Theorem 16.

The final, and crucial, axiom establishes the needed relation between the concatenation structures on $A_1$ and $A_2$ and the conjoint ordering relation $R$ on $A_1 \times A_2$. Note that Axiom F is needed to state it since $i^m$ is a rational when $i$ is an integer and $m$ is a negative integer.

**Axiom C-E.** *There exist non-zero integers $m$ and $n$ such that for all positive integers $i$ and $j$, all $a$ in $A_1$, and all $p$ in $A_2$,*

$$(i^m a, j^n p)I(j^m a, i^n p).$$

*Moreover, $m$ is positive or negative according as $R_2{}^* = R_2$ or $R_2{}'$; $n$ is positive or negative according as $R_1{}^* = R_1$ or $R_1{}'$.*

**Lemma 3.** *Suppose that Axioms* C1–C3, E1–E6, C–E, *and* F *hold. Then for any positive rationals $r$ and $s$, any $a$ in $A_1$, and any $p$ in $A_2$,*

$$(r^m a, s^n p)I(s^m a, r^n p).$$

*Proof.* Let $r = i/j$ and $s = k/l$. Then

$$\begin{aligned}
(r^m a, s^n p) & I \left\{(il)^m \left[\frac{a}{(jl)^m}\right], (kj)^n \left[\frac{p}{(jl)^n}\right]\right\} && \text{(Lemma 2 and Axiom F)} \\
& I \left\{(kj)^m \left[\frac{a}{(jl)^m}\right], (il)^n \left[\frac{p}{(jl)^n}\right]\right\} && \text{(Axiom C–E)} \\
& I \left[\left(\frac{k}{l}\right)^m a, \left(\frac{i}{j}\right)^n p\right] && \text{(Lemma 2)} \\
& I(s^m a, r^n p).
\end{aligned}$$

The next theorem shows that Suppes's Archimedean axiom can be derived from the axioms stated.

**Theorem 2.** *Suppose that Axioms* C1–C4, E1–E6, C–E, *and* F *hold. For any $a$, $b$ in $A_\rho$, there exists a positive integer $j$ such that $jbR_\rho{}^*a$.*

*Proof.* Let $m$ and $n$ be the integers whose existence is asserted in Axiom C–E. With no loss of generality, suppose $\rho = 1$. Let $p$ be in $A_2$. Since $m, n \neq 0$, $2^m, 2^n \neq 1$, and so by Lemma 1 not $(2^m bI_1 b)$ and not $(2^n pI_2 p)$. Moreover by Axiom C–E,

$$(2^m b, p)I(b, 2^n p).$$

Thus, by Theorem IX J, [1], there exists a nontrivial *dss* $\{b_i, p_i\}$ that passes through $b$, $2^m b$, $p$, and $2^n p$. We now show that $b_i I_1(2^{mi}b)$ and $p_i I_2(2^{ni}p)$. Since *dss's* with two terms fixed on each component are unique up to indifference, it is sufficient to show that these quantities satisfy the three conditions of Def. 2.

i. By Lemma 2,

$$[2^{m(i+1)}b]I_1[2^m(2^{mi}b)].$$

By Lemma 1 and the fact that $2^m \neq 1$,

$$\text{not}\{[2^m(2^{mi}b)]I_1(2^{mi}b)\},$$

and so by the definition of $b_i$,

$$\text{not}(b_{i+1}I_1 b_i).$$

A similar argument shows that

$$\text{not}(p_{i+1}I_2 p_i).$$

| | |
|---|---|
| ii. $(b_{i+1}, p_i)I[2^{m(i+1)}b, 2^{ni}p]$ | (Def. of $b_i$, $p_i$; Theorem V L [1]). |
| $I[2^{mi}b, 2^{n(i+1)}p]$ | (Lemma 3) |
| $I(b_i, p_{i+1})$ | (Def. of $b_i$, $p_i$; Theorem V L). |
| iii. $(b_{i+1}, p_{i-1})I[2^{m(i+1)}b, 2^{n(i-1)}p]$ | (Def. of $b_i$, $p_i$; Theorem V L). |
| $I[2^{mi}(2^m b), 2^{n(i-1)}p]$ | (Lemma 2) |
| $I[2^{m(i-1)}(2^m b), 2^{ni}p]$ | (Lemma 3) |
| $I(2^{mi}b, 2^{ni}p)$ | (Lemma 2) |
| $I(b_i, p_i)$ | (Def. of $b_i$, $p_i$; Theorem V L). |

By Lemma XII A [1], there exist integers $\mu$ and $\nu$ such that $b_\nu R_1 a R_1 b_\mu$, and by what we have just shown $b_i I_1(2^{mi}b)$. If $R_1{}^* = R_1$, let $j$ be the next integer larger than $2^{m\nu}$; and if $R_1{}^* = R_1'$, let $j$ be the next integer larger than $2^{m\mu}$. By Lemma 1 and the transitivity of $R_1{}^*$ (Theorem 1), $jbR_1{}^*a$.

**3. The Principal Theorem.** From Suppes, [2], we know that Axioms E1-E6 plus Theorem 2 are sufficient to show that there exists a positive real-valued function $v_\rho$ on $A_\rho$ such that for $a$ and $b$ in $A_\rho$

Ei. $aR_\rho{}^*b$ if and only if $v_\rho(a) \geqslant v_\rho(b)$,

Eii. $v_\rho(ao_\rho b) = v_\rho(a) + v_\rho(b)$, and

Eiii. If $v_\rho{}'$ is any other function satisfying Ei and Eii, then there exists a constant $\delta_\rho > 0$ such that $v_\rho{}' = \delta_\rho v_\rho$.

From Luce and Tukey, [1], we know that Axioms C1–C4 are sufficient to show that there exist positive real-valued functions $u_\rho$ on $A_\rho$ such that for $a$ and $b$ in $A_1$ and $p$ and $q$ in $A_2$

Ci. $aR_1b$ if and only if $u_1(a) \geqslant u_1(b)$ and
$pR_2q$ if and only if $u_2(p) \geqslant u_2(q)$,

Cii. $(a, p)R(b, q)$ if and only if $u_1(a)u_2(p) \geqslant u_1(b)u_2(q)$, and

Ciii. If $u_\rho'$, $\rho = 1, 2$, is any other pair of functions satisfying Ci and Cii, then there exist constants $\alpha > 0$ and $\beta_\rho$ such that

$$u_1' = \beta_1 u_1^\alpha \quad \text{and} \quad u_2' = \beta_2 u_2^\alpha.$$

The principal theorem shows that when Axioms C–E and F hold the functions $u_\rho$ and $v_\rho$ are very simply related.

**Theorem 3.** *Suppose that Axioms* C1–C4, E1–E6, C–E, *and* F *hold. If $u_\rho$ are conjoint measures for $\langle A_1 \times A_2, R\rangle$ and $v_\rho$ are extensive measures for $\langle A_\rho, R_\rho^*, o_\rho\rangle$, then there exist constants $\alpha > 0$ and $\beta_\rho$ such that*

$$u_1 = \beta_1 v_1^{\alpha n}, \qquad u_2 = \beta_2 v_2^{\alpha m}.$$

*Proof.* It suffices to work out the proof for $A_1$, the proof for $A_2$ being completely parallel. From the conjoint measurement representation applied to Axiom C–E, we have for positive integers $i$ and $j$, $a$ in $A_1$, and $p$ in $A_2$,

$$\frac{u_1(i^m a)}{u_1(j^m a)} = \frac{u_2(i^n p)}{u_2(j^n p)}.$$

Clearly, this ratio is independent of either $a$ or $p$ and depends only on $i$ and $j$. Denote its value by $g(i, j)$.

First, observe that there exists a positive real-valued function $f$ on the rationals such that $f(i/j) = g(i, j)$. For let $r = i/j = k/l$ be rational. Since there exists an integer $t$ such that $k = ti$ and $l = tj$,

$$g(k, l) = g(ti, tj) = \frac{u_1[(ti)^m a]}{u_1[(tj)^m a]} = \frac{u_1[i^m(t^m a)]}{u_1[j^m(t^m a)]} = g(i, j).$$

So we define $f(r) = g(i, j)$ for $r = i/j$.

Second, $f$ is strictly monotonic. Suppose $i/j = r > s = k/l$, then $il > jk$. There are two cases:

i) $mn > 0$. If $m > 0$ and $n > 0$, then $R_1^* = R_1$ and $(il)^m > (jk)^m$. By Theorem 17 of Suppes,

$$[(il)^m a]P_1[(jk)^m a].$$

By property Ci of conjoint measurement, this implies

$$u_1[(il)^m a] > u_1[(jk)^m a],$$

and so

$$\begin{aligned} f(r) &= u_1(i^m a)/u_1(j^m a) \\ &= u_1[(il)^m a]/u_1[(jl)^m a] \\ &> u_1[(jk)^m a]/u_1[(jl)^m a] \\ &= u_1[k^m(j^m a)]/u_1[l^m(j^m a)] \\ &= f(s). \end{aligned}$$

If $m < 0$ and $n < 0$, then $R_1^* = R_1'$ and $(il)^m < (jk)^m$. By Theorem 17,

$$[(jk)^m a]P_1^*[(il)^m a],$$

hence

$$[(il)^m a]P_1[(ik)^m a].$$

The remainder of the argument is unchanged, so $f$ is strictly increasing if $mn > 0$.

ii) $mn < 0$. A similar argument shows that $f$ is strictly decreasing.

Third, $f$ satisfies the functional equation $f(rs) = f(r)f(s)$. Let $r = i/j$ and $s = k/l$, then

$$\begin{aligned} f(rs) &= u_1[(ik)^m a]/u_1[(jl)^m a] \\ &= \frac{u_1[i^m(k^m a)]}{u_1[j^m(k^m a)]} \frac{u_1[k^m(j^m a)]}{u_1[l^m(j^m a)]} \\ &= f(r)f(s). \end{aligned}$$

By Lemma 4, stated and proved following the completion of this proof, there exists a constant $\gamma \neq 0$ such that $f(r) = r^\gamma$. Note that by choosing $j = 1$ above, this means that

$$u_1(i^m a) = i^\gamma u_1(a).$$

Moreover, we may write $\gamma = \alpha mn$, where $\alpha > 0$, since $\gamma \gtrless 0$ if and only if $mn \gtrless 0$.

Since, by property Ei, $v_1$ preserves the order $R_1{}^*$ and, by property Ci, $u_1$ preserves the order $R_1$, there exists a strictly monotonic function $h_1$ such that $u_1 = h_1(v_1.)$ Note that $h_1$ is increasing or decreasing according as $R_1{}^* = R_1$ or $R_1{}'$. By an appropriate choice of units, there is no loss of generality in assuming $h_1(1) = 1$. To determine the form of $h_1$, let $r = i/j$ be a rational and let $a$ be in $A_1$. By what was just shown and Def. 4,

$$i^\gamma u_1(a) = u_1(i^m a) = u_1[j^m(r^m a)] = j^\gamma u_1(r^m a),$$

and so

$$u_1(r^m a) = r^\gamma u_1(a).$$

By properties Ei and Eii,

$$i^m v_1(a) = v_1(i^m a) = v_1[j^m(r^m a)] = j^m v_1(r^m a),$$

and so

$$v_1(r^m a) = r^m v_1(a).$$

Therefore,

$$\begin{aligned} r^\gamma h_1[v_1(a)] &= r^\gamma u_1(a) \\ &= u_1(r^m a) \\ &= h_1[v_1(r^m a)] \\ &= h_1[r^m v_1(a)]. \end{aligned}$$

Define $\phi(x) = h_1(x^m)$, then for $r$ rational and $x^m$ in the range of $v_1$,

$$\phi(rx) = r^\gamma \phi(x).$$

By the choice of units, $x = 1 = x^m$ is in the range of $v_1$, so $\phi(r) = r^\gamma$ since $\phi(1) = h_1(1) = 1$.

Observe that by the relation of the monotonicity of $h_1$ to $n$, $\phi$ is strictly monotonic increasing if $mn > 0$ and decreasing if $mn < 0$. From this and the fact that $\phi(r) = r^\gamma$, it is easy to show that $\phi(x) = x^\gamma$. For example, suppose $\phi$ is decreasing and $\phi(x) > x^\gamma$. Then we may choose a rational $r$ such that $\phi(x) > \phi(r) = r^\gamma > x^\gamma$. Since $\phi$ is decreasing, $x < r$; but also $\gamma < 0$, and so $x^\gamma > r^\gamma$, which is contrary to choice. Writing $\gamma = \alpha mn$, $\alpha > 0$,

$$h_1(x) = \phi(x^{1/m}) = x^{\alpha n}.$$

A parallel argument shows that

$$h_2(x) = x^{\alpha m}.$$

**Lemma 4.** *Suppose that $f$ is a strictly monotonic, positive, real-valued function defined on the rationals such that for $r$ and $s$ rational $f(rs) = f(r)f(s)$. Then there exists $\gamma \neq 0$ such that $f(r) = r^\gamma$.*

*Proof*[3]. Let $f$ be any function that satisfies the assumption. If $f$ is increasing we extend $f$ to the reals by the following definition: for $\lambda$ real,

$$f(\lambda) = \sup_{r \leqslant \lambda} f(r).$$

This function is also strictly monotonic increasing since when $\lambda < \eta$, there exists a rational $s$ with $\lambda < s < \eta$, and so for $r \leqslant \lambda$, $f(r) < f(s)$. Thus, $f(\lambda) < f(\eta)$. Furthermore, the extended $f$ satisfies the same functional equation since

$$\begin{aligned} f(\lambda\eta) &= \sup_{r \leqslant \lambda\eta} f(r) \\ &\geqslant \sup_{s \leqslant \lambda, t \leqslant \eta} f(st) \\ &= \sup_{s \leqslant \lambda} f(s) \sup_{t \leqslant \eta} f(t) \\ &= f(\lambda) f(\eta). \end{aligned}$$

The inequality also holds in the other direction since if $r \leqslant \lambda\eta$, we can choose rationals $s$ and $t$ such that $r \leqslant st$, $s \leqslant \lambda$, and $t \leqslant \eta$. It is well known that under these conditions $f(\lambda) = \lambda^\gamma$, $\gamma > 0$. If $f$ is decreasing, a similar proof yields the same result with $\gamma < 0$.

**4. Discussion.** There are three aspects of Axiom C-E, the only new axiom, that need discussion. First, as was mentioned, this axiom is a necessary property if the representation is assumed to hold. By the representation and Eii,

$$\begin{aligned} u_1(i^m a) u_2(j^n p) &= \beta_1 v_1(i^m a)^{\alpha n} \beta_2 v_2(j^n p)^{\alpha m} \\ &= \beta_1 i^{m\alpha n} v_1(a)^{\alpha n} \beta_2 j^{n\alpha m} v_2(p)^{\alpha m} \\ &= \beta_1 j^{m\alpha n} v_1(a)^{\alpha n} \beta_2 i^{n\alpha m} v_2(p)^{\alpha m} \\ &= \beta_1 v_1(j^m a)^{\alpha n} \beta_2 v_2(i^n p)^{\alpha m} \\ &= u_1(j^m a) u_2(i^n p), \end{aligned}$$

from which the axiom follows by Cii.

Second, the question must be raised why this axiom has been stated with integer exponents $m$ and $n$. The consequence of this formulation is that the ratio of exponents in the representation has to be rational: $\alpha n / \alpha m = n/m$. In point of fact, this happens to be the case in all representations of this general type in classical physics of which I am aware, and so the theory as stated is adequate to handle the cases of interest. Within the theory itself, however, the reason for requiring $m$ and $n$ to be integers is the fact that we have only defined the notation $ra$ for rational $r$, not irrational. Were we to suppose that $m$ and $n$ are rationals and to continue to restrict $i$ and $j$ to be integers, there would be trouble because, for example, $2^{1/2}$ is irrational.

[3] I am indebted to David Krantz for suggesting this method of proof.

It is clear, therefore, that if we wish a representation theorem without any restriction on the exponents, it is necessary to increase the density of elements so that $\lambda a$ can be defined for all irrational $\lambda$. One way to proceed is as follows. Before Axiom C–E, introduce the following axiom that insures the existence of "irrational" elements in $A$.

**Axiom I.** *For every $a$ in $A_\rho$ and every positive irrational number $\lambda$, there exists an element, $\lambda a$, in $A_\rho$ which is unique up to indifference ($I_\rho$) and is such that for all rationals $r$ and $s$ with $r < \lambda < s$,*

$$(sa)P_\rho(\lambda a)P_\rho(ra).$$

By standard arguments, one can then show that

$$\text{If} \quad \lambda > \eta, \quad (\lambda a)P_\rho(\eta a),$$

$$\lambda(\eta a)I_\rho(\lambda\eta)a,$$

and

$$v(\lambda a) = \lambda v(a),$$

where $v$ is the measure from the theory of extensive measurement. Then, Axiom C–E is modified to read:

**Axiom C-E*.** *There exist non-zero numbers $\mu$ and $v$ such that for all positive real numbers $\lambda$ and $\eta$ and all $a$ in $A_1$ and $p$ in $A_2$*

$$(\lambda^\mu a, \eta^\nu p)I(\eta^\mu a, \lambda^\nu p).$$

*Moreover, $\mu$ is positive or negative according as $R_2{}^* = R_2$ or $R_2{}'$, and $\nu$ is positive or negative according as $R_1{}^* = R_1$ or $R_1{}'$.*

These assumptions lead again, with slight modifications in the proofs, to Theorems 2 and 3 with, of course, $m$ and $n$ replaced by $\mu$ and $\nu$. Lemma 4 is not used in the proof of Theorem 3 since $f$ is now defined on the positive reals, not just the rationals.

The third, and final, point about Axiom C–E is that it is a qualitative formulation of a class of laws that, traditionally, have been formulated only in terms of relations among numerical scales constructed by means of extensive measurement theories. It is, perhaps, of some philosophical significance to have a purely qualitative equivalent to the standard numerical formulas.

## REFERENCES

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