The Analysis of Singular Spacetimes[†]

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Abstract

Much controversy surrounds the question of what ought to be the proper definition of "singularity" in general relativity, and the question of whether the prediction of such entities leads to a crisis for the theory. I argue that there is no single canonical definition of such a things, and that none is required—various definitions present themselves, respectively suitable for different sorts of investigations. In particular, I argue that a definition in terms of curve incompleteness is adequate for most purposes, though the idea that singularities correspond to "missing points" has insurmountable problems. I conclude that singularities *per se* pose no serious problem for the theory, but their analysis does bring into focus several problems of interpretation at the foundation of the theory often ignored in the philosophical literature.

[†]I presented an earlier, adumbrated version of this paper at the Philosophy of Science Association's biannual meeting in 1998. That version was subsequently published in the proceedings of the conference (Curiel 1999).

[‡]This paper began life as a small criticism of a few points John Earman makes in chapter 2 of his book *Bangs, Crunches, Whimpers and Shrieks: Singularities and Acausalities in Relativistic Spacetimes*, and grew as I grew to realize more fully the complexity and subtlety of the issues involved. I will not always point out where I am in agreement or disagreement with Earman, much less always discuss why this is so, though I will try to regarding the most important points. The reader ought to keep in mind, though, that Earman's book is the constant foil lurking in the background. In intellectual matters, the sincerest form of flattery is not imitation, but rather the attempt to understand and improve upon—in writing this paper, I grew to understand even more keenly than I had before not only how difficult the issues here are, but also what a very good book Earman has written on it. If the reader needs more background in the subject matter than I provide in this paper, I strongly urge him or her to consult it.

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1 Introduction

I suspect that, for many, talk of a singularity in the context of general relativity conjures up the image of something like a rent in the fabric of spacetime.[1] Perhaps unbounded curvature from the self-gravitational collapse of a massive body tore the fabric, or perhaps the cloth was simply ill-woven from the start, but in any case the idea of a flaw in the fabric of spacetime naturally accompanies the word 'singularity'. This metaphor, evocative as it may be, is perhaps misleading: a web of cloth exists in space and time, and one naturally would rely (implicitly, at least) upon this fact were one to define what one meant in saying the cloth were rent. For instance, one might define a cloth to have a hole if one could thread a string through the cloth, tie the ends of the string together and have the string touching disjoint components of the edge of the cloth.[2] When thinking of spacetime, though, one does not have the luxury of imagining it embedded in any physically meaningful way in a larger space with respect to which one can try to define what one means by saying there is a hole.

One can think about holes in cloth in (at least) two ways: no cloth has been removed, but parts of the cloth have simply been separated from each other (torn) for a length; a bit of the cloth has actually been excised and removed. In the former case, all the points (bits) of the cloth are still there but the topology has changed, whereas in the latter case there are actually points (bits) that once were part of the cloth now missing from it. I wager that people usually conceive of singular spacetimes in a way analogous to the latter idea when they think in a vague, intuitive way: there are points missing from spacetime. For example, in thinking about the self-gravitational collapse of a massive body, one might imagine the "point" in which all the matter in the body becomes

[1] By 'spacetime', I will always mean a smooth, 4-dimensional, connected, paracompact manifold endowed with a fixed, smooth metric of Lorentz signature.

[2] More rigorously, this amounts to saying a 2-dimensional compact topological manifold has a hole if and only if it has a boundary not homeomorphic to $\mathbb{S}^1$. Thus the torus does not have a hole, since it has no boundary; neither the spherical shell with a small cap excised nor the Möbius strip has a hole, since the boundary of each is homeomorphic to $\mathbb{S}^1$ (the spherical shell with a cap excised is homeomorphic to the planar disk); the finite cylinder has a hole, since its boundary is homeomorphic to the disjoint union of $\mathbb{S}^1$ with itself (the finite cylinder is homeomorphic to a planar annulus). I thank James Geddes for illuminating discussion on this question.

eventually concentrated. In a normal collapse, the curvature of spacetime will in some sense become unboundedly large as one approches this "point", so, again loosely, one will not be able to define the spacetime metric at that "point"—and now one sees why I have been enclosing 'point' in scare-quotes, for spacetime comprises solely points of a manifold with a pseudo-Riemannian metric of Lorentz signature defined thereat. The "point" to which all the matter collapsed is missing from the spacetime.[3]

On a manifold endowed with a positive-definite Riemannian metric, one can give a precise characterization, according quite well with our intuitions, of what it is for there to be missing points. Turn the manifold into a pointwise-metric space (*i.e.*, one possessing a true distance-function on the space of ordered pairs of its points) via the usual construction: define the distance between any two points to be the infimum of the lengths, with respect to the Riemannian metric, of all smooth curves connecting them. The manifold has no missing points if and only if it is Cauchy complete with respect to the constructed pointwise-metric. Intuitively speaking, if a sequence of points begins to accumulate, there ought to be a place at which they actually do accumulate. If there are missing points, one may take the Cauchy completion of the manifold in its guise as a pointwise-metric space to fill in the gaps, as it were.

On a manifold with a pseudo-Riemannian metric of Lorentz signature, such as a spacetime in general relativity, there is no natural way to construct a true pointwise-metric measuring the distance between points of the manifold, so one cannot employ this technique to test whether a spacetime has missing points. By the Hopf-Rinow-de Rham theorem, the manifold in the Riemannian case is Cauchy complete with respect to the constructed pointwise-metric if and only if it is geodesically complete with respect to the Riemannian metric.[4] This naturally suggests that we define a spacetime to have missing points if and only if it is geodesically incomplete with respect to the spacetime pseudo-Riemannian metric. Now one faces a severe problem, which lies at the heart of the difficulty in giving a precise and intuitively satisfying definition of singular structure as a point missing from spacetime: there is no natural way to take a Cauchy-like completion of a spacetime manifold having incomplete geodesics so as to give substance to the idea that there really "are" points that in some sense ought to have been included in the spacetime in the first place.[5] In the Riemannian case, roughly speaking, one constructs the missing points by taking equivalence classes of incomplete curves that get arbitrarily close to one another as measured by the constructed pointwise-metric. In the pseudo-Riemannian case there is no natural way to measure how close two curves come to one another, so, *a fortiori*, there is no natural way to define missing points as the equivalence classes of incomplete curves that come arbitrarily close to each other.[6]

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[3]More precisely, a point of a spacetime manifold ought to be considered a point of spacetime itself if and only if, on the bundle of pseudo-Riemannian metrics over the manifold, the cross-section representing the spacetime's metric is well defined in the fiber over the point in question.

[4] See Spivak (1979, ch. 9) for a precise statement and proof of the theorem, and for more information on the constructed pointwise-metric and the Cauchy completion of a manifold endowed with a Riemannian metric.

[5]The scare-quotes now come from the fact that it is not clear in the slightest what sense may accrue to the attachment of an existential quantifier to "points that are possibly spacetime points but in the event are not". I touch on this issue in §5. I will dispense with scare-quotes from hereon, the point having been made.

[6]Cauchy completeness of a Riemannian manifold with respect to the constructed distance-function happens also

The usual tack taken at this point in the physics literature is simply to bracket the question of missing points and define a spacetime to be singular if and only if it contains incomplete, inextendible curves of a certain specified type, and the spacetime manifold itself satisfies a few collateral conditions. The commonly accepted schema for fixing a rigorous definition of a singular spacetime, then, is:[7]

> A spacetime ($\mathcal{M}$, g_{ab}) satisfying ____ is *singular* if and only if there exists a curve γ incomplete in the sense that ____.

Such a conception of singular structure actually has a lot to say for itself, as capturing the idea that singular structure is somehow physically *outré*, even if one is not able to hook it up cleanly to an idea of missing points. As Hawking and Ellis (1973, p. 258) put it with regard to one particular way of filling in the blanks,

> Timelike geodesic incompleteness has an immediate physical significance in that it presents the possibility that there could be freely moving observers or particles whose histories did not exist after (or before) a finite interval of proper time. This would appear to be an even more objectionable feature than infinite curvature and so it seems appropriate to regard such a space as singular.

The current paper has several concrete aims: to investigate particular ways that have been proposed to fill in the blanks of the schematic definition with an eye to determining whether they capture the spirit of the idea that an incomplete curve corresponds to singular structure; to examine the relation between curvature pathology and singular structure so defined; to argue that the idea of missing points ought not be central in thought about singular structure; and to argue that the reasons most often given for condemning singular structure as unphysical do not withstand scrutiny.

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to be equivalent to the following condition: every bounded (with respect to the constructed distance-function) subset of the Riemannian manifold is relatively compact (Kobayashi and Nomizu 1963, p. 172). So far as I know, no one has tried to parlay this equivalence into a definition of "missing points" in the pseudo-Riemannian case. Again, since there is no distance-function in the pseudo-Riemannian case, there is no natural candidate for what ought to count as bounded subsets of the manifold. A first stab might be: points are missing from the manifold if and only if, for some p and q in the manifold, $J^+(p) \cap J^-(q)$ is not relatively compact, where $J^+(p)$ ($J^-(p)$) is the causal future (past) of the point p. If this could be made to work, it would have the great virtue of "localizing" the missing point—when asked, "where is the point missing from?", one could point to the salient $J^+(p) \cap J^-(q)$, and say, "from that region." The obvious problem with this candidate is that it fails to categorize Schwarzschild spacetime as having a missing point, whereas one might have thought that Schwarzschild was the paradigm of a spacetime with a missing point, *viz.*, the "point" into which all the matter from a body undergoing self-gravitational collapse squeezes itself. In fact, the first stab fails to categorize any globally hyperbolic spacetime as having missing points, since all globally hyperbolic spacetimes by definition satisfy the proposed condition (Wald 1984, p. 209). Even though a fairly obvious first candidate fails, it still might be interesting to explore whether one could propose a reasonable analogue of "bounded subset" for the pseudo-Riemannian case and use this to define missing points. Of course, because of the known examples of compact, geodesically incomplete spacetimes (*cf.*, *e.g.*, Misner 1963), one should expect that any such characterization based on the relative compactness of bounded subsets would be bound to differ in what it counts as singular from the traditional characterization in terms of geodesic completeness.

[7]See, for example, Hawking and Ellis (1973, pp. 256–61), Wald (1984, pp. 212–6), Clarke (1993, p. 10), and Joshi (1993, pp. 161–2).

It also has one overarching, more nebulous aim: to try to give a sense of the philosophical riches still waiting to be mined from thorough investigation of the foundations of general relativity—which is to say, a sense of how little of this theory we even now comprehend, and how much we stand in need of that comprehension if we wish to understand the world.

2 Curve Incompleteness

The path-breaking work of the mid-1960's demonstrating the existence of singular structure in generic solutions to the Einstein field equation invoked timelike or null geodesic incompleteness as a sufficient condition for classifying a spacetime as singular,[8] in so far as timelike and null geodesics represent possible world-lines of particles and observers and, *prima facie*, it appears physically suspect for an observer or a particle to be allowed to pop in or out of existence right in the middle of spacetime, so to speak. There was, however, no consensus on what ought to count as a necessary condition. In particular, workers at the time were unclear on the role played by curvature pathology in singular structure. Hawking, for example, in his very early work, distinguished between the mere incompleteness of the spacetime manifold (as characterized by the existence of incomplete geodesics) and what he referred to as a "physical singularity" apparently meaning a spacetime region wherein, in one of a number of technical senses, the magnitude of the curvature grows without bound: "Penrose has shown that either a physical singularity must occur or space-time is incomplete if there is a closed trapped surface. . . ."[9] Context makes clear that Hawking relates the existence of a trapped surface with the existence of pathology in the behavior of the curvature. It is worth remarking that, based on a careful reading of Penrose (1965), to which Hawking here refers, it is not at all clear that Penrose himself would have endorsed this statement of his result. In an apparent lightning-fast sequence of changes of mind that strikingly illustrates the uncertain and fluid nature of the idea of singular structure in the field at the time, in April, 1966, Hawking proposed the prediction of singular structure (which, note, meant only the existence of incomplete timelike or null geodesics) as a possible test of the validity of general relativity,[10] whereas by February of the very next year he concludes that the singularity theorems proved up to that point "probably" indicate not that singular structure actually occurs in the universe but rather that general relativity breaks down in the strong field regime![11]

The field was ripe for a little sober reflection, happily provided by Geroch (1968b), who gave the first extended discussion of the difficulty of framing a satisfactory definition of a singular space-

[8] *Cf.* Geroch (1966, 1967), Penrose (1965), and Hawking (1965, 1966a, 1966b, 1966c, 1966d, 1967).

[9] Hawking (1965, p. 689).

[10] Hawking (1966b, p. 511).

[11] Hawking (1967, p. 189). The dates referred to in the text (as opposed to those of the citations proper) are those on which the journal recieved the papers for review, as indicated in the published versions. The second viewpoint seems to represent Hawking's settled opinion on the matter—*cf.* Hawking and Ellis (1973, §10.2) and Hawking and Penrose (1996, p. 20). I do not take this oscillation of Hawking's from position to position as an act worthy of derogation, far from it. Rather, he seems to me to have been engaged in the practice of a good scientist: entertaining all the decent possibilities presenting themselves so as to test them by use in his investigations, in order to see which bear fruit and which do not.

time.[12] Geroch's discussion begins in earnest with a Galilean dialogue, a form, as Earman (1995, p. 27) notes, nicely suited for displaying the unsettled state of the topic.[13] After concluding that one can use neither the physical components of the Riemann curvature tensor nor any of the scalar curvature-invariants to define precisely what one means by, and construct necessary and sufficient conditions for, saying a spacetime contains regions wherein the curvature grows without bound in a physically accessible manner,[14] the discussants in the dialogue settle on simple geodesic incompleteness as the criterion for singular structure, conceding that the definition is perhaps overly inclusive, but better to brand 10 innocents than to allow one guilty man unmarked. The possible innocents include spacetimes all of whose timelike and null geodesics are complete but that possess incomplete spacelike geodesics (null and timelike complete and spacelike incomplete, for short). Spacelike incompleteness (in the absence of the other two types of incompleteness) sets off no serious alarms, or so thought commonly goes, for an incomplete spacelike geodesic seems to represent structure of the spacetime not physically accessible to any observer in a direct way.[15] Moreover, not only does geodesic incompleteness lock up a few possible innocents but, as Geroch proceeds to show, it almost certainly fails to nab a few clever guilty parties, for a spacetime can be geodesically complete and yet possess an incomplete timelike curve of bounded total acceleration—that is to say, an inextendible curve traversable by a rocket expending only a finite amount of fuel, along which an observer could experience only a finite amount of proper time.

Because of these problems, null and timelike geodesic incompleteness continued to be used as a sufficient condition for declaring a spacetime singular, but was (and still is) considered inadequate as a definition.[16] To analyze the structure of non-geodetic curves in the search for a necessary condition, we require a method for characterizing their completeness. The following appears tempting at first glance: an inextendible curve is incomplete just in case it has finite proper length. Even if one puts aside for the moment the fact that every null curve has zero proper length, one still faces the following problem with any approach based on proper time: every spacetime, including Minkowski

[12]Kundt (1963), Misner (1963) and Hawking (1967), among others, had already broached in a cursory manner several of the topics Geroch discussed in the paper.

[13]Only Sagredo and Salviati discuss the issue, by Geroch's report, with no word from Simplicio. I can speculate only that the issue was too difficult for Simplicio's limited capacities.

[14]The physical components of the Riemann tensor are its components relative to any pseudo-orthonormal tetrad; roughly speaking, a scalar curvature-invariant is a scalar "function" of the metric, the Riemann tensor and its covariant derivatives that is preserved under diffeomorphisms of the spacetime. See Ehlers and Kundt (1962) for details. I will discuss in §3 why none of these suffice for constructing necessary conditions.

[15]See, *e.g.*, Synge (1960, ch. I, §14, pp. 24–6) for a discussion of the physical content the measurement of spacelike intervals in general relativity may possess. In a similar vein, one may also consult Geroch (1981). It would be of some interest to investigate whether one could parlay discussions such as those two into arguments for the "physicality" of incomplete spacelike geodesics.

[16]Hawking, in Hawking and Penrose (1996, p. 15), defines a singular spacetime as one which is timelike or null geodesically incomplete, but I believe this is not meant as a serious attempt at a strict definition, merely an easy criterion to work with in light of the fact that all known singularity theorems prove the existence of incomplete timelike or null geodesics. It would be of interest, again, to investigate the question whether there exists a set of conditions that "physically reasonable" spacetimes ought to satisfy, having as a consequence the existence of an incomplete spacelike geodesic.

space, has inextendible timelike curves of finite total proper length, *viz.*, those of unbounded total acceleration that go zooming off to infinity, so to speak, asymptotically approaching the speed of light. Surely one does not want to classify Minkowski space as singular, and anyhow, if an observer is able to reach infinity, as it were, even in a finite amount of time, the prevailing sentiment in the physics community at large seems to be that such structure ought not qualify as singular.[17] One wants a method of winnowing such acceptably finite, inextendible curves from unacceptable ones.

Schmidt (1971) appears to have been the first to propose using so-called generalized affine parameters to define the completeness of general curves. Let $\mathcal{M}$ be an n-dimensional manifold with an affine connection, $\gamma(t)$ a curve through $p = \gamma(0)$, and $\{\overset{i}{\xi}{}^a(0)\}_{i=1\ldots n}$ a basis for the tangent space at p. One can now write $\gamma^a(0)$, the vector tangent to γ at p, as a linear combination of the elements of the chosen basis with coefficients $\gamma_i(0)$:

$$\gamma^a(0) = \sum_{i=1}^{n} \gamma_i(0)\ \overset{i}{\xi}{}^a(0)$$

If one parallel-transports the chosen basis along $\gamma(t)$, one gets a similar expression at every point on $\gamma(t)$. The *generalized affine parameter* $\theta(t)$ of $\gamma(t)$ associated with this basis is defined by:

$$\theta(t) \equiv \int_0^t \left(\sum_{i=1}^{n} (\gamma_i(t'))^2 \right)^{\frac{1}{2}} dt'$$

In effect, one treats the parallel-transported basis of vectors as though they were the orthonormal basis of a Riemannian metric and then defines the "length" of $\gamma(t)$ accordingly. The generalized affine parameter of a curve does not depend on the basis chosen in one crucial respect: whether or not the generalized affine parameter of the curve increases without bound. Furthermore, any curve of unbounded proper length automatically has an unbounded generalized affine parameter, but not vice-versa—any inextendible timelike curve of unbounded total acceleration and finite total proper time in Minkowski space, for example, has an unbounded generalized affine parameter. A spacetime in which every inextendible curve has an unbounded generalized affine parameter will be referred to as *b-complete*.[18] This sort of completeness promises to distinguish precisely what wanted distinguishing, and works just as well for null as for timelike or spacelike curves. Thus, one has what Earman (1995, p. 36) refers to as the "semioffical view": a spacetime is said to be singular if and only if it is b-incomplete.[19] This definition is more general than geodesic completeness, in that it implies, but is not implied by, the latter, as Geroch's example demonstrates.

[17] I think this sentiment represents a hypocrisy on the part of the community, as I will discuss briefly just below and in more detail in §6.

[18] '*b*' for 'bundle': with this construction one tacitly defines a natural, albeit basis-dependent, Riemannian metric on the bundle of frames of the spacetime manifold to define completeness of curves.

[19] Strictly speaking, this is not the standardly accepted definition, since I have not mentioned anything about the maximality of the spacetime in question, whether, that is, it can be embedded in (thought of as merely a part of) a larger spacetime in such a way as to make previously incomplete, inextendible curves extendible. I will take up this issue in §6.

It is difficult to think of a more comprehensive criterion of completeness than b-completeness, and I suspect its popularity arises from that fact,[20] but that it sits comfortably with some of the intuitions that drove the search for a definition of singular structure in the first place is not so clear on reflection. That it counts some timelike curves of total finite proper time as complete (*viz.*, some of those of unbounded total acceleration) is perhaps its most unsettling feature, if one of the intuitions driving the search for a definition of singular structure is the impropriety of having entities that can exist for only a finite period of time. It is also a cumbersome and technically awkward criterion to deploy in practice. In fact, perhaps the most damning fact about b-completeness is that, so far as I know, it is never used in the statement or demonstration of any result of physical interest. All the singularity theorems, for instance, demonstrate only the existence of null or timelike geodesics, and are formulated only in those terms. For the moment, I will waive these qualms and accept b-incompleteness as the definition of singular structure—when I refer to 'incomplete curves', unless I explicitly state otherwise I will mean b-incomplete, inextendible curves. I will return to some of these questions below in §6.

3 Explosive Curvature Growth along Incomplete Curves

While curve incompleteness seems to capture one aspect of the intuitive picture of singular structure, it completely ignores a different aspect, curvature pathology. One may measure the growth and diminution of the magnitude of spacetime curvature in various ways, but it turns out that the unbounded growth of curvature according to any of these measures is neither necessary nor sufficient for the existence of incomplete, inextendible curves. To get an idea of the independence of the existence of incomplete curves from the presence of curvature pathology, consider the striking ease with which examples of a spacetime with everywhere vanishing Riemann tensor and incomplete geodesics can be constructed: excise from 2-dimensional Minkoswki space a closed set in the shape of Ingrid Bergman. This example may strike one as cheating, since one has only to restore the excised set to restore geodesic completeness (or, in fancier terms, to restore completeness one has only to isometrically embed our mutilated spacetime by the natural inclusion map back into Minkowski spacetime). So a slightly more sophisticated example: for some $0 < \phi_0 < \frac{\pi}{2}$, excise from Minkowski space, represented in polar coordinates, the wedge consisting of all points with azimuthal coordinate $0 < \phi < \phi_0$; identify the corresponding points on the hyperplanes $\phi = 0$ and $\phi = \phi_0$. By a suitable redefinition of the coordinate neighborhoods of the points on $\phi = 0$, the resulting space can be given the manifold structure of $\mathbb{R}^4$, and the Minkowski metric can be smoothly extended to the points at $\phi = 0$, $r > 0$. It cannot be smoothly extended to the points $r = 0$, however, and so those points

[20] Schmidt (1971) claims that b-completeness is for pseudo-Riemannian metrics the "natural" generalization of completeness with respect to a Riemannian metric, in so far as it is equivalent to completeness with respect to a Riemannian metric when used on a Riemannian manifold: geodesic completeness with respect to a Riemannian metric is logically equivalent to b-completeness as defined by its affine connection. I do not know what 'natural' signifies in this context, because the criterion in the Riemannian case may be formulated without the use of components of geometric objects in an arbitrary coordinate system, but Schmidt's method in the pseudo-Riemannian case cannot be formulated without it.

must be excised from the spacetime. The Riemann tensor of this spacetime vanishes everywhere, but any geodesic that previously passed through the line $r = 0$ will now be incomplete; there is, moreover, no other spacetime into which this spacetime can be embedded and in which the metric can be smoothly extended.[21] This sort of structure is known as a 'conical singularity', since the singular structure has many of the same characteristics as that accruing to the two-dimensional real plane with a wedge removed and the edges pasted together, so as to form a cone.

Perhaps this example will also strike the reader as too artificial, too contrived, to have any physical relevance.[22] I believe that on a matter such as the global topological structure of spacetime, about which we have so very little hard data and so little prospect of gathering any for the foreseeable future, one should be wary of ignoring certain sorts of examples on the ground that they appear artificial. That judgment has its roots in the schooling our intuitions have received in our contemplation of well worked out examples of physical theories, which by and large tend to include mathematical structures that strike us as "simple" and "natural". This ought not escape our notice: most such examples of physical theories are demonstrably false (Newtonian mechanics and classical Maxwell theory) or have at the moment insuperable problems of interpretation (quantum mechanics) or experimental accessibility (general relativity). We should beware of relying too much on intutions trained in such schools—especially when one also recalls how much of our contemplation of those theories involves models of systems with physically unrealistic perfect symmetries and vaguely jusified approximations, simplifications and idealizations. It may turn out, for all we know, that spacetime instantiates just such topological structure as $\mathbb{R}^4$ with a closed set excised (assuming, for the moment, that we can even make sense in a physically substantive and cogent way of the idea of the global topological structure of spacetime). Perhaps the most important point to notice, though, is that "$\mathbb{R}^4$ with certain closed sets excised" is a *misleading* description of such a manifold. It suggests that we built that manifold from a more fundamental one, *viz.*, $\mathbb{R}^4$. But that manifold *simply is* a manifold all on its own, with no intrinsic reference to $\mathbb{R}^4$, or indeed any other manifold. Because of certain facts about how we practice mathematics, the most convenient presentation of that manifold happens to be "$\mathbb{R}^4$ with certain closed sets excised". One could as legitimately present $\mathbb{R}^4$ as that manifold glued together with certain other manifolds-with-boundary. There are no good grounds I can see for suspecting that the universe heeds our preferred methods for organizing mathematical structures.

In the event, I am happy to report that I do not need to rely on these constructions and considerations to demonstrate, for those unmoved by my sermon, that curvature pathology has no necessary connection to the existence of incomplete curves. Examples more acceptable to everyone present themselves. The two most commonly used methods of measuring the growth of curvature intensity are the behavior of scalar curvature-invariants along some particular curve through the region of

[21] This example is from Wald (1984, p. 214). See Ellis and Schmidt (1977, pp. 921–3) for further discussion of this sort of singular structure.

[22] Ellis and Schmidt (1977, p. 932) exemplify this sort of simplicity-chauvinism: "We know lots of examples of [flat singular spacetimes], all constructed by cutting and gluing together decent space-times; and because of this construction, we know that these examples are not physically relevant." See §5 for further remarks on this issue.

interest, and the behavior of the physical components of the Riemann tensor as measured by a frame parallel-propagated along some particular curve through the region of interest.[23] (If any of the physical components grow without bound in such a frame on a particular curve, or oscillate endlessly without settling down to a fixed, limiting value, then they will do so in all such frame-fields on that curve). In accordance with customary usage, I will refer to the existence of an incomplete curve along which the physical components of the Riemann tensor in a parallel-propagated frame do not approach a finite, limiting value as *p.p.-singular structure*, and I will refer to the same of some scalar curvature-invariant along an incomplete curve as *s.p.-singular structure* ('s.p.' for 'scalar polynomial'). Also in accord with custom, I will call the existence of an incomplete curve along which the physical components of the Riemann tensor in parallel-propagated frames and all its scalar invariants converge to finite values *quasi-regular singular structure*.[24] Note that curvature pathology on these definitions occurs not only if some feature of the curvature grows without bound along an incomplete curve, but also if it oscillates indefinitely (even if only within finite bounds), never settling down to a limiting value.

I believe there are two primary motivations for using a parallel-propagated frame in the terms of which to express the components of the Riemann tensor. First, one naturally expects the presence of curvature pathology to show itself, at the least, in misbehavior of the tidal forces an observer would experience along his or her worldline.[25] The intensity of tidal force, as measured in any frame, is directly proportional to the components of the Riemann tensor in that frame, as one can see by inspection of the equation of geodesic deviation. In a back-of-the-envelope sort of way, the unbounded growth of the components of the Riemann tensor in a parallel-propagated frame would seem to indicate that an observer traversing that curve would experience unbounded tidal forces as well. Second, Clarke (1973) demonstrated that an incomplete curve in a singular spacetime has a local extension if and only if the relevant incomplete curve constitutes quasi-regular singular structure. A local extension is an isometric embedding of an open subset containing the incomplete curve from the spacetime manifold into another spacetime in which the (image of the) curve can be extended. Local extensions can exist even when the singular spacetime as a whole is not embeddable as a proper open submanifold into a larger spacetime in which the (images of the) incomplete curves can be extended.[26] Many take the existence of local extensions to indicate that nothing *local*, such as curvature pathology (narrowly construed), goes wrong in quasi-regular singular spacetime, but rather some global structure impedes the extension of spacetime.

[23] A *frame* is a pseudo-orthonormal complete set of basis vectors for the tangent plane over a point of a manifold. A *frame-field* is an assignment of frames to points in some specified region, *e.g.*, along a curve or in an open set.

[24] Quasi-regular singular structure is perhaps the most psychologically disturbing, since it can be absolutely inobservable until one runs into it, so to speak, creating a hair-raising hazard for spacetime navigation.

[25] Tidal force is generated by the differential in intensity of the gravitational field, so to speak, at neighboring points of spacetime. For example, when I stand, my head is farther from the center of the Earth than my feet, so it feels a (practically negligible) smaller pull downward than my feet, inducing (practically negligible) stress on my body. For a graphic illustration of the effects of tidal forces on observers in strong gravitational fields, see the description, in Misner, Thorne, and Wheeler (1973, §32.6), of what would happen to a person standing on the surface of a collapsing star—not for the faint of heart or weak of stomach.

[26] Ellis and Schmidt (1977, pp. 928–9).

The motivation for using the behavior of scalar curvature-invariants as a criterion for the existence of curvature pathology is somewhat more straightforward. First, all the points made with regard to the components of the Riemann tensor in parallel-propagated frames hold as well for scalar invariants. Even better, though, a scalar curvature-invariant at a point does not depend on what curve through that point or what frame on a curve through that point one uses to probe the point: it is, as the name suggests, invariant. Unbounded growth of a scalar curvature-invariant, moreover, is logically equivalent to the unbounded growth of the components of the Riemann tensor as measured in *every* frame-field along the curve, parallel-propagated or not. S.p.-singular structure, then, implies, but is not implied by, p.p.-singular structure. In fact, all scalar curvature-invariants can be zero and yet the Riemann tensor not be equal to zero, as in certain plane gravitational wave spacetimes.[27] Spacetimes with colliding, thick gravitational waves provide examples of p.p.-singular structure in regions where all scalar curvature-invariants are well behaved; more strikingly, spacetimes containing colliding sandwich plane gravitational waves can exhibit p.p.-singular structure and yet all scalar curvature-invariants remain identically zero.[28] Finally, there are spacetimes containing colliding plane gravitational wave having incomplete curves in regions of a spacetime in which the Riemann tensor itself vanishes identically. These, the claim goes, provide examples of the existence of quasi-regular singular structure less artificial than that of the conical singularity above.[29] Thus the existence of incomplete curves does not *ipso facto* necessitate any sort of curvature pathology as conventionally quantified. That the misbehavior of the physical components of the Riemann tensor in a parallel-propagated frame or of a scalar curvature-invariant in the limit as one traverses a curve does not suffice to ensure that the curve be b-incomplete follows from examples of spacetimes produced by Sussmann (1988) in which scalar curvature-invariants diverge asymptotically along complete timelike and null geodesics.

Though there is no necessary connection of any sort between the existence of incomplete curves and curvature pathology as quantified in the standard ways sketched above, Ellis and Schmidt (1977) used b-completeness as a criterion to construct a classification of singular spacetimes according to the behavior of the curvature along the incomplete curves, as quantified in those standard ways. The classification has a binary, branching structure: first, an incomplete curve is said to constitute *essential singular structure* if there is no larger spacetime into which the singular spacetime can be embedded as a proper open submanifold, in which the (image of the) incomplete curve is extendible; otherwise it is said to be *inessential*. Essential singular structure is then sub-divided into quasi-regular and p.p.-singular structure; finally, p.p.-singular structure is subdivided into s.p.-singular

[27]Penrose (1960, p. 189).

[28]Konkowski and Helliwell (1992).

[29]There is something odd about this claim—though it is accepted without question in the physics and in the philosophy literature—especially in comparison with the contrary claim concerning flat spacetimes with conical singularities: no observations we have made or can make in the foreseeable future rule out the possibility of the existence of conical singularities in the actual spacetime in which we reside, but innumerable observations we have already made demonstrate with a hard finality that our spacetime cannot contain radiation of *any* sort in a form approximating to that of plane-waves, not even in the most extravagant and inaccurate of approximations. Our spacetime is too lumpy. Oughtn't this make spacetimes containing plane-waves "more unphysical" than those containing conical singularities?

and non-s.p.-singular structure. The thought behind the putative importance of the classification scheme seems to be as follows. Very little is known about singular structure at the present time, in part due to the difficulty of the mathematics involved in analyzing singular structure rigorously and in part due to the vanishingly small amount of experimental access we can get to singular structure in the foreseeable future. Nevertheless, the singularity theorems indicate that the spacetime we actually inhabit is singular, so it behooves us to try to understand such structure as much as possible. Classifying singular structure appears to be a way for us to organize and begin to get a grip on such a daunting task, and the scheme proposed by Ellis and Schmidt does seem to have many desirable features, such as clarity and simplicity. Earman (1995, pp. 37, 43–4) goes so far as to proclaim one of the most seminal virtues of the definition of singular structure in terms of b-completeness that it allows for a classification of this sort.

To be appropriate for such a task, I submit, the mathematically different species of singular structure ought to exhibit sorts of physical behavior *prima facie* different from each other in a physically significant way, as near as one can judge that sort of thing with the crude tools at our disposal; otherwise it will be difficult to see the physical relevance of this so far purely mathematical classification. As already noted, in a spacetime with s.p.-singular structure, the Riemann tensor components will behave badly as expressed in *any* frame-field along the relevant incomplete curve, and, moreover, will do so in general along any curve close enough, as it were, to the incomplete curve.[30] The tidal forces a body will suffer along its worldline are naturally measured in a spacelike 3-frame fixed rigidly in the body, orthogonal to the timelike unit vector tangent to the curve, itself used to fill out the full 4-frame. Based on what has already been said, one might expect that the state of motion of the observer along the curve, whether the observer is slowing down and speeding up somewhat, or spinning on his or her axis, would have no effect on how the observer experiences the curvature pathology: when a scalar curvature-invariant grows without bound along a curve, after all, the tidal forces as measured in *any* frame along the curve also will grow without bound. Interestingly enough, however, the state of motion of the observer as it traverses an incomplete curve, in the person of so-called inertial effects, can be decisive in determining the physical response of an object to the curvature pathology. Whether the object is spinning on its axis or not, for example, or accelerating slightly in the direction of motion, may determine whether the object gets crushed to zero volume along an s.p.-singular curve or whether it survives (roughly) intact all the way along the curve.[31]

The effect of the observer's state of motion on his or her experience of tidal forces can be even more pronounced in the case of p.p.-singular structure that is not s.p.-singular, which is precisely the existence of an incomplete curve along which there is a frame-field (necessarily not parallel-

[30]More precisely, in general there will exist an open neighborhood of the incomplete curve such that every curve completely contained in the open neighborhood has Riemann components that are as badly-behaved as one likes in all frames along the curve. The 'in general' hedges against the case where the scalar curvature-invariant oscillates wildly along the incomplete curve; in this case, it may be possible for nearby curves to weave cleverly around the incomplete curve in such a way as to avoid the peaks of oscillation, and so have well behaved Riemann tensor components. No hard results are known either way in such cases.

[31]Ellis and Schmidt (1977, p. 944–7).

propagated) relative to which the components of the Riemann tensor approach definite, finite limiting values along the curve.[32] In such a case, the frame-field in which the physical components of the Riemann tensor stably approach a limit is related to any parallel-propagated frame-field by a Lorentz transformation that, in an appropriate sense, behaves pathologically in the limit along the curve. For a non-geodetic curve, the proper mode of transport along a curve of a frame rigidly fixed in the body of an object traversing that curve is not parallel-propagation but Fermi-transport.[33] A Fermi-transported frame is related to a parallel-propagated frame by a continuously varying Lorentz transform. It can happen, therefore, that an observer cruising along a p.p.-singular curve that is not s.p.-singular would experience unbounded tidal forces and so be torn apart while another observer, in a certain technical sense approaching the same limiting point as the first observer, accelerating and decelerating in just the proper way, would experience perfectly well behaved tidal force, though he would approach as near as one likes to the other poor fellow in the midst of being ripped to shreds. Again, certain gravitational plane wave spacetimes provide good examples of this phenomenon: an observer travelling along the incomplete timelike geodesic constituting the singular structure would experience unbounded tidal acceleration, whereas *any* observer travelling arbitrarily close by would not.[34]

Things can get stranger still. An incomplete geodesic contained entirely within a compact subset of a spacetime, with accumulation point p, satisfying a certain genericity condition, necessarily constitutes p.p.-singular structure, so that an observer freely falling along such a curve would be torn apart by unbounded tidal forces as he or she approaches p; it can easily be arranged in such circumstances, though, that a separate observer, who actually travels through p, will experience perfectly well behaved tidal forces.[35] Here we have an example of an observer being ripped apart by unbounded tidal forces right in the middle of spacetime, as it were, while other observers cruising peacefully by could reach out to touch him or her in solace in the final throes of agony. This discussion points to a startling conclusion: curvature pathology, as standardly quantified, is not in any physical sense a well defined property of a region of spacetime *simpliciter*; rather, whether or not phenomena both physically pathological itself and attributable directly to pathology in the behavior of the Riemann tensor manifest themselves may sensitively depend on the sorts of devices, including their states of motion, with which one probes the region of interest. These matters are far more subtle and complicated than many, including Earman (1995), would lead one to believe.

Ellis and Schmidt (1977, p. 918) say, *vis-à-vis* their classificatory scheme (the canonical one outlined above):

> It is not claimed here that the singularities discussed are *likely* to occur in physically realistic situations, but rather that only when we understand which singularities can occur (a) in general space-times, and (b) in space-times with the field equations satisfied for particular matter content, can we hope to discuss fruitfully their occurrence, equations

[32] Ellis and Schmidt (1977, p. 939).
[33] Hawking and Ellis (1973, pp. 80–1).
[34] Ellis and Schmidt (1977, p. 937).
[35] Hawking and Ellis (1973, pp. 290–2).

of motion, and so on.

I do not mean to argue with the motivation for their classificatory scheme, but they beg a serious question with their 'which' in the phrase "when we understand which singularities can occur": clearly the correlative demonstratives of this relative interrogative refer to the different species of their classification, but why ought one think that their classification picks out physically relevant differences among all possible singular structures? This question becomes more poignant when one reflects on the fact that curvature pathologies provide the differentiæ for their speciation, and, as I have attempted to show, curvature pathology as customarily quantified is not a straightforward concept with clear and unambiguous physical content. I believe there is far more work to be done straightening out the physical consequences of the existence of singular structure. The mathematics has outrun the physics, but still masquerades as such.

Taub (1979) is the only person I know who shares in print[36] my qualms about the physical significance of the canonical classification scheme:

> I have difficulty understanding the usefulness of the classification scheme of singularities proposed... by Ellis and Schmidt.... I think that the important work on singularities now being done would become much more important if it turned toward learning how to deal with the physics associated with singularities...

He appears to be saying that one ought to concentrate first on trying to work out the behavior associated with various singular structures we are more or less familiar with in a clear and unambiguous way, and only then should one feel confident enough to begin classifying singular structures, based on that clear physical knowledge, not on a purely mathematical scheme that becomes murky as soon as one tries to think about it in physical terms. I heartily concur.[37]

4 Missing Points

We now have a precise definition of a singular spacetime, and some ideas about what such structure implies and does not imply about the curvature of spacetime, but, as Earman notes, "it is not true to an idea that is arguably a touchstone of singularities in relativistic spacetimes: spacetime singularities correspond to missing points."[38] For those who would argue missing points ought to be such a touchstone, Earman sketches what seems to me the most promising position, that, though the idea of missing points and that of curve incompleteness lead to *prima facie* different concepts of

[36] Geroch has told me in conversation that he does not see the use of the classification scheme either, on the grounds that he cannot see what physical content it has.

[37] A physically unambiguous sense of curvature pathology occurs in, *e.g.*, the FRWL (Friedmann-Roberston-Walker-Lemaître) metrics, wherein physical quantities such as the mass-density of ponderable matter grow without bound along incomplete curves and thus scalar curvature-invariants correlatively grow without bound as well. This sort of idea is developed nicely in a not very well known paper (to judge by its citation record) by Thorpe (1977). I think it would be of interest to see whether a classification scheme based on some of Thorpe's ideas could be constructed and compared to the canonical classification.

[38] Earman (1995, p. 40).

singular structure, they are extensionally equivalent in all physically reasonable singular spacetimes, and so the two concepts are for all practical purposes in agreement.[39] I will argue with this: missing points ought not be a touchstone of discussion of singular structure in relativistic spacetimes.

Missing points, could they be defined, would correspond to a boundary for a singular spacetime—actual points of an extended spacetime at which incomplete curves would terminate.[40] My argument therefore will alternate between speaking of missing points and speaking of boundary points, with no difference of sense intended. In many cases of physical interest, such as the FRWL and Schwarzschild metrics, one can attach boundary points by hand, so to speak, by visual inspection of the metric expressed in an appropriate global coordinate system, though different coordinate systems can lead to different topological structure for the boundary points. If one is to have a general notion of missing points, corresponding to the existence of incomplete curves, determined by nothing more than the metric structure of spacetime, clearly what is wanted is a method of attaching boundary points that does not depend on the choice of coordinate system, and which, moreover, can be used for any singular spacetime, not just the ones with 'simple' metric structure and global topology.

Before I begin examining the primary attempts to define boundary points for singular spacetimes,[41] it is well to note two oddities of the situation. In the case of a manifold with a Riemannian metric, Cauchy completion provides a well defined notion of missing points, and, by the Hopf-Rinow theorem, no points are missing from the manifold if and only if all geodesics of the Riemannian metric are complete (see footnote 4). We have already seen that any definition of missing points for a spacetime may—perhaps ought—not satisfy this condition: a spacetime can be geodesically complete yet still be b-incomplete, as Geroch's example illustrates. This already suggests that, even were one able to come up with a satisfactory definition of missing points in the context of Lorentzian metrics, it may not be extensionally equivalent to the existence of incomplete curves of the physically relevant type. The second, and more striking, circumstance strengthens this suspicion: compact spacetimes can contain incomplete, inextendible geodesics, as shown by a simple example due to Misner (1963). In a sense that can be made precise, compact sets, from a topological point of view, "contain every point they could possibly be expected to contain",[42] one consequence of which is that a compact manifold cannot be embedded as an open submanifold of any other manifold, a necessary pre-requisite for attaching a boundary to a singular spacetime—a manifold-with-boundary

[39] Earman (1995, p. 42). Earman continues on to say that even were one to grant this claim, the concept of singular structure as based on the idea of missing points is still conceptually distinct from that based on incomplete curves, and deserves in its own right to be examined. In reading the rest of the book one wishes that Earman had sketched a little more what he had in mind here. In particular, in later chapters, where Earman maps out issues and problems associated with the existence of singular structure, he never clarifies which conception of singular structure he is working with, and how opting for one or the other of the two conceptions would alter the character of the issue or problem at hand.

[40] Strictly speaking, such a space would not be a manifold in the usual sense of the term, but a manifold with boundary. See Spivak (1979) for a discussion of manifolds with boundary.

[41] I will not consider in this paper the 'ideal-point' boundary construction of Geroch, Kronheimer, and Penrose (1972), as it requires the singular spacetime to be past- and future-distinguishing, a fairly strong causality condition. I intend to sidestep all questions about the physical plausibility or necessity of such conditions.

[42] See Geroch (1985, §30) for a discussion of this precise sense.

minus its boundary is embeddable by the identity map as an open submanifold into itself, and, in the case when the manifold-with-boundary has a (pseudo-)Riemannian metric, the embedding can be made an isometry from the manifold *cum* metric on the full manifold to the interior of that manifold endowed with the natural restriction of the full metric. We ought not expect then that any definition of a boundary for singular spacetimes will cover every possible kind of singular structure, unless we are willing to swallow *outré* topological structure.[43]

Schmidt (1971) produced the most well known boundary construction for singular spacetimes, the so-called *b*-boundary based on the *b*-completeness criterion. An affine connection on a manifold allows one to define in a natural way a family of Riemannian metrics on the frame bundle over that manifold equivalent in the sense that they yield the same topology for the bundle-manifold, the natural topology of the frame bundle. It follows that the bundle-manifold is Cauchy complete with respect to one of these metrics if and only if it is so with respect to all. Schmidt showed, moreover, that the bundle-manifold is Cauchy complete in this sense if and only if the spacetime manifold is itself *b*-complete. To complete a singular spacetime in this scheme, then, one lifts all the incomplete curves from the spacetime manifold to the frame bundle, takes the Cauchy completion of the frame bundle with respect to one of the family of natural, topological, Riemannian metrics, and "projects down" the constructed boundary from the frame bundle to form a boundary for spacetime.

The relativity community at first embraced Schmidt's contruction with enthusiasm, to judge by the remarks in chapter 8 of Hawking and Ellis's canonical work *The Large Scale Structure of Space-Time*. Shortly thereafter, however, Bosshard and Johnson separately showed that the *b*-boundary had undesirable properties in the most physically relevant spacetimes known, the FRWL spacetimes, which to a quite high degree of approximation accurately model the large scale structure of the actual universe, and the Schwarzschild spacetimes, which represent the neighborhood of spherically symmetric isolated bodies, such as stars.[44] For closed FRWL spacetimes, the *b*-boundary consists of a single point (the same for the big bang as for the big crunch) that is not Hausdorff-separated from any point in the interior of the spacetime. Not only does one reach the same point, then, by travelling either forward or backward in time, but that point is, in a certain sense, arbitrarily near

[43]As an aside, this discussion highlights the fact that the interplay between metric and topological structure in the case of manifolds endowed with pseudo-Riemannian metrics is a far more delicate matter than it is, in general, acknowledged to be, and, in any event, *far* more delicate than in the case of a Riemannian manifold. This point does not seem to command the attention I think it ought to, in the philosophical as well as the physical literature; indeed, the matter often seems to be handled with a surprisingly cavalier attitude. It is not uncommon, for example, for a derivation of the Schwarzschild solution to yield, as the solution, the standard Schwarzschild coordinates, after which the deriver, almost always without comment, takes the topology of the spacetime to be $\mathbf{R}^2 \times \mathbf{S}^2$ as a matter of course. The same often happens with derivations of the FRWL spacetimes as well. Of course, a presentation of the metric in a particular coordinate-system does *not* determine the global topological structure of the manifold. It determines only *local* geometry. It makes as much (or as little) sense to take the topology of a spacetime whose metric can be represented in the Schwarzschild coordinates to be $\mathbf{R}^4$ as it does to take it to be $\mathbf{R}^2 \times \mathbf{S}^2$. Compare this state of affairs with that holding for a Riemannian manifold. The induced pointwise-metric (distance function) on the manifold defines a natural topology on the manifold in the standard way: one demands that the family of open balls of all radii centered on all points form a sub-basis for a topology. Conversely, a topological manifold selects a preferred family of Riemannian metrics one may impose on it, those that yield the already given topology.

[44]*Cf.* Bosshard (1976), Johnson (1977), Bosshard (1979) and Johnson (1979).

every single spacetime event! Similarly, the *b*-boundary of a Schwarzschild spacetime consists of a single point not Hausdorff-separated from any interior point of the spacetime. This certainly will not do for the advocates of missing points.[45]

A second (albeit temporally prior) method of constructing a boundary for singular spacetimes due to Geroch (1968a) fares much better with physically relevant spacetimes.[46] In this construction, the so-called *g-boundary*, geodesic incompleteness rather than *b*-incompleteness defines singular structure, and one defines a boundary point to be an equivalence class of incomplete geodesics under the equivalence relation 'approach arbitrarily close to each other' (in a certain technical sense). The set of boundary points can be given a topology and, in many cases of physical interest, can even be given a differentiable and metric structure, so that one can locally analyze the structure of spacetime at a 'singularity' rather than mess around with troublesome limits along incomplete curves.[47] The *g*-boundary construction, moreover, yields the boundaries one might have expected on physical grounds in spacetimes of particular physical interest: the *g*-boundary of a Schwarzschild spacetime is a spacelike 3-surface, topologically $\mathbb{S}^2 \times \mathbb{R}$, and that of a closed FRWL spacetime is the disjoint union of two spacelike $\mathbb{S}^3$'s. Pathological topology rears its head here as well, though, in the case of Taub-NUT spacetime:[48] the *g*-boundary of this spacetime contains a point that again is not Hausdorff-separated from any point in the interior of the spacetime.

The advocate of missing points who wants to hold on to the *g*-boundary may at this point retort that Taub-NUT spacetime hardly constitutes a physically relevant spacetime for other reasons, namely that it violates strong causality, which is to say that it contains causal curves that come arbitrarily close to intersecting themselves. While I do not think this reply carries much weight,[49] I have a better example at hand. Geroch, Can-bin, and Wald (1982) construct a geodesically incomplete spacetime with no causal pathology for which a very large class of boundary constructions, including the *b*- and the *g*-boundary, will yield pathological topology in the completed spacetime. The conditions that a boundary construction must satisfy to fall prey to this example are quite weak: each incomplete geodesic of a singular spacetime must terminate at some boundary point;

[45]The reactions to these problems vary widely. Clarke (1993), for instance, still embraces the *b*-boundary contruction, and defines a singularity to be a point on the *b*-boundary of a singular spacetime (§3.4). He barely mentions these problems, noting only in passing that the topological structure of the singular spacetime with boundary can be "very strange,"(p. 40) which I do not think qualifies as an adequate address of the issue. Wald (1984), on the other hand, does not like the *b*-boundary construction precisely because of these problems (*cf.* pp. 213–4), and Joshi (1993) does not even mention the possibility of attaching boundaries to singular spacetimes, speaking only of incomplete curves.

[46]Hawking (1966c) apparently proposed a similar construction, but, as the essay was never published (I learned of it from the bibliography of Hawking and Ellis (1973)), I have not been able to get a hold of it for examination.

[47]In certain contrived examples, there is an ambiguity in choice of topology for the *g*-boundary, but I will waive this concern for the sake of argument. I have bigger fish to fry.

If one suspects that this use of 'contrived' represents a hypocrisy now on my part—well, it may and it mayn't. To quote Geach quoting Whitman, "Do I contradict myself? Very well, I contradict myself. I am large, I contain multitudes."

[48]*Cf.* Hawking and Ellis (1973, §5.3) for a thorough account of Taub-NUT spacetime.

[49]Earman (1995, chs. 6–7) explains better than I could why a violation of strong causality *simpliciter* does not constitute an argument for the unphysicality of a spacetime.

and, in a certain technical sense, the boundary points corresponding to incomplete geodesics that are 'close together' must also be 'close together'. The advocate of missing points may point out that the example appears artificial and contrived, with closed sets excised here and conformal factors plastered on there, and in short has no physical relevance. I would reply with the lesson of my sermon from §3, and a remark that Geroch, Can-bin, and Wald (1982, p. 435) make: "The purpose of [a boundary] construction, after all, is merely to clarify the discussion of various physical issues involving singular space-times: general relativity as it stands is fully viable with no precise notion of 'singular points.'" When we contemplate potential phenomena that we have little or no observational access to, I submit that the standards for what can count as a *physical* account of a situation ought to be priggishly severe, if we are not unwittingly to degenerate into pure mathematical discourse.[50] A boundary-construction that yields topological pathology, and contains no precise criteria for what ought to count as a 'physically relevant' spacetime, does nothing to clarify discussion of the physical issues involved in analyzing singular spacetimes.

The abstract-boundary construction, or *a-boundary*, proposed by Scott and Szekeres (1994) appears at first glance to have the most promise for those wanting a natural, workable definition of missing points for singular spacetimes.[51] It also nicely exemplifies a feature of all missing point constructions I know of or can easily imagine, their dependence on a prior characterization of incomplete curves. For these two reasons, I will consider it in a little more detail than the previous two. An *envelopment* of a manifold $\mathcal{M}$ is an ordered pair $(\mathcal{N}, \phi)$ consisting of a manifold $\mathcal{N}$ and an embedding ϕ into $\mathcal{N}$ of $\mathcal{M}$ as a proper open submanifold of the same dimension.[52] Scott and Szekeres propose that singular structure always arises by the deletion of points from an envelopment of a singular manifold. Given an envelopment $(\mathcal{N}, \phi)$ of $\mathcal{M}$, a subset of its topological boundary in $\mathcal{N}$ will be called a *boundary set*. Now, as it clearly is possible to envelop a given manifold in many ways (if the manifold has any envelopment at all), one does not want to consider merely boundary sets of manifolds under particular envelopments, but rather equivalence classes of boundary sets under some appropriate equivalence relation. To this end, Scott and Szekeres propose the following:

Definition 4.1 *A boundary set B of $\mathcal{M}$ in an envelopment $(\mathcal{N}, \phi)$ is said to* cover *the boundary set B' of $\mathcal{M}$ in an envelopment $(\mathcal{N}', \phi')$ if for every open neighborhood U' in $\mathcal{N}'$ of B' there exists an open neighborhood U in $\mathcal{N}$ of B such that*

$$\phi \circ \phi'^{-1}[U' \cap \phi'[\mathcal{M}]] \subset U.$$

A boundary set B may cover another boundary set B' while B' does not cover B. One easily sees, however, that defining B and B' to be equivalent if they mutually cover each other does in fact

[50]Geroch stressed this point to me in a conversation in which he also dismissed the adequacy of his own g-boundary construction *merely because* it gave unphysical results in the admittedly contrived (!) example of Geroch, Can-bin, and Wald (1982). It gives very nice results in almost all other known types of examples.

[51]Whether the a-boundary construction satisfies the conditions of Geroch, Can-bin, and Wald (1982), and so necessarily leads to pathological topology for certain spacetimes, is not clear, for as of yet Scott and Szekeres have not defined a topology on the relevant entities of their construction. From the structure of the construction, I suspect that any topology one could more or less naturally define for it would satisfy Geroch, Can-bin, and Wald's conditions.

[52]When it can cause no confusion, I will often identify $\mathcal{M}$ with its image under the envelopment mapping.

yield an equivalence relation; the equivalence class of the boundary set B under this relation will be written '$[B]$' and called an *abstract boundary set*. An equivalence class that contains a singleton as a representative member will be called an *abstract boundary point*. The collection of all abstract boundary points is the *abstract* or *a-boundary*, written '$\mathcal{B}[\mathcal{M}]$'.

Although $\mathcal{B}[\mathcal{M}]$ by itself is defined without reference to any particular geometrical structure on $\mathcal{M}$, such as a pseudo-Riemannian metric or an affine connection, which Scott and Szekeres take to be one of its cardinal virtues, to define singular structure they must select a class of curves $\mathcal{C}$ on $\mathcal{M}$ satisfying what they call the bounded-parameter property: roughly speaking, the curves in $\mathcal{C}$ must cover the manifold and must be such that the parameter along any of the curves grows without bound if and only if it grows without bound along every "nice" reparametrization of the curve. The class of geodesics on a manifold with affine connection and the class of C^1 curves parametrized by generalized affine parameter on a manifold with affine connection provide two examples of classes of curves satisfying the bounded-parameter property. The idea is that curves in $\mathcal{C}$ will be used to probe the boundary to distinguish points 'at infinity' from points that can be reached in a finite parameter interval and hence are candidate singular points. The details of the construction and definitions hereon out become quite complicated, so I will sketch only the most salient points.

First, for a candidate singular spacetime $\mathcal{M}$, Scott and Szekeres wish to remove from consideration all abstract boundary points that have a representative singleton boundary point in some envelopment through which, in a certain technical sense, the spacetime metric can be smoothly extended. In this case, the thought is, the original spacetime simply had not been made as 'large' as it reasonably could have. Such points will be called *regular*, and need not apply as potential singular points. Next, one fixes the class of curves $\mathcal{C}$, and defines the $\mathcal{C}$-*boundary* to be the class of a-boundary points that have, in some envelopment, a singleton representative that is the limit point of a curve in $\mathcal{C}$; such points are also referred to as *approachable*. All other a-boundary points are *unapproachable*. It is straightforward to show that the property of being approachable or unapproachable is invariant under the defining a-boundary equivalence relation, but one must keep in mind that it depends entirely on the class $\mathcal{C}$ chosen. Given an envelopment $\mathcal{N}$ of $\mathcal{M}$, a non-regular boundary-point that is not the limit point of any curve of bounded parameter in $\mathcal{C}$ will be called a *point at infinity*; if, moreover, it cannot be covered by any regular boundary set of another envelopment, it will be called an *essential* point at infinity. This property is clearly invariant under the a-boundary equivalence relation, and so one speaks of a-boundary points at infinity. A non-regular boundary point p that is the limit point of some curve in $\mathcal{C}$ of bounded parameter will be called a *singular point*. If there exists a non-singular boundary set of another envelopment that covers p, then it is said to be *removable*; otherwise it is *essential*. Again, this property is invariant under the a-boundary equivalence relation, so one says that $[p]$ is an essentially singular a-boundary point. These, finally, are the missing points Scott and Szekeres aimed to construct.

The most obvious problem facing the a-boundary approach is its physical significance, or lack thereof. First off, a 'point' of the a-boundary is not a point in any usual sense of the term: an individual boundary point of one envelopment of a manifold can always be made to cover an uncountable number of boundary points in another envelopment. It is the case that, given any envelopment, the representative boundary set of an a-boundary point in that envelopment must be compact, but it is not even true that every compact boundary set is a representative of some a-boundary point, nor does the a-boundary point equivalence relation preserve connectedness and simple-connectedness—ought one think of a candidate singularity as a single point or as a non-simply connected, non-connected compact set? Then there is the unapproachability of some a-boundary points: it can happen, for instance, that regular a-boundary points of a pseudo-Riemannian manifold are not approachable by any geodesic of the metric. The existence of such extraneous points makes one wonder about the physical relevance of those boundary points that are approachable by curves in the spacetime. It is not also not clear what relevance the 'covering' relation they define has to anything physical: for a given $\mathcal{C}$, $\mathcal{C}$-boundary sets may cover unapproachable boundary sets; non-regular unapprochable boundary sets may cover approachable regular boundary sets; essential boundary points at infinity may cover anything except singular boundary sets and may be covered by anything except regular points; essential singular points may cover any kind of boundary set. Given the promiscuity of possible covering relations, I believe an argument is needed why this definition captures any physically relevant information, an argument they do not provide.

Neither do Scott and Szekeres broach a technical point that raises a serious difficulty for their approach at the very initial stages: some spacetimes, such as Taub spacetime, have two incomplete curves such that the spacetime can be extended so as to make either one or the other curve extendible, but no extension of the spacetime exists that makes both curves simultaneously extendible.[53] On Scott and Szekeres's account, both of these curves run into regular boundary points, and so neither will be counted as possible singularities, even though there is no actual envelopment of the spacetime in which both curves are simultaneously extendible.

Finally, on this view, incomplete curves wholly contained in compact regions of spacetime cannot count as singular structure, trivially so since compact manifolds cannot be embedded as proper open submanifolds of another manifold. Scott and Szekeres not only gamely swallow this consequence, but actually claim that it is a "*sine qua non* of any successful theory of singularities,"[54] and cite Shepley and Ryan (1978) as evidence for this claim.[55] This is not only a contentious view, at best, which they do not bother to argue for, and not only seems to run counter to the spirit of most considerations forwarded in discussions of singular structure, which revolve around incomplete curves, but seems seriously to conflict with their own stated criterion for selecting those points of the a-boundary that will be singular points, *viz.*, limit points of curves of bounded parameter, *i.e.*, curves that are, in some sense or other with (presumed) physical significance, incomplete.

This last point brings out my final consideration against the idea of missing points as touchstones in the investigation of singular spacetimes: the definition of singular spacetimes by incomplete curves is logically prior to the construction of missing points for singular spacetimes. All the missing point constructions I know of, and all the ways I can more or less easily imagine trying to concoct a new one, rely on probing the spacetime with curves of some sort or other to discover where points may be

[53]See, *e.g.*, Ellis and Schmidt (1977, p. 920) and Hawking and Ellis (1973, §5.8).
[54]Scott and Szekeres (1994, p. 34).
[55]In fact, Shepley and Ryan provide only the briefest and most tendentious of justifications for this position.

thought of as missing, just as in the Riemannian case one cannot complete a manifold until one knows which Cauchy sequences do not have a limit point, or equivalently which geodesics are incomplete. Even Scott and Szekeres, who make much of the fact that the construction of their a-boundary *per se* does not depend on the existence of any particular geometrical structure on a manifold, such as an affine connection and incomplete curves, cannot define singular points, which after all was the point of the whole affair, without probing their boundary with some specified class of curves.[56] One, however, does not need any conception of a missing point, much less a definition of such a thing, to define and investigate the existence of incomplete curves on a manifold. In sum, I disagree with the gist of much of the discussion of Earman (1995, ch. 2), wherein he suggests that unclarity plagues the semi-official definition of a singular spacetime, in terms of b-incompleteness, in so far as, on the face of it, one does not know how it relates to the idea of missing points. Incomplete curves seem to me a fine definition of singular structure on their own. I will try to make these considerations more precise in the following section.

5 Local *vs.* Global Properties of a Manifold

There is at least one *prima facie* good reason why it would be useful to have a precise characterization of points missing from singular spacetimes: one would then be able to "paste the points to the boundary of the spacetime manifold" and so analyze the structure of the spacetime locally at the singularity, instead of taking troublesome, perhaps ill-defined limits along incomplete curves. The power and elegance of Penrose's construction of conformal infinity for asymptotically flat spacetimes lies precisely in the ability one gains to perform such analysis locally at infinity, without relying on limits.[57] The example of Geroch, Can-bin, and Wald (1982) already discussed makes the prospects for a reasonable boundary construction for singular spacetimes grim. I believe this should not have been surprising.

In desiring a boundary so as to have a place to analyze structure locally, one ought to be clear on what one means by 'locally'. One sometimes hears talk of a global, as opposed to a local, feature of a spacetime, but I know of no precise characterization of the difference. I believe this distinction plays a crucial role in a proper understanding of the standardly proposed definitions of a singular spacetime in terms of incomplete curves. I therefore offer the following precise definition of this distinction. I formulate it initially for topological properties both for the sake of generality and because I think it easier to get a feel for the definition in the sparser arena of topological structure than in the more cluttered arena of differentiable manifolds with an affine structure.

Consider the class $\mathfrak{T}$ of all topological spaces. A *topological property* $\mathfrak{P}$ is a subclass of this class. A topological space $\mathcal{S}$ has the property $\mathfrak{P}$ if $\mathcal{S} \in \mathfrak{P}$.

Definition 5.2 *A topological property* $\mathfrak{P}$ *is* local *if it has the following feature: a given topological space* $\mathcal{S}$ *has the property* $\mathfrak{P}$ *if and only if* $\mathcal{S}$ *is such that every neighborhood of every point has a*

[56] I thus think that Earman (1995, p. 42) was off-base when he suggested that the a-boundary might be used to "do justice" to the idea of missing points for singular spacetimes.

[57] See Wald (1984, §11.1) for an account of Penrose's construction.

subneighborhood that, considered as a topological space in its own right, with the restriction topology, has the property $\mathfrak{P}$.[58]

Roughly speaking, a local property must hold in arbitrarily small neighborhoods of every point of a topological space, but not necessarily in every neighborhood of every point of the space; and conversely, if the property holds in arbitrarily small neighborhoods of every point of a space, it must hold for the entire space for it to be local.

Definition 5.3 *A topological property is* global *if and only if it is not local.*[59]

One could be sure of ascertaining for a given topological space whether the local property $\mathfrak{P}$ held or not by checking for $\mathfrak{P}$ at individual points of the space (quite a few points, to be sure), whereas a global property cannot be checked by examining the structure of the space at any collection of points. As one should expect, local compactness, local connectedness and local simple connectedness for example all come out to be local on this definition, whereas compactness, paracompactness, connectnedness and simple connectedness come out to be global.[60]

In an analogous manner, one can now straightforwardly characterize properties of differentiable manifolds and of differentiable manifolds with an affine connection as either local or global. Non-trivial examples of local properties for a manifold include any structure residing entirely on the tangent planes over every point. For our purposes, the most important fact about a manifold with an affine connection arising from a pseudo-Riemannian metric is that both the property of geodesic completeness and of geodesic incompleteness come out to be global properties, again as one should expect. One might initially have thought that geodesic incompleteness, at least, ought to have been a local property—if a geodesic came to an end abruptly, as it were, surely one ought to be able to pinpoint where this happens. If one could do this, however, then it also would seem that one could continue the geodesic. If there were a point on the manifold where the incomplete geodesic terminated, one could, around that point, take a chart diffeomorphic to some open set of $\mathbb{R}^n$ (assuming the manifold does not already have a boundary), push the geodesic and the connection down to $\mathbb{R}^n$, where the geodesic obviously would be extendible, and pull the extended version back to the manifold, contradicting the hypothesis that the geodesic could not be continued. This cannotbe done, however, for incomplete geodesics of a pseudo-Riemannian metric. All attempts to construct "missing points" founder on this rock.

A point of spacetime, in the usual way of thinking of these matters, represents an *event*, a highly localized occurrence in spacetime such as a snapping of fingers or the collision of two billiard balls.

[58]This sense of 'local' has nothing to do with that often bandied about in discussions about the foundations of quantum mechanics.

[59]By this 'not', I do not mean the logical negation of the definition of 'local' but rather the class complement of the class of local properties in the class of all topological properties—interestingly enough, these do not come to the same thing. Were the logical negation of the definition of 'local' used to define 'global', this would entail that a space with the global property $\mathfrak{P}$ would have a point and a neighborhood of that point such that *every* subneighborhood of that neighborhood did *not* have $\mathfrak{P}$. Compactness is clearly not a local property, and yet does not satisfy the logical negation of the definition of 'local'.

[60]*Cf.* Hocking and Young (1988) for definitions of these topological properties.

It represents an instant of some ponderable object, the specious 'now' of some sentient being. When thinking on cosmic scales, the sun, at a certain instant, can profitably be thought of as occupying a single point of spacetime. In short, spacetime points pertain to discrete objects, very broadly construed, that can be localized in an intuitive sense. There is no *a priori* reason to suspect that the existence of an incomplete curve, a global phenomenon, could be tied in any natural or reasonable way to the existence of a particular point in an extended manifold. Incomplete curves are not discrete, localizable objects in the appropriate sense.

A detractor will likely balk at this line of thought, pointing to the case of Riemannian manifolds, wherein incomplete curves can be naturally associated with points of an extended manifold. I would reply that it is merely a happy accident in the Riemannian case that one can arrange this. One has no grounds for suspecting that one will be able to do this in the general case, and in fact, as I endeavored to show, one has reasons to suspect that in general one will not be able to do this, since curve incompleteness is global and a missing point is, well, a point, and so *prima facie* "local". Of course, even for Lorentzian manifolds, in certain cases, one will be able to associate to an incomplete curve a missing point in a natural way—*e.g.*, in Minkowski spacetime (in some global coordinate system) with the origin removed, to continue all the geodesics aimed at the missing origin one pastes the origin back into the space and continues the geodesics through that point—in general, though, one ought not expect the two to have anything to do with each other.

The demand that singular structure be localized at a *place* bespeaks an old Aristotelian substantivalism that invokes the maxim, "To exist is to exist in space and time."[61] When I speak of 'Aristotelian substantivalism' here, I refer to the fact that Aristotle thought that everything that exists is a substance and that all substances can be qualified by the Aristotelian categories, two of which are location in time and location in space. In particular, not only substantivalists but also relationalists in debates about the nature of spacetime points could (and often do, I think) consistently fall prey to this particular brand of substantivalism. By focusing attention on the way that spacetimes can have actual features that do not rely on the existence or absence of any particular point, and are not instantiated at any particuar point, I suspect that this distinction between global and local properties of spacetime could have a salutary effect on the moribund debate between substantivialists and relationalists. To lay my cards on the table, I suspect one could parlay these considerations into a persuasive argument for the most salutary (to my mind) of effects on that debate, its dismissal as a *Scheineproblem*.[62]

Geroch, Can-bin, and Wald (1982, p. 435) deserve the last word on this subject: "Perhaps the localization of singular behavior will go the way of 'simultaneity' and 'gravitational force.'"

[61] This formulation of the maxim is due to Earman (1995, p. 28).

[62] I would base such considerations and argument on those delivered by Stein (1989), against the cogency of the traditional debate between realists and anti-realists, so called.

6 The Finitude of Existence

> The mind of man, by nature a monist, cannot accept *two* nothings; he knows there has been *one* nothing, his biological inexistence in the infinite past, for his memory is utterly blank, and *that* nothingness, being, as it were, past, is not too hard to endure. But a second nothingness—which perhaps might not be so hard to bear either—is logically unacceptable.
>
> *Ada*
> V. Nabokov

In this paper I have examined the standard characterizations of singular spacetimes and rejected attempts to link singular structure to the existence of missing points, arguing that the characterization of singular structure in terms of incomplete curves is adequate for the purposes of all known sorts of physical investigations touching on the subject. In the end, this is the only criterion I know of that ought to matter when the issue is the cogency and cognitive content of a proposed physical notion and concomitant methods of physical investigation and argumentation framed in the terms of that notion. Before concluding, I turn to examine whether singular structure as thus characterized is objectionable on physical or interpretive grounds, and whether one is forced to or ought to take them as indicating the breakdown of classical general relativity, as some would have it. In the process, I will examine whether *b*-completeness is wholly consistent with some of the explicit sentiments behind using curve incompleteness as a criterion for singular structure.

Two types of worries, one psychological, the other physical, give rise to the dissatisfaction with the existence of incomplete curves in relativistic spacetimes. Trying to imagine the experience of an observer traversing one of the incomplete curves provokes the psychological anxiety, for that observer would, of necessity, be able to experience only a finite amount of proper time's worth of observation, even were he, in Earman's evocative conceit, to have drunk from the fountain of youth. The physical worry arises from the idea that particles could pop in and out of existence right in the middle of a singular spacetime, and spacetime itself could simply come to an end, as it were, though no fundamental physical mechanism or process is known that could produce such effects. These two types of worries are not always clearly distinguished from each other in discussions of singular structure, but I think it important to keep in mind that in fact there are two distinct types of problems envisaged for incomplete curves, requiring to some degree two separate sorts of response.

The existence of incomplete spacelike curves is often felt not to be so objectionable as that of incomplete timelike or null curves, on the grounds that it represents structure beyond the direct experience of any observer.[63] I submit that, on this criterion, neither ought one be so bothered by the existence of incomplete timelike or null curves, *for an observer travelling along such a curve will never directly experience the fact that he has only a finite amount of proper time to exist*—there is

[63] See, *e.g.*, Hawking and Ellis (1973, §8.1).

no spacetime point, no event in spacetime, that corresponds to the observer's ceasing to exist. This is not to say that the person traversing this worldline cannot surmise the fact, perhaps based on observation of the curvature in his immediate neighborhood, that he has only a finite amount of time to exist; the claim, rather, is that there will never be an instant when the observer experiences himself as dissipating, popping out of existence as it were. To disarm possible misunderstanding, I emphasize that I am referring to, not the "popping out of existence" due to the observer's possibly being torn apart by unbounded tidal acceleration or being shot in the midst of his experiments by a Luddite lunatic, but the "popping out of existence" that would come about because the observer actually reached the "end" of his worldline, so to speak—for there is no end of the worldline to reach! We may be unable to conceive of experiencing such a state of affairs, but this reflects limitations in our psychological constitution, not an inherent flaw in general relativity.[64]

These considerations suggest as well a tension between the definition of singular structure by b-incompleteness on the one hand and the intuitions that drove some to look to incomplete curves as marks of singular structure in the first place on the other. Only the finitude of proper time matters so far as the experience of a possible observer goes—a generalized affine parameter has no clear physical significance—but, while a curve's being b-incomplete implies that the curve is of finite total proper time, the converse is not true: timelike curves of unbounded total acceleration in Minkowski space can be of finite total proper time and yet be b-complete. I would even say that such a curve should be more disturbing on reflection to those with such intuitions than an incomplete null geodesic, for the concept of 'proper time' does not apply to null curves at all, even though they are the possible paths of massless particles. The few people who even remark on the tension usually mouth a few vague generalities about particles' "reaching infinity", the implication seeming to be that, in so far as particles tracing out such worldlines are able to accomplish this *recherché* feat, one should have no qualms about the discomfiture they may feel in having only a finite amount of proper time in which to exist.

I speculate, with no hard evidence, that people have not wanted to count such curves as constituting singular structure in large part because of vague worries about energy conservation—an observer would require an "infinite amount of energy" to traverse a curve of unbounded total acceleration. In general relativity, however, there is no rigorous, generic notion of energy conservation, not globally or locally—there is not even a rigorous, generic, invariant definition of 'energy'.[65] Indeed, the structure of general relativity offers up no *a priori* reason to suspect that it in any way excludes

[64] I remark in passing that those disturbed by the prospect of an observer's having only a finite amount of proper time in which to exist into the future ought to be troubled by the Big Crunch, if there is to be one, but I have found no discussion of this point in the vast literature on singular structure, even by those relativists who display the germane intuitions. The thought seems to be that one ought to abhor singular structure in the 'interior' of spacetime, because one could imagine 'encountering' it on a walk through the park, so to speak. I do not think this a consistent stance, though.

Also, there seems to be a feeling among workers in the area that incomplete spacelike curves are not so bad in so far as they will have no observable effect on possible experiments one could perform in such a spacetime. I do not have space or time to go into it here, but I do not think this view is correct. For a sketch of the grounds for my reasons for saying so, see the account in Synge (1960, ch. 1, §14) of the physical significance of spacelike intervals.

[65] See, *e.g.*, Curiel (2000b) and Curiel (2000a).

a particle's getting shot out asymptotically "to infinity" in finite total proper time, having started from perfectly regular (in whatever sense of that term one likes) initial data. After all, it is not even difficult to construct solutions to Maxwell's equation on Minkowski spacetime in which a charged test particle gets shot off "to infinity" in finite total proper time.[66]

An example of a spacetime that was b-complete for all timelike curves of bounded total acceleration but not for timelike curves of unbounded total acceleration would clarify some of these issues, and I conjecture that examples of such spacetimes exist. Those who would not want to count such a spacetime as singular would be forced to give up b-incompleteness as the criterion for singular structure—which, given the lack of a clear physical interpretation of b-incompleteness in general, as opposed to incompleteness with respect to total proper time, I would not mind.[67] Really, so long as the idea of "reaching infinity" is given no precise content, and no argument is made to show why such a thing ought to alleviate anxiety about observers having only a finite amount of proper time in which to exist, there seems no reason to hold on to b-completeness. Of course, if incomplete timelike curves of unbounded total acceleration constituted singular structure, then every solution to Einstein's field equations would be singular. Many would reject this conclusion out of hand, but it does not seem intolerable to me. Singular structure would simply be one more type of global structure that all spacetimes necessarily had, along with, *e.g.*, paracompactness. Once so much was settled, then one could further classify spacetimes, according to the needs of the project at hand, by satisfaction of various more restrictive types of curve-completeness in order to produce more restricted, physically significant types of singular structure, as the compactness of a spacetime is a more restrictive type of paracompactness.

On physical grounds, curve incompleteness has been objected to because it seems to imply that particles could be "annihilated" or "created" right in the middle of spacetime, with no known physical force or mechanism capable of performing such a virtuosic feat of prestidigitation.[68] The demand that a spacetime be maximal, *i.e.*, have no proper extension, often rests on similar considerations: Clarke (1975, pp. 65–6) and Ellis and Schmidt (1977, p. 920) conjecture that maximality is required by the lack of a physical process that could cause spacetime to draw up short, as it were, and not continue on as it could have, were it to have an extension. This sort of argument, though, relies (implicitly) on a certain picture of physics that does not sit so comfortably with general relativity: that of the dynamical evolution of a system. From a certain quite natural point of view in general relativity, spacetime does not evolve at all. It just sits there, sufficient unto itself, very like the Parmenidean One. A solution to Einstein's equation, after all, is an entire spacetime

[66]To see how one might do this, consider solutions to Laplace's equation that exert along some particular, fixed direction a force constantly increasing as one moves along that direction. Perhaps one places positive charges at regular intervals along a geodesic starting from a given point, such that the magnitudes of the charges increase exponentially as one moves along the geodesic away from the point. If one then fires off a negatively charged particle essentially tangent to the given geodesic in the direction of increasing charge, the electric field of the charges will send it shooting off with an exponentially increasing acceleration in the direction orthogonal to the geodesic.

[67]It is one of the few major shortcomings of Earman (1995) that he does not analyze the physical significance of the various sorts of curve incompleteness.

[68]*Cf.*, *e.g.*, Hawking (1967, p. 189).

simpliciter—and the topology may be naturally suggested by the form of the metric, as in the case of the Schwarzschild coordinates, but one can always put the 'same' metric on a space with an entirely different topology, and still have a solution to the field equations. From this point of view, the question of a physical mechanism capable of causing the spacetime manifold not to have all the points it could have had, as it were—which is essentially a topological question in the first place—becomes less poignant, perhaps even misleading.[69] Of course, an opponent of this point of view could argue that such a move could foreclose the possibility of deterministic physics, to which I would whole-heartedly agree, for we already know that general relativity does not guarantee deterministic physics: there may be no Cauchy surface in our spacetime, or there may even be so called naked singularities.[70]

Perhaps a more serious worry is that such a viewpoint would seem to deny that certain types of potentially observable physical phenomena require explanation, when on their face they would look puzzling, to say the least. Were we to witness particles popping in and out of existence, the mettle of physics surely would demand an explanation. I would contend in such a case, however, that a perfectly adequate explanation was at hand: we would be observing singular structure. If there were no curvature pathology around, such a response might appear to be ducking the real issue, *viz.*, why is there this anomalous singular structure when all our strongest intuitions and most dearly held metaphysical principles tell us it should be impossible?[71] Far from ducking the issue, the viewpoint I advocate is the only one I know of that gives us a toehold in looking for precise answers to such questions—or, more precisely, in making such questions precise in the first place. Note that those who balk at this viewpoint ought to be equally as troubled by the singular structure associated with the Big Bang as they are by the example under discussion, for it just as surely lacks an explanation. From the viewpoint I advocate, questions about what happened "before" the Big Bang, or why the universe "came into being", can come from their former nebulosity into sharper definition, for

[69] The invocation of problems arising from the principle of sufficient reason (if one thinks of these as problems at all!) in postulating maximality makes the same assumption: that a "creative force", in Earman's words, would create spacetime piece by piece, and not simply have it be there all at once, so to speak, in whatever form was desired. Under such a conception, one might wonder why the creative force would stop at any particular point and not continue on to 'complete' the spacetime. Such problems do not arise in the viewpoint I propound. Demanding maximality may lead to Buridan's Ass problems anyhow, for, as mentioned earlier, it can happen that global extensions exist in which one of a given set of incomplete curves is extendible, but no global extension exists in which every curve in the set is simultaneously extendible. Also, there may exist several physically quite different global extensions: the spacetime covered by the usual Schwarzschild coordinates for $r > 2M$, for instance, can be extended analytically to Kruskal-Schwarzschild spacetime, or it can be extended to a solution representing the interior of a massive spherical body. The three criteria usually invoked in choosing an extension are: analyticity (as in the Kruskal extension); preservation of a symmetry group (as in the interior Schwarzschild solution); limiting the Bondi news-function, *e.g.*, incoming radiation, in the extended spacetime. None seems very compelling. For example, why limit incoming radiation when relativity treats radiation as every bit as real as ponderable matter, in the sense of contributing to the stress-energy tensor and so to metric structure, irrespective of the presence of other fields or matter that may be considered sources of the radiation? Indeed, I know of no convincing, rigorous way to distinguish "radiation" from "ponderable matter" in general relativity.

[70] See Earman (1995, ch. 3) for a discussion of these phenomena.

[71] Speaking of which, I would love to have someone explain to me in a way that will not make me cringe what the difference is between a philosophical intuition and a metaphysical principle.

they become questions about the presence of certain global structure in the spacetime manifold, in principle no different from paracompactness, connectedness or the existence of an affine connection, and one can at least envisage possible forms of an answer to the (precise) question, "Are there any factors that necessitate spacetime's having such and such global structure?" And were we actually to observe particles popping in and out of existence, we could formulate and begin trying to answer the analogous questions.

The most serious problem I can imagine for the viewpoint I advocate is that of representing our subjective experience, experience that seems inextricably tied up with ideas of evolution and change. As I suggested earlier, this problem is not an idiosyncracy of the viewpoint I advocate, but in fact arises from the character of general relativity itself: 'dynamical evolution' and 'time' are subtle and problematic concepts in the theory no matter what viewpoint one takes, as attested by the most notorious and seemingly intractable problem in the drive to 'quantize' gravity, the so-called problem of time.[72] My viewpoint has the virtue of calling attention to this very fact, that, to judge by the preponderant mass of literature in both physics and philosophy, is often overlooked: general relativity, in its own way, requires us to refashion the conceptual apparatus we use to comprehend the physical world, to a rethink in a profound way several dearly held, deeply related concepts and the relations among them, just as quantum mechanics has.

It has become fashionable of late to say that such problems point to the need to find an "interpretation" of general relativity in the same sense in which the measurement problem in quantum mechanics is taken to require that that theory be interpreted. Belot (1996), for instance, reaches this conclusion on the basis of an investigation into the problems encountered in trying to develop a quantum theory of gravity. I think this is a serious misunderstanding. Quantum mechanics demands an interpretation because it is not clear how to model physical phenomena, how to model the outcomes of experiments *simpliciter*: the predictions of standard quantum theory are in some sense in contradiction with the outcomes of experiments, but not in such a way as to invalidate the theory—an extraordinary state of affairs. There is no analogous problem in general relativity. We know how to model in the terms of the theory experiments that manifest and probe every phenomena suggested or predicted by the theory, with no inconsistency of any kind, for we know with no ambiguity what are the fundamental, physical terms and principles of the theory in which one articulates these models and draws conclusions on their basis. In a similar vein, the comprehension of special relativity's dismissal of the idea of absolute simultaneity did not require an interpretation of the theory, in any sense of the term; it required only that investigators come to terms with the fact that the fundamental terms of the theory does not allow for the rigorous, physically relevant articulation of the fundamental terms of Newtonian physics. In quantum mechanics, we do not even know what the fundamental terms and principles—'measurement'? 'observable'?—ought to be.

In a paper on the foundations of quantum mechanics, discussing the lack in general relativity of an explicit representation of our experience of a privileged instant in our history, the "now", Stein (1984, p. 645) makes a remark most *à propos* to the present case: "…although relativity does not

[72]See Ashtekar (1991, §12.3) for a brief discussion of this problem.

give us a *representation* of that experience[, the psychologically privileged status of the "now"], there is no *incompatibility* between the experience and the theory: a gap is not a contradiction."[73] There is a gap between the raw materials the theory provides us and the rich content of our experience to be explained—but it is no flaw of or lacuna in general relativity—it is not in virtue of the lack of an "interpretation"—that the theory does not illuminate the psychological experience we imagine will accrue to an observer in any particular circumstance the theory predicts, no more than Newtonian mechanics fell short in so far as it did not show why I understand by certain irritations of my eardrum from perturbations in the ambient air pressure the import of the spoken word 'gap'. It cannot be an argument against general relativity that it predicts phenomena we find it difficult to envisage, when we also know perfectly well how to model experiments that manifest and probe the phenomena. On those grounds, I submit, every revolutionary physical theory ever proposed would have been DOA, in light of the historical evidence concerning the reception by contemporaneous scientists of every one of them.[74]

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[73]The italics are Stein's. The point Stein makes with this remark is somewhat different than the point I wish to extract from it, but they are close enough for my profitable use of it.

[74]I thank R. Geroch and D. Malament for stimulating conversations on all these topics. I'm also grateful to M. Dorato for writing a review of Earman (1995) that made me realize the need to reread it and think more about singular structure, and to the History and Philosophy of Science Department at Pittsburgh, where I presented an earlier, briefer, version of this paper in a colloquium, for stimulating questions.

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