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Deriving Special Relativity from the Theory of Subsonic Compressible Aerodynamics

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Abstract Mathematical transformations that convert the convected wave equation in subsonic compressible flow to one based on incompressible flow have profound implications in understanding the physical basis of the transformations used in the theory of special relativity. The evolution of using incompressible flow solutions for airfoil design in lieu of conducting high speed wind tunnel tests is briefly reviewed. This in turn evokes the forgotten history of aerodynamicists using the Prandtl-Glauert method of spatial contraction as a substitute for compressibility effects before WWII. Matrix expressions identical in form to those representing relativistic velocity, acceleration, and mass are developed from linear transformations relating compressible versus incompressible flow systems and fixed-to-vehicle versus fixed-in-space coordinate reference frames. The mathematical intersection of special relativity and compressible flow theory is generally not understood nor appreciated outside the field of subsonic aerodynamics, making it a compelling subject for us to explore.

Keywords Compressible flow, Convected wave equation, Incompressible flow, Lorentz factor, Prandtl-Glauert factor, Special relativity

1. Introduction

The development of a comprehensive theory of flight has been a slow, frustrating study in progress. An interesting account of its complex history up to the 1930s in Europe is given by [1]. Briefly stated, the question of interest revolves around using the approximate, ideal fluid theories of Bernoulli and Euler versus using the more exact, non-linear Navier-Stokes equations. By the beginning of the 1940's, a lot of low speed data had been collected from wind tunnel testing of standardized airfoil sections for subsonic aerodynamic research by the National Advisory Committee for Aeronautics [2, 3]. Using incompressible flow theory, the wind tunnel data were processed and formulated into tables and graphs suitable for aircraft development and design. The near universal assumption of incompressible flow for manned flight was quite reasonable at that time considering air speeds less than 200 miles per hour were typically involved, with only the tips of high speed propellers approaching the speed of sound. With the development of faster planes and jets, such as the 500 mph Messerschmitt Me262, it became imperative that methods be devised that would allow the older low speed, incompressible based tabulations to be used with simple correction factors so that the effect of compressibility could be accounted for when designing aircraft for higher speeds. The alternative was to build faster wind tunnels and to painstakingly redo the airfoil tests and tabulations.

Analytical methods to compensate incompressible based lift calculations are called compressibility corrections. The Prandtl-Glauert method is one such approach that assumes two-dimensional, incompressible flow parallel to an airfoil cross section and then reduces the length of the airfoil chord in the lift and moment equations to account for compressibility [4]. The effect of mathematically compressing the parallel axis coordinate can be easily accounted for in two dimensions. However, in three dimensions, stretching and compressing the parallel axis coordinate results in lateral coordinate effects that produce non-linear changes in wing forces and moments [5, 6]. To the aeronautical engineer, contracting chord length and other aircraft dimensions as a function of speed in expressions formulated on the basis of incompressible flow is not controversial. It will be shown that these spatial contractions are mathematical artefacts induced while performing transformations from one coordinate system to another.

Interest expressed by the aeronautical community in using coordinate transformation methods to convert incompressible to compressible flow has substantially decreased since the 1950s. It has been replaced with methods that directly solve the fluid dynamic equations of compressible flow using fast computers and software based

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on computational fluid dynamics. None the less, there is still interest in special applications such as for steady-state flow problems. Examples of this are seen by the use of a power series expansions in terms of the Mach number [7] and using a variable density and flow angle for mapping in two dimensions [8].

A link between special relativity and compressible fluid dynamics is not a new concept [9, 10]. But it is rarely pursued outside the shadow of unconventional physics. The purpose of this article is to systematic examine the mappings between compressible and incompressible flow systems in different coordinate frames and to show the profound similarity between special relativity and the convective wave equation expressions.

We shall assume air is a continuous fluid with fluid properties that are irrotational, inviscid, barotropic, and isentropic. There are also two flow systems considered that affect the form of the wave equation when solving for the perturbation velocity potential. In both cases, the X-axis of the coordinate system is aligned with the direction of the free-stream velocity vector. The x=0 coordinate is located at the airfoil's leading edge and coordinate values increase towards the trailing edge. Compressible flow refers to the representation of the wave equation for the perturbation velocity potential in which there are cross-derivative terms between the X, Y, Z spatial coordinates and the time coordinate. Incompressible flow is defined as flow within which the fluid density remains constant with pressure. It is represented with a wave equation for the perturbation velocity potential in which there are no cross-derivative terms. These two flow systems are linked by coordinate transformations.

Two reference frames are also considered. There is the non-inertial, fixed-to-vehicle (*FTV*) reference frame with coordinates attached to the leading edge of the vehicle. The vehicle remains at rest as the fluid medium flows against the leading edge with free-stream velocity $\mathbf{V}_\infty$, where $\mathbf{V}_\infty = \hat{e}_1 V_{1\infty} + \hat{e}_2 V_{2\infty} + \hat{e}_3 V_{3\infty}$. The other is the inertial, fixed-in-space (*FIS*) reference frame with the fluid medium initially at rest and the vehicle moving with speed $\left|\mathbf{V}_\infty\right|$.

2. Convected Wave Equation

2.1. Transient Case

Consider the fluid dynamics of a moving fluid medium that is compressible. The adiabatic compression of a fluid $\hat{\rho}\kappa_s$ can be defined as the relative change in the local fluid density $\hat{\rho}$ and adiabatic compressibility coefficient κ_s in response to a change in the local fluid pressure $\hat{p}$, such that $\hat{\rho}\kappa_s = (\partial\hat{\rho}/\partial\hat{p})_s$. In addition, the free-stream squared-speed of sound c_∞ is inversely proportional to the adiabatic compression $c_\infty = 1/\hat{\rho}\kappa_s$ evaluated at the free-stream fluid density ρ_∞, such that:

$$c_\infty^2 = \left(\partial\hat{p}/\partial\hat{\rho}\right)_{\hat{\rho}=\rho_\infty} \tag{1}$$

The Cauchy momentum equation can be written in the absence of viscous and external forces (e.g., gravity) as follows:

$$\hat{\rho}\, D\hat{\mathbf{u}}/Dt = \nabla\hat{p} \tag{2}$$

Expression (2) basically states that a pressure gradient at any point in the medium produces an acceleration of the fluid. The convective derivative term $D\hat{\mathbf{u}}/Dt$ is written out as the sum of two contributions, where $\nabla\hat{\mathbf{u}}$ is a tensor derivative of the fluid velocity $\hat{\mathbf{u}}$:

$$D\hat{\mathbf{u}}/Dt = \partial\hat{\mathbf{u}}/\partial t + \partial\mathbf{x}/\partial t\cdot\nabla\hat{\mathbf{u}} \tag{3}$$

The vector term $\partial\mathbf{x}/\partial t$ is defined as the fluid velocity $\hat{\mathbf{u}}$ for a stationary medium. However, when the medium is also moving with a free-stream velocity $\mathbf{V}_\infty$, then the free-stream contribution can be separated from the perturbation velocity component $\mathbf{u}$, such that $\partial\mathbf{x}/\partial t = \mathbf{V}_\infty + \mathbf{u} \quad = \hat{\mathbf{u}}$.

The convective form of the continuity equation for the conservation of mass can be expressed in terms of the divergence of the fluid velocity, such that:

$$D\hat{\rho}/Dt = -\hat{\rho}\, Div\,\hat{\mathbf{u}} \tag{4}$$

The gradient of a velocity potential Φ_c can be expressed in terms of the fluid velocity for irrotational flow, such that:

$$\hat{\mathbf{u}} = \nabla\Phi_c \tag{5}$$

The velocity vector of the fluid becomes tangent to the streamline for streamline flow conditions [3, 11], such that in Cartesian coordinates:

$$dy/dx = \hat{u}_2/\hat{u}_1;\ dz/dy = \hat{u}_3/\hat{u}_2;\ \&\ dx/dz = \hat{u}_1/\hat{u}_3 \tag{6}$$

Consider the case for unsteady compressible flow expressed in terms of a velocity potential, with a *FTV* (or laboratory) reference frame, irrotational (i.e., $curl\,\mathbf{u} \equiv 0$), inviscid (i.e., $\mu_c \equiv 0$), barotropic (i.e., $\rho_c \equiv (\rho_c)$), isentropic (i.e., constant entropy) flow conditions, and in the absence of external forces (e.g., gravity), such that [12, 13]:

$$\begin{aligned} c_a^2\,\nabla^2\Phi_c = {} & \partial^2\Phi_c/\partial t^2 + \partial\left(\nabla\Phi_c\cdot\nabla\Phi_c\right)/\partial t \\ & + \frac{1}{2}\nabla\Phi_c\cdot\nabla\left(\nabla\Phi_c\cdot\nabla\Phi_c\right) \end{aligned} \tag{7}$$

The subscript "c" indicates compressible flow conditions and c_a is the local speed of sound. The wave described by (7) continues onwards to infinity since no viscosity terms are included in the formulation to dissipate the wave. Terms in

(7) can be rearranged such that the second-order derivatives of the velocity potential are combined to form the following unsteady wave equation in Cartesian coordinates for compressible fluid flow [14, 15]:

$$\Phi_{c\,xx}\left(1-\left(\Phi_{c\,x}/c_a\right)^2\right)+\Phi_{c\,yy}\left(1-\left(\Phi_{c\,y}/c_a\right)^2\right)+\Phi_{c\,zz}\left(1-\left(\Phi_{c\,z}/c_a\right)^2\right)=\frac{1}{c_a^2}\Phi_{c\,tt}+2\frac{1}{c_a^2}\left(\Phi_{c\,x}\Phi_{c\,xt}+\Phi_{c\,y}\Phi_{c\,yt}+\Phi_{c\,z}\Phi_{c\,zt}\right)+2\frac{1}{c_a^2}\left(\Phi_{c\,x}\Phi_{c\,y}\Phi_{c\,xy}+\Phi_{c\,x}\Phi_{c\,z}\Phi_{c\,xz}+\Phi_{c\,y}\Phi_{c\,z}\Phi_{c\,yz}\right) \tag{8}$$

Poisson [16] was apparently the first to derive the one-dimensional version of (8). Some of the earliest discussions concerning the steady-state, two-dimensional version can be found in [17, 18]. The cross-derivative terms such as $\Phi_{c\,xt}$ and $\Phi_{c\,xy}$ in the wave equation (8) represent nonlinearities generated after differentiating squared-velocity quantities in (7). The cross-derivative terms vanish as the flow speed goes to zero but otherwise remain non-zero valued as the speed increases. Hence, any attempt to eliminate the cross derivatives by means of coordinate transformations will also make the transformed fluid a *fictitious fluid*. The non-dimensional variable M_j is defined as the Mach number in the j[th] direction of flow. It is evaluated as the quotient of the fluid velocity in the j[th] direction and the local speed of sound c_a, such that:

$$M_{ja}=\frac{1}{c_a}\frac{\partial\Phi_c}{\partial x_j} \tag{9}$$

Substitute the Mach number expressions from (9) back into the unsteady velocity potential equation (8) for a compressible fluid medium with a *FTV* reference frame, such that in Cartesian coordinates:

$$\Phi_{c\,xx}\left(1-M_{1a}^2\right)+\Phi_{c\,yy}\left(1-M_{2a}^2\right)+\Phi_{c\,zz}\left(1-M_{3a}^2\right)=2\frac{1}{c_a}\left(M_{1a}\Phi_{c\,xt}+M_{2a}\Phi_{c\,yt}+M_{3a}\Phi_{c\,zt}\right)+\frac{1}{c_a^2}\Phi_{c\,tt}+2\left(M_{1a}M_{2a}\Phi_{c\,xy}+M_{1a}M_{3a}\Phi_{c\,xz}+M_{2a}M_{3a}\Phi_{c\,yz}\right) \tag{10}$$

Of course the solution to (10) is not unique until both initial conditions and boundary conditions are prescribed for the velocity potential. Discussion of these are outside the scope of the paper but flow around aircraft vehicles is typically assumed to be tangent to all solid surfaces. Expression (10) holds for subsonic and supersonic flow conditions but not transonic flow since additional nonlinear terms representing compression shock and temperature loses have not been included. It should also be clear that a singularity, called a *Prandtl-Glauert singularity*, occurs when one of the j[th] Mach numbers approach the value of one. It is a mathematical singularity, rather than a physical singularity, arising from the absence of additional nonlinear terms in (10).

An additional assumption can be introduced to further simplify the compressible flow expression (10) for barotropic and isotropic flow conditions. Consider the case where a solid body disturbing the flow field is slender and is traveling at either subsonic or supersonic velocities. Using a Legendre transformation, replace the velocity potential term Φ_c with one representing a free-stream or undisturbed flow speed $|\mathbf{V}_\infty|=V_{c1\infty}$ (i.e., speed of the solid body) in the X-direction and that of a small perturbed velocity potential component φ_c. Under these conditions, the following linearized expansion using a perturbation velocity potential φ_c for compressible flow conditions can be used, such that:

$$\frac{\partial\Phi_c}{\partial x_c}=V_{c1\infty}+\frac{\partial\varphi_c}{\partial x_c}=V_{c1\infty}+u_c\,;\ \frac{\partial\Phi_c}{\partial y_c}=\frac{\partial\varphi_c}{\partial y_c}=v_c\,;\ \frac{\partial\Phi_c}{\partial z_c}=\frac{\partial\varphi_c}{\partial z_c}=w_c\,;\ \frac{\partial\Phi_c}{\partial t_c}=\frac{\partial\varphi_c}{\partial t_c} \tag{11}$$

Expression (10) further simplifies if the velocity potential derivatives are much smaller than the free-stream speed $|\mathbf{V}_\infty|$ and if flow is aligned along the X-axis. The resultant unsteady perturbation velocity potential equation reduces as follows:

$$\varphi_{c\,xx}\left(1-M_{1a}^2\right)+\varphi_{c\,yy}+\varphi_{c\,zz}=\frac{1}{c_a^2}\varphi_{c\,tt}+\frac{2}{c_a}M_{1a}\varphi_{c\,xt} \tag{12}$$

2.2. Steady-State Case

The steady Prandtl-Glauert equation is obtained from the general velocity potential equation (10), such that:

$$\Phi_{c\,xx}\left(1-M_{1a}^2\right)+\Phi_{c\,yy}\left(1-M_{2a}^2\right)+\Phi_{c\,zz}\left(1-M_{3a}^2\right)=2\left(M_{1a}M_{2a}\Phi_{c\,xy}+M_{1a}M_{3a}\Phi_{c\,xz}+M_{2a}M_{3a}\Phi_{c\,yz}\right) \tag{13}$$

The steady-state expression (13) will dramatically simplify if the free-stream velocity $\mathbf{V}_\infty$ is also aligned along the X-axis, such that:

$$\Phi_{c\,xx}\left(1-M_{1\infty}^2\right)+\Phi_{c\,yy}+\Phi_{c\,zz}=0 \tag{14}$$

The simplified form of (14) does not hold near the leading edge of an airfoil where the velocity perturbation is of the same order-of-magnitude as the free-stream velocity, unless the free-stream Mach number is itself small. It forms the basis for many aerodynamic analysis methods.

The presence of cross-derivative terms in the convected wave equation (10) still makes its numerical computation a challenging process despite the recent advantage in having fast computers and more efficient algorithms.

3. Incompressible Flow

The more general problem of describing the convective wave equation in a compressible fluid medium will now be simplified for subsonic speeds. A brute force selection procedure will be presented to transform the coordinate system to various fictitious, incompressible flow systems. This approach will demonstrate that the resultant transformations are mathematical constructs that only depend on satisfying the desired form of the partial differential equations being transformed.

Convert coordinate time intervals $t_c - t_{c0}$ and $t_{ic} - t_{ic0}$ with initial times t_{c0} and t_{ic0} to distance intervals τ_c and τ_{ic} by introducing the following change in variables:

$$\tau_c = \left(t_c - t_{c0} \right) V_{c1\infty} \quad \text{and} \quad \tau_{ic} = \left(t_{ic} - t_{ic0} \right) V_{ic1\infty} \tag{15}$$

The linearized, unsteady, elliptic *convected wave equation* of velocity potential for a compressible fluid medium previously given in expression (10) reduces as follows for Cartesian coordinates upon application of the linearization expansion procedure given in (11) [13, 14, 19]:

$$\begin{aligned} &\frac{\partial^2 \varphi_c}{\partial x_c^2}\left(1 - M_{c1\infty}^2\right) + \frac{\partial^2 \varphi_c}{\partial y_c^2} + \frac{\partial^2 \varphi_c}{\partial z_c^2} = \\ &M_{c1\infty}^2 \frac{\partial^2 \varphi_c}{\partial \tau_c^2} + 2 M_{c1\infty}^2 \frac{\partial^2 \varphi_c}{\partial x_c \partial \tau_c} \end{aligned} \tag{16}$$

Note that the Mach number $M_{c1\infty}$ in the convected wave equation (16) is defined in (9) and (11) as a function of the local perturbation velocity $\mathbf{u}_c$ for a compressible fluid medium.

The wave equation (16) for a compressible fluid with coordinate system $\left(x_c, y_c, z_c, \tau_c\right)$ will now be transformed to an equivalent incompressible flow coordinate system $\left(x_{ic}, y_{ic}, z_{ic}, \tau_{ic}\right)$ in Cartesian coordinates. Incompressible terms are indicated by using the subscript "ic". A subsonic free-stream velocity $V_{c1\infty}$ and free-stream speed of sound $c_{c\infty}$ in the compressible flow system will be assumed, such that:

$$\varphi_{ic} = \beta_c \varphi_c \tag{17}$$

$$\beta_c = \sqrt{1 - M_{c1\infty}^2} \quad iff \quad \left| M_{c1\infty} \right| < 1 \tag{18}$$

$$\left.\begin{aligned} x_{ic} &= a_c x_c + b_c \tau_c \\ y_{ic} &= y_c \\ z_{ic} &= z_c \\ \tau_{ic} &= c_c x_c + d_c \tau_c \end{aligned}\right\} \tag{19}$$

$iff \ -\infty < x_c < +\infty \ \ \& \ -\infty < t_c < +\infty$. Note that the X and T line coordinates must be defined over an infinite domain in order for these transformation methods to be valid. The coefficient β_c in (17) is called the *Prandtl-Glauert factor* in aerodynamics and the *Lorentz contraction factor* in the theory of relativity [20]. It should be obvious from (19) that the new variable τ_{ic} is no longer a real time variable since its value is translated by the X space coordinate term $c_c x_c$. This means the newly introduced x_{ic}, τ_c and τ_{ic} coordinates are part of a fictitious mathematical construct that will be used to simplify the partial differential equations. Term Det is set equal to the Jacobian determinant of the $\left(x_c, \tau_c\right)$ transformation matrix given in (19):

$$Det = a_c d_c - b_c c_c \tag{20}$$

Take the second-order partial derivatives of the perturbation velocity potential φ_c with respect to the compressible Cartesian coordinate x_c and the compressible time variable τ_c. Replace the coordinate derivatives in (16) using the chain rule on (19), such that:

$$\begin{aligned} \frac{\partial^2 \varphi_c}{\partial x_c \partial x_c} = {} & \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial x_{ic}} a_c^2 + 2 \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial \tau_{ic}} a_c c_c \\ & + \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial \tau_{ic} \partial \tau_{ic}} c_c^2 \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial^2 \varphi_c}{\partial x_c \partial \tau_c} = {} & \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial x_{ic}} a_c b_c + \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial \tau_{ic}} a_c d_c \\ & + \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial \tau_{ic} \partial x_{ic}} c_c b_c + \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial \tau_{ic} \partial \tau_{ic}} c_c d_c \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial^2 \varphi_c}{\partial \tau_c \partial \tau_c} = {} & \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial x_{ic}} b_c^2 + 2 \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial x_{ic} \partial \tau_{ic}} b_c d_c \\ & + \frac{\partial \varphi_c}{\partial \varphi_{ic}} \frac{\partial^2 \varphi_{ic}}{\partial \tau_{ic} \partial \tau_{ic}} d_c^2 \end{aligned} \tag{23}$$

Group all of the XX partial derivative terms upon substitution of (21), (22), and (23) into the compressible convected wave equation (16). Then set the resultant expression equal to the corresponding XX partial derivative term given below in (28) that represents the incompressible wave equation with a *FTV* reference frame. Repeat the process of grouping the XT and TT partial derivatives. Solve the resultant grouped coefficient expressions and the Jacobian determinant (20), such that:

$$\beta_c^2 a_c^2 - 2M_{c1\infty}^2 a_c b_c - M_{c1\infty}^2 b_c^2 = Sgn_{XX} \tag{24}$$

$$2\beta_c^2 a_c c_c - 2M_{c1\infty}^2 \left(a_c d_c + c_c b_c \right) - M_{c1\infty}^2 2 b_c d_c = 0 \tag{25}$$

$$\beta_c^2 c_c^2 - 2M_{c1\infty}^2 c_c d_c - M_{c1\infty}^2 d_c^2 = - Sgn_{TT} M_{i\,c1\infty}^2 \tag{26}$$

$$a_c d_c - b_c c_c = Sgn_{Det} \tag{27}$$

The Y and Z axis components to the transformation coefficients will always equal to one since the free-stream velocity $\mathbf{V}_{c\infty}$ vector is assumed to be aligned parallel with the X axis. Hence, the transformation problem reduces to finding the remaining coefficients for just the X and T components. There remain four unknown coefficients a_c, b_c, c_c, d_c and four constraint equations in (24) to (27). The value of the three *sign* terms used in (24) to (27) are unknown but they are restricted to plus or minus one. The end product of the transformation process is a partial differential equation that represents the wave equation of the perturbation velocity potential for an incompressible fluid medium with a *FTV* reference frame, such that:

$$Sgn_{XX}\, \varphi_{i c x_{ic} x_{ic}} + \varphi_{i c y_{ic} y_{ic}} + \varphi_{i c z_{ic} z_{ic}} = Sgn_{TT}\, M_{ic1\infty}^2\, \varphi_{i c \tau_{ic} \tau_{ic}} \tag{28}$$

The expression (28) is also known as the acoustic wave equation in fluid dynamics and the d'Alembert equation for electromagnetic waves in physics when the two Sgn terms equal plus one [21].

3.1. Brute Force Algorithm

The more general problem to be addressed here is to present a brute force algorithm that is used to perform a systematic search of coefficients for the linear homogeneous transformation with just X and T components. These coefficients will convert the convected wave equation for a compressible fluid medium with a *FTV* reference frame into a wave equation for an incompressible fluid medium with either a *FTV* or *FIS* reference frame. The algebraic matrix used in this search is as follows:

$$\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix} = \begin{pmatrix} \dfrac{C_a}{E} & \dfrac{C_b}{E} \\ \dfrac{C_c}{E} & \dfrac{C_d}{E} \end{pmatrix} \cdot \begin{pmatrix} x_c \\ \tau_c \end{pmatrix} \tag{29}$$

The transformation coefficients first presented in (19) are related to the generic case of (29) as follows: $a_c = C_a / E$; $b_c = C_b / E$; $c_c = C_c / E$; and $d_c = C_d / E$. A search was conducted using eleven trial formulas that were substituted into the numerator coefficients C_a, C_b, C_c, and C_d. The eleven trial formulas are as follows: 0, 1, -1, M_∞, $-M_\infty$, β, $-\beta$, β^2, $-\beta^2$, M_∞^2, & $-M_\infty^2$. In addition, seven trial formulas were systematically substituted into the denominator coefficient E. The seven trial formulas are as follows: 1, β, M_∞, $M_\infty^{3/2}$, $\beta^{3/2}$, M_∞^2, & β^2. A systematic search using the above combination of numerator and denominator trial formulas resulted in a total of 102,487 combinations that were tested. The three *sign* terms used in (24) to (27) are only allowed to have values equal to plus or minus one: $Sgn_{XX} = \pm 1$; $Sgn_{TT} = \pm 1$; & $Sgn_{Det} = \pm 1$. The search using these three sign coefficients requires an additional eight times of effort, resulting in a total of 819,896 searches. The actual search is easily performed using a numerical algorithm by setting the Mach number M_∞ to an arbitrary subsonic value, such as 0.3, looping through all of the different formulas and signs, and saving only those trial formulas that exactly satisfy the four constraining expressions (24) to (27). Only six unique transformation sets were found during these searches. These are listed in Table 1. Term Det is the Jacobian determinant of the transformation matrix shown in the third column. Three of the transformations have a Jacobian determinant equal to plus one and three have a Jacobian determinant equal to minus one.

It should be obvious that making changes in the sign of the Jacobian determinant and in the partial differential equation terms of (28) will result in profound differences in the physical system being represented by the incompressible equation and in any boundary conditions associated with the original compressible flow system given by (16).

All six of the transformations listed in Table 1 start with the partial differential equation for the linearized, unsteady, convected wave equation for a compressible fluid medium with a *FTV* reference frame given in (16) for subsonic flow conditions. During the forward process, the transformed system represents various partial differential equations for the linearized, unsteady, wave equation for an incompressible fluid medium with a *FTV* reference frame. Both the original convected wave equation (16) and the six transformed wave equations of Table 1 are classified as elliptic partial differential equations. The remainder of the paper will only focus on the first two solutions given in Table 1.

The matrix shown on *line a* in Table 1 is the inverse matrix for a Galilean transformation. The matrix shown on *line b* of Table 1 will be labeled the Miles transformation matrix and is described in [13]. The Lorentz transformation can be obtained by pre-multiplying the Galilean matrix with the Miles matrix:

$$\frac{1}{\beta}\begin{pmatrix} 1 & 0 \\ M_\infty^2 & \beta^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & 1 \\ M_\infty^2 & 1 \end{pmatrix} \tag{30}$$

The complete family of transformations will be described shortly.

There are at least four different ways to write the unsteady wave equation for perturbation velocity potential [13] in Cartesian coordinates when the free-stream velocity vector is made parallel to the X-axis. These come about from transforming between compressible versus incompressible flow systems and from transforming between *FTV* and *FIS* reference frames. The four partial differential expressions are listed in Table 2. Of these four, only *line a* in Table 2 represents air as a real fluid by using a compressible *FTV* wave equation with cross-derivative terms. The other three represent the equations for different fictitious fluids. Obviously the boundary conditions of the wave equations, which are not discussed here, would also be transformed accordingly. The fourth column of Table 2 gives a two letter abbreviation to distinguish each of the flow and coordinate frame assumptions.

Table 1. Summary of the six coordinate transformations that are unique and that satisfy the four constraints given in (24) to (27)

	Transformation	*Det*	Transformed PDE
a	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	1	$\varphi_{icX_{ic}X_{ic}} + \varphi_{icY_{ic}Y_{ic}} + \varphi_{icZ_{ic}Z_{ic}} = M_\infty^2\,\varphi_{ic\mathfrak{T}_{ic}\mathfrak{T}_{ic}}$
b	$\frac{1}{\beta}\begin{pmatrix} 1 & 0 \\ M_\infty^2 & \beta^2 \end{pmatrix}$	1	$\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$
c	$\frac{1}{M_\infty}\begin{pmatrix} 0 & 1 \\ -M_\infty^2 & M_\infty^2 \end{pmatrix}$	1	$-\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = -M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$
d	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	-1	$\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$
e	$\frac{1}{\beta}\begin{pmatrix} -1 & 0 \\ M_\infty^2 & \beta^2 \end{pmatrix}$	-1	$\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$
f	$\frac{1}{M_{1\infty}}\begin{pmatrix} 0 & 1 \\ M_\infty^2 & -M_\infty^2 \end{pmatrix}$	-1	$-\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = -M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$

Note that the Mach number M_∞ is defined as $M_\infty = (\hat{\mathbf{e}}_1 \cdot \mathbf{V}_\infty)/c_\infty$. The free-stream speed of sound c_∞ and vehicle speed $|\mathbf{V}_\infty|$ are assumed to be the same in all reference frames. The compressible and incompressible subscripts of these and associated speed variables are henceforth dropped.

Table 2. Partial differential equations of the unsteady, perturbation velocity potential in Cartesian coordinates

	Flow Type	Frame	Label	Coordinates	Unsteady Wave Equation
a	compressible	*FTV*	*CV*	(x_c, τ_c)	$\varphi_{cx_cx_c}(1-M_\infty^2) + \varphi_{cy_cy_c} + \varphi_{cz_cz_c} = M_\infty^2\varphi_{c\tau_c\tau_c} + 2M_\infty^2\,\varphi_{cx_c\tau_c}$
b	incompressible	*FTV*	*IV*	(x_{ic}, τ_{ic})	$\varphi_{icx_{ic}x_{ic}} + \varphi_{icy_{ic}y_{ic}} + \varphi_{icz_{ic}z_{ic}} = M_\infty^2\,\varphi_{ic\tau_{ic}\tau_{ic}}$
c	compressible	*FIS*	*CS*	$(X_c, \mathfrak{T}_c)$	$\varphi_{cX_cX_c} + \varphi_{cY_cY_c} + \varphi_{cZ_cZ_c} = M_\infty^2\,\varphi_{c\mathfrak{T}_c\mathfrak{T}_c}$
d	incompressible	*FIS*	*IS*	$(X_{ic}, \mathfrak{T}_{ic})$	$\varphi_{icX_{ic}X_{ic}} + \varphi_{icY_{ic}Y_{ic}} + \varphi_{icZ_{ic}Z_{ic}} = M_\infty^2\,\varphi_{ic\mathfrak{T}_{ic}\mathfrak{T}_{ic}}$

Given four partial differential equations, with abbreviates *CV*, *CS*, *IV*, & *IS* listed in Table 2, there are a total of twelve ways to inter-transform them. These transformation combinations are as follows: $IV \rightarrow \{ IS, CS, CV \}$; $CV \rightarrow \{ CS, IS, IV \}$; $IS \rightarrow \{ IV, CS, CV \}$; and $CS \rightarrow \{ CV, IS, IV \}$. The coefficients for these are derived in the same brute force manner previously described or by using simple substitution or inversions between matrices. These twelve transformation matrices are listed in Table 3.

In a manner similar to that developed for (15), the coordinate time variables T_c and T_{ic} for a *FIS* reference frame are related to units of distances $\mathscr{T}_c$ and $\mathscr{T}_{ic}$ through the following change in variables:

$$\mathscr{T}_c = \left(T_c - T_{c0} \right) V_{1\infty} \quad \text{and} \quad \mathscr{T}_{ic} = \left(T_{ic} - T_{ic0} \right) V_{1\infty} \tag{31}$$

What is most striking about the brute force approach described above is that it can be used to sculpture virtually any transformation matrix between two different partial differential equations of the same order. This should be seen as a cautious warning that coordinate transformations are convenient mathematical constructs that create a fictitious set of equations with fictitious fluid properties that are simpler to solve. It does not imply the resultant fluid or flow system exists or is even plausible.

3.2. Summary of Subsonic Coordinate Transformations

A list of the twelve transformation matrices used to convert the convected wave equation for subsonic speeds from compressible to incompressible flow conditions or the reverse process are listed in Table 3. Two letter abbreviations of the reference frame are listed in the second and fourth columns and the classical name associated with the transformation matrix is listed in the last column.

It is worth repeating that only the compressible *FTV* (i.e., *CV*) wave equation with cross derivatives in Table 2 represent air as a real fluid. The other three represent the wave equations for three different fictitious fluids. The brute force algorithm used to produce Table 3 demonstrates, from a heuristic point of view, that the transformation matrices enforce a particular linkage between time and space coordinates. This linkage eliminates the generation of cross-derivative terms involving time when transforming between the compressible wave equation *CV* and the three fictitious wave equation *CS*, *IV*, and *IS*. As stated at the bottom of Table 3, the brute force algorithm assumed the speeds $c_{c\infty}$ and $c_{ic\infty}$ to be the same in all reference frames, i.e., $c_{c\infty} = c_{ic\infty}$. However, the constraints of (24) to (27) do not require this speed constant to be the maximum allowed speed.

Consider the $IV \rightarrow CV$ transformation formulas $dx_c = \beta \, dx_{ic}$ for the X-coordinate and $d\tau_c = d\tau_{ic} / \beta$ for the T-coordinate given on *line f* of Table 3. These formulas convert spatial and temporal changes measured in an incompressible flow system to changes in a compressible flow system. The Prandtl-Glauert factor β is always less than one for subsonic speeds. It then follows that an observer working in a compressible flow system might be tempted to conclude that spatial measurement dx_{ic} decreases in length and that temporal measurement $d\tau_{ic}$ increases in duration with an increase in the free-stream speed $|\mathbf{V}_\infty|$. Of course we know this is only a mathematical artefact arising from the decision that someone previously had used a theoretical model based on the assumption of incompressible flow. This is exactly the case faced by aeronautical engineers before the availability of high speed wind tunnels and fast computers to run computational fluid dynamic software before WWII. No discrepancy in measurement occurs with speed if the observer had been consistent in comparing measurements against a theoretical model based on compressible flow. However, the situation is much more difficult and paradoxical to resolve if the observer were to deny the very existence of a compressible flow system. Space and time would then be concluded to be warped or bent with speed.

3.3. Velocity Transforms between Reference Frames

This section will describe how the perturbed velocity components developed in one reference frame can be related to those in another reference frame using the various coordinate transformations introduced in the previous section. The basic approach of [20] will be followed.

Let there be two different reference frames, called the K and K' systems. Assume the K' system moves relative to the K system with velocity $\mathbf{V}_\infty$ in a direction that is parallel to the X-axis of both systems. Define the X-axis velocity component $u = dx/dt$ as the particle velocity for the K system and the X-axis velocity component $u' = dx'/dt'$ of the same particle in the K' system, such that in general:

$$\left. \begin{aligned} u &= dx/dt \\ v &= dy/dt \\ w &= dz/dt \end{aligned} \right\} \tag{32}$$

$$\left. \begin{aligned} u' &= dx'/dt' \\ v' &= dy'/dt' \\ w' &= dz'/dt' \end{aligned} \right\} \tag{33}$$

Table 3. Summary of coordinate transformations for subsonic flow conditions

Reference frame on left side of the equal sign		Transformation	Reference frame on right side of the equal sign	
a	*IV*	$\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & 1 \\ M_\infty^2 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix}$	*IS*	Lorentz
b	*IV*	$\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & 1 \\ M_\infty^2 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix}$	*CS*	Lorentz
c	*IV*	$\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & 0 \\ M_\infty^2 & \beta^2 \end{pmatrix}\cdot\begin{pmatrix} x_c \\ \tau_c \end{pmatrix}$	*CV*	Miles
d	*CV*	$\begin{pmatrix} x_c \\ \tau_c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix}$	*CS*	Galilean
e	*CV*	$\begin{pmatrix} x_c \\ \tau_c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix}$	*IS*	Galilean
f	*CV*	$\begin{pmatrix} x_c \\ \tau_c \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} \beta^2 & 0 \\ -M_\infty^2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix}$	*IV*	Miles inverse
g	*IS*	$\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & -1 \\ -M_\infty^2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix}$	*IV*	Lorentz inverse
h	*IS*	$\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix}$	*CS*	Unit
i	*IS*	$\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_c \\ \tau_c \end{pmatrix}$	*CV*	Galilean inverse
j	*CS*	$\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_c \\ \tau_c \end{pmatrix}$	*CV*	Galilean inverse
k	*CS*	$\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} X_{ic} \\ \mathcal{T}_{ic} \end{pmatrix}$	*IS*	Unit
l	*CS*	$\begin{pmatrix} X_c \\ \mathcal{T}_c \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & -1 \\ -M_\infty^2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x_{ic} \\ \tau_{ic} \end{pmatrix}$	*IV*	Lorentz inverse

Note that the expressions in Table 3 make the assumption that free-stream velocity components $V_{c1\infty} = V_{ic1\infty} = V_{1\infty}$ and speed of sound values $c_{c\infty} = c_{ic\infty} = c_\infty$ are equal. It then follows that the Mach numbers $M_{c1\infty} = M_{ic1\infty} = M_\infty$ and Prandtl-Glauert factors $\beta_c^2 = \beta_{ic}^2 = \beta^2 = 1 - M_\infty^2$ are equal.

3.4. Transforming Velocities from *FIS* to *FTV* Reference Frames

The transformation between *FIS* and *FTV* reference frames is represented by the matrix on *line a* of Table 3 for incompressible flow systems, such that upon taking incremental differences in the coordinates:

$$\beta^2 = 1 - M_\infty^2 \quad iff \quad \left| M_\infty \right| < 1 \tag{34}$$

$$\left.\begin{aligned} dx_{ic} &= \frac{1}{\beta}\left(dX_{ic} + dT_{ic}V_{1\infty} \right) \\ dy_{ic} &= dY_{ic} \\ dz_{ic} &= dZ_{ic} \\ dt_{ic}V_{1\infty} &= \frac{1}{\beta}\left(M_\infty^2 dX_{ic} + dT_{ic}V_{1\infty} \right) \end{aligned}\right\} \tag{35}$$

The incremental changes in time variable τ_{ic} associated with the *FTV* reference frame and $\mathcal{T}_{ic}$ associated with the *FIS* reference frame are related to incremental changes in coordinate time variables t_{ic} and T_{ic} by the relationships (15) and (31), such that:

$$d\tau_{ic} = dt_{ic}V_{1\infty} \quad \text{and} \quad d\mathcal{T}_{ic} = dT_{ic}V_{1\infty} \tag{36}$$

Divide the incremental spatial variable expressions in the first three lines of (35) by the expression for dt_{ic} from the fourth line in (35), such that:

$$\frac{dx_{ic}}{dt_{ic}} = \frac{V_{1\infty}}{\left(M_\infty^2 \dfrac{dX_{ic}}{dT_{ic}} + V_{1\infty} \right)} \left(\frac{dX_{ic}}{dT_{ic}} + V_{1\infty} \right) \tag{37}$$

$$\frac{dy_{ic}}{dt_{ic}} = \frac{\beta V_{1\infty}}{\left(M_\infty^2 \dfrac{dX_{ic}}{dT_{ic}} + V_{1\infty} \right)} \frac{dY_{ic}}{dT_{ic}} \tag{38}$$

$$\frac{dz_{ic}}{dt_{ic}} = \frac{\beta V_{1\infty}}{\left(M_\infty^2 \dfrac{dX_{ic}}{dT_{ic}} + V_{1\infty} \right)} \frac{dZ_{ic}}{dT_{ic}} \tag{39}$$

Define the following perturbation flow velocity $\{u_{ic}, v_{ic}, w_{ic}\}$ of the *FTV* reference frame for an incompressible flow system using expressions from (5) and (37):

$$u_{ic} = \frac{\partial \varphi_{ic}}{\partial x_{ic}} \equiv \frac{dx_{ic}}{dt_{ic}} \tag{40}$$

$$v_{ic} = \frac{\partial \varphi_{ic}}{\partial y_{ic}} \equiv \frac{dy_{ic}}{dt_{ic}} \tag{41}$$

$$w_{ic} = \frac{\partial \varphi_{ic}}{\partial z_{ic}} \equiv \frac{dz_{ic}}{dt_{ic}} \tag{42}$$

Define the following perturbation flow velocity $\{U_{ic}, V_{ic}, W_{ic}\}$ of the *FIS* reference frame for an incompressible flow system:

$$U_{ic} \equiv \frac{dX_{ic}}{dT_{ic}} \tag{43}$$

$$V_{ic} \equiv \frac{dY_{ic}}{dT_{ic}} \tag{44}$$

$$W_{ic} \equiv \frac{dZ_{ic}}{dT_{ic}} \tag{45}$$

Substitute the perturbation velocities from (40) through (45) back into (37), (38), & (39), replace the Mach number $M_{1\infty}$ with $M_\infty = V_{1\infty} / c_\infty$, and rearrange the resultant terms, such that for incompressible flow systems:

$$u_{ic} = \frac{\left(U_{ic} + V_{1\infty} \right)}{\left(1 + U_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)} \tag{46}$$

$$v_{ic} = \frac{\beta V_{ic}}{\left(1 + U_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)} \tag{47}$$

$$w_{ic} = \frac{\beta W_{ic}}{\left(1 + U_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)} \tag{48}$$

for $\beta^2 = 1 - M_\infty^2$ & $M_\infty = V_{1\infty} / c_\infty$. The expressions in (46) through (48) are collectively called the *velocity-addition formulas* [20] or the *composition law for velocities*. They are mathematical manifestations of the Lorentz transformation used in (30) to transform the *FIS* reference frame to a *FTV* reference frame in incompressible flow systems. It is easy to show with numerical simulation that the magnitude of the perturbed velocity set $\{u_{ic}, v_{ic}, w_{ic}\}$ in formulas (46) to (48) will always vary between 0 and the fluid's characteristic speed c_∞ when the magnitude of the *FIS* perturbation velocity set $\{U_{ic}, V_{ic}, W_{ic}\}$ varies between 0 and c_∞; and when the magnitude of the free-stream velocity $V_{1\infty}$ is restricted to vary between 0 and c_∞.

Consider for a moment the simple coordinate modification for the *FTV* and *FIS* velocity representations in the X-Y plane of the incompressible flow system: $u_{ic} = u\,Cos\theta$, $v_{ic} = u\,Sin\theta$, $U_{ic} = U'\,Cos\theta'$, and $V_{ic} = U'\,Sin\theta'$. Substitute these trigonometric relationships into the Y-axis component of (47):

$$v_{ic} = u\,Sin\theta$$
$$= U'\sqrt{1-V_{1\infty}^2\Big/c_\infty^2}\;Sin\theta'\Big/\left(1+U'\,Cos\theta'\,V_{1\infty}\Big/c_\infty^2\right) \tag{49}$$

Perform a binomial series expansion of the square root and denominator terms in (49) when the ratio $V_{1\infty}/c_\infty$ is much less than one such that: $u\,Sin\theta - U'\,Sin\theta' = -\,U'^2\,Sin\theta'\,Cos\theta'\,V_{1\infty}\Big/c_\infty^2$. Further assume the special case where the velocity components u and U' are equal to speed c_∞ : $Sin\theta - Sin\theta' = -\,Sin\theta'\,Cos\theta'\,V_{1\infty}/c_\infty$, *iff* $u \equiv c_\infty$ & $U' \equiv c_\infty$. Substitute in the sine subtracting function $Sin\theta' - Sin\theta = 2Cos\left((\theta'+\theta)/2\right)\,Sin\left((\theta'-\theta)/2\right)$, such that: $2\,Cos\left((\theta'+\theta)/2\right)\,Sin\left((\theta'-\theta)/2\right) = Sin\theta'\,Cos\theta'\,V_{1\infty}/c_\infty$. Define $\Delta\theta_{ic} = \theta' - \theta$ as the difference between the velocity coordinate angles θ' and θ in the X-Y plane of the incompressible flow system. The $Cos\,((\theta'+\theta)/2)$ and $Cos\theta'$ terms cancel each other when $\Delta\theta_{ic}$ goes to zero, such that: $\Delta\theta_{ic} = Sin\,\theta'\,V_{1\infty}\big/c_\infty$, where $V_{1\infty}/c_\infty \ll 1$, $u \equiv c_\infty$, $U' \equiv c_\infty$, and $\Delta\theta_{ic} \to 0$. This final expression for $\Delta\theta_{ic}$ is called the *aberration of light* formula in relativistic physics [20] when c_∞ is interpreted as the speed of light in vacuum.

Now consider the relationship between the speed of the *FTV* perturbation velocity set $\{u_{ic}, v_{ic}, w_{ic}\}$ and the speed of the *FIS* perturbation set $\{U_{ic}, V_{ic}, W_{ic}\}$. Define the *FTV* speed $\left|\mathbf{u}_{ic}\right|$ and *FIS* speed $\left|\mathbf{U}_{ic}\right|$ for incompressible flow as follows:

$$\left|\mathbf{u}_{ic}\right| = \sqrt{u_{ic}^2 + v_{ic}^2 + w_{ic}^2} \tag{50}$$

$$\left|\mathbf{U}_{ic}\right| = \sqrt{U_{ic}^2 + V_{ic}^2 + W_{ic}^2} \tag{51}$$

Substitute the velocity relationships derived in (46) to (48) into the *FTV* expression for the squared-speed $\left|\mathbf{u}_{ic}\right|^2$ in (50), such that:

$$\left|\mathbf{u}_{ic}\right|^2 = \frac{\left(U_{ic}^2 + 2U_{ic}V_{1\infty} + V_{1\infty}^2 + V_{ic}^2 + W_{ic}^2 - \left(V_{ic}^2 + W_{ic}^2\right)\dfrac{V_{1\infty}^2}{c_\infty^2}\right)}{\left(1 + U_{ic}\dfrac{V_{1\infty}}{c_\infty^2}\right)^2} \tag{52}$$

Substitute squared-speed $\left|\mathbf{u}_{ic}\right|^2$ from (52) into the formula $1 - \left|\mathbf{u}_{ic}\right|^2\Big/c_\infty^2$ and take the square root of the resultant expression, such that for incompressible flow:

$$\sqrt{1 - \frac{\left|\mathbf{u}_{ic}\right|^2}{c_\infty^2}} = \beta_{ic}\sqrt{1 - \frac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1} \tag{53}$$

The derivation of the velocity addition formulas will not be given here for the reverse transformation of going from a *FTV* to a *FIS* reference frame in an incompressible flow system. The final solution is given by the matrix on *line g* of Table 4. In addition, the squared-speed $\left|\mathbf{U}_{ic}\right|^2$ can be derived with the help of (51) as follows:

$$\left|\mathbf{U}_{ic}\right|^2 = \frac{\left(u_{ic}^2 - 2u_{ic}V_{1\infty} + V_{1\infty}^2 + v_{ic}^2 + w_{ic}^2 - \left(v_{ic}^2 + w_{ic}^2\right)\dfrac{V_{1\infty}^2}{c_\infty^2}\right)}{\left(1 - u_{ic}\dfrac{V_{1\infty}}{c_\infty^2}\right)^2} \tag{54}$$

The square root of the formula for $1 - \left|\mathbf{U}_{ic}\right|^2\Big/c_\infty^2$ is given by:

$$\sqrt{1 - \frac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}} = \beta\sqrt{1 - \frac{\left|\mathbf{u}_{ic}\right|^2}{c_\infty^2}}\left(1 - u_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1} \tag{55}$$

Rearrange (53) and (55) for the Prandtl-Glauert factor β and multiply the resultant expressions together, such that:

$$\beta^2 = \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)\left(1 - u_{ic}\frac{V_{1\infty}}{c_\infty^2}\right) \tag{56}$$

3.5. Summary of Subsonic Velocity Transforms

This section summarizes in Table 4 all twelve of the velocity relationships developed from the twelve coordinate transformations listed in Table 3 for subsonic speeds.

Table 4. Summary of subsonic velocity transformations when using *FTV* versus *FIS* reference frames and compressible versus incompressible flow systems

Left side of The equal sign		Formula	Right side of the equal sign
a	*IV*	$\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix} = \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1}\left(\begin{pmatrix} V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}\cdot\begin{pmatrix} U_{ic} \\ V_{ic} \\ W_{ic} \end{pmatrix}\right)$	*IS*
b	*IV*	$\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix} = \left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1}\left(\begin{pmatrix} V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}\cdot\begin{pmatrix} U_{c} \\ V_{c} \\ W_{c} \end{pmatrix}\right)$	*CS*
c	*IV*	$\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix} = \left(\beta^2 + u_c\frac{V_{1\infty}}{c_\infty^2}\right)^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}\cdot\begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix}$	*CV*
d	*CV*	$\begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix} = \begin{pmatrix} V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} U_{c} \\ V_{c} \\ W_{c} \end{pmatrix}$	*CS*
e	*CV*	$\begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix} = \begin{pmatrix} V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} U_{ic} \\ V_{ic} \\ W_{ic} \end{pmatrix}$	*IS*
f	*CV*	$\begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix} = \beta\left(1 - u_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1}\begin{pmatrix} \beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix}$	*IV*
g	*IS*	$\begin{pmatrix} U_{ic} \\ V_{ic} \\ W_{ic} \end{pmatrix} = \left(1 - u_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1}\left(\begin{pmatrix} -V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}\cdot\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix}\right)$	*IV*
h	*IS*	$\begin{pmatrix} U_{ic} \\ V_{ic} \\ W_{ic} \end{pmatrix} = \begin{pmatrix} U_{c} \\ V_{c} \\ W_{c} \end{pmatrix}$	*CS*
i	*IS*	$\begin{pmatrix} U_{ic} \\ V_{ic} \\ W_{ic} \end{pmatrix} = \begin{pmatrix} -V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix}$	*CV*
j	*CS*	$\begin{pmatrix} U_{c} \\ V_{c} \\ W_{c} \end{pmatrix} = \begin{pmatrix} -V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_{c} \\ v_{c} \\ w_{c} \end{pmatrix}$	*CV*

Left side of The equal sign		Formula	Right side of the equal sign
k	*CS*	$\begin{pmatrix} U_c \\ V_c \\ W_c \end{pmatrix} = \begin{pmatrix} -V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_c \\ v_c \\ w_c \end{pmatrix}$	*IS*
l	*CS*	$\begin{pmatrix} U_c \\ V_c \\ W_c \end{pmatrix} = \left(1 - u_{ic}\dfrac{V_{1\infty}}{c_\infty^2}\right)^{-1}\left(\begin{pmatrix} -V_{1\infty} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}\cdot\begin{pmatrix} u_{ic} \\ v_{ic} \\ w_{ic} \end{pmatrix}\right)$	*IV*

The subsonic velocity-addition formulas are shown to be entirely based on the characteristics of the partial differential equations listed in Table 2 using linearized expressions derived from classical fluid dynamic principals, the Navier-Stokes equation for compressible flow, and Galilean, Miles, and Lorentz transformations. The formulas (46) to (48) and the matrix on *line a* in Table 4 are also identical to those developed for special relativity using clocks, rods, the Lorentz transformation, and first-order expansions [22]. However, it was also pointed out that these velocity-addition formulas are only valid when used with fictitious fluid systems of incompressible flow where one embeds the compressibility effects directly into the definition of the coordinates when using the Lorentz transformation.

Even though the Lorentz transformation is almost unanimously associated today with special relativity, it can be derived, as shown above, entirely with classical hydrodynamic principals for compressible fluids.

3.6. Transforming Velocities from *FIS* to *FTV* Reference Frames

This section describes how the perturbed acceleration components developed in one reference frame can be related to those in another using the various transformations introduced in the previous sections. Let there be two different reference frames, called the K and K' systems. Assume the K' system moves relative to the K system with constant velocity $\mathbf{V}_\infty$ in a direction that is parallel to the X-axis of both systems. Define the X-axis acceleration component $a_u = du/dt$ as the particle acceleration for the K system and the X-axis acceleration component $a'_u = du'/dt'$ of the same particle in the K' system, such that in general:

$$\left.\begin{array}{rcl} a_u &=& du/dt \\ a_v &=& dv/dt \\ a_w &=& dw/dt \end{array}\right\} \tag{57}$$

$$\left.\begin{array}{rcl} a'_u &=& du'/dt \\ a'_v &=& dv'/dt \\ a'_w &=& dw'/dt \end{array}\right\} \tag{58}$$

3.6.1. Transforming Accelerations from FIS to FTV Reference Frames

Make an incremental change in the velocity terms $\mathbf{u}_{ic}$ and $\mathbf{U}_{ic}$ in the matrix of *line a* in Table 4, keeping the free-stream velocity $\mathbf{V}_\infty$ and speed c_∞ constant, such that:

$$du_{ic} = \beta^2\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} dU_{ic} \tag{59}$$

$$\begin{aligned} dv_{ic} &= \beta\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1} dV_{ic} \\ &\quad - \beta\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} V_{ic}\frac{V_{1\infty}}{c_\infty^2}dU_{ic} \end{aligned} \tag{60}$$

$$\begin{aligned} dw_{ic} &= \beta\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1} dW_{ic} \\ &\quad - \beta\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} W_{ic}\frac{V_{1\infty}}{c_\infty^2}dU_{ic} \end{aligned} \tag{61}$$

Divide the three incremental velocity expressions in (59) on the left side of the equal sign by $dt_{ic}V_{1\infty}$ from the fourth line in (35) and on the right side of the equal sign by $dT_{ic}\left(M_\infty^2\, dX_{ic}/dT_{ic} + V_{1\infty}\right)/\beta$, such that for incompressible flow:

$$\frac{du_{ic}}{dt_{ic}} = \beta^3 \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3} \frac{dU_{ic}}{dT_{ic}} \tag{62}$$

$$\begin{aligned}\frac{dv_{ic}}{dt_{ic}} = {} & \beta^2 \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} \frac{dV_{ic}}{dT_{ic}} \\ & - \beta^2 \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3} V_{ic}\frac{V_{1\infty}}{c_\infty^2}\frac{dU_{ic}}{dT_{ic}}\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{dw_{ic}}{dt_{ic}} = {} & \beta^2 \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} \frac{dW_{ic}}{dT_{ic}} \\ & - \beta^2 \left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3} W_{ic}\frac{V_{1\infty}}{c_\infty^2}\frac{dU_{ic}}{dT_{ic}}\end{aligned} \tag{64}$$

Define the following perturbation flow acceleration components $\{a_{uic}, a_{vic}, a_{wic}\}$ of the *FTV* reference frame for an incompressible flow system, such that:

$$a_{uic} \equiv \frac{du_{ic}}{dt_{ic}} \tag{65}$$

$$a_{vic} \equiv \frac{dv_{ic}}{dt_{ic}} \tag{66}$$

$$a_{wic} \equiv \frac{dw_{ic}}{dt_{ic}} \tag{67}$$

Define the following perturbation flow acceleration components $\{A_{uic}, A_{vic}, A_{wic}\}$ of the *FIS* reference frame for an incompressible flow system, such that:

$$A_{uic} \equiv \frac{dU_{ic}}{dT_{ic}} \tag{68}$$

$$A_{vic} \equiv \frac{dV_{ic}}{dT_{ic}} \tag{69}$$

$$A_{wic} \equiv \frac{dW_{ic}}{dT_{ic}} \tag{70}$$

Substitute the accelerations terms of (65) to (70) back into the expressions (62) through (64). The final expressions are listed on *line a* of Table 5 [23].

Table 5. Summary of subsonic acceleration transformations as a function of using fixed-to-vehicle versus fixed-in-space reference frames and compressible versus incompressible flow systems

Left side of the equal sign		Formula	Right side of the equal sign
a	*IV*	$a_{uic} = \beta^3 A_{uic}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$ $a_{vic} = \beta^2 A_{vic}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} - \beta^2 V_{ic} A_{uic}\frac{V_{1\infty}}{c_\infty^2}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$ $a_{wic} = \beta^2 A_{wic}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} - \beta^2 W_{ic} A_{uic}\frac{V_{1\infty}}{c_\infty^2}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$	*IS*
b	*IV*	$a_{uic} = \beta^3 A_{uc}\left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$ $a_{vic} = \beta^2 A_{vc}\left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} - \beta^2 A_{uc} V_{c}\frac{V_{1\infty}}{c_\infty^2}\left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$ $a_{wic} = \beta^2 A_{wc}\left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-2} - \beta^2 A_{uc} W_{c}\frac{V_{1\infty}}{c_\infty^2}\left(1 + U_{c}\frac{V_{1\infty}}{c_\infty^2}\right)^{-3}$	*CS*

c	*IV*	$a_{uic} = \beta^2 a_{uc} \left(\beta^2 + u_c \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $a_{vic} = \beta^2 a_{vc} \left(\beta^2 + u_c \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} - \beta^2 a_{uc} v_c \frac{V_{1\infty}}{c_\infty^2} \left(\beta^2 + u_c \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $a_{wic} = \beta^2 a_{wc} \left(\beta^2 + u_c \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} - \beta^2 a_{uc} w_c \frac{V_{1\infty}}{c_\infty^2} \left(\beta^2 + u_c \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$	*CV*
d	*CV*	$\mathbf{a}_c = \mathbf{A}_c$	*CS*
e	*CV*	$\mathbf{a}_c = \mathbf{A}_{ic}$	*IS*
f	*CV*	$a_{uc} = \beta^3 a_{uic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $a_{vc} = \beta^2 a_{vic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} v_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $a_{wc} = \beta^2 a_{vic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} w_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$	*IV*
g	*IS*	$A_{uic} = \beta^3 a_{uic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $A_{vic} = \beta^2 a_{vic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} v_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $A_{wic} = \beta^2 a_{wic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} w_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$	*IV*
h	*IS*	$\mathbf{A}_{ic} = \mathbf{A}_c$	*CS*
i	*IS*	$\mathbf{A}_{ic} = \mathbf{a}_c$	*CV*
j	*CS*	$\mathbf{A}_c = \mathbf{a}_c$	*CV*
k	*CS*	$\mathbf{A}_c = \mathbf{A}_{ic}$	*IS*
l	*CS*	$A_{uc} = \beta^3 a_{uic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $A_{vc} = \beta^2 a_{vic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} v_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$ $A_{wc} = \beta^2 a_{vic} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-2} + \beta^2 a_{uic} w_{ic} \frac{V_{1\infty}}{c_\infty^2} \left(1 - u_{ic} \frac{V_{1\infty}}{c_\infty^2} \right)^{-3}$	*IV*

3.7. Transforms of Momentum and Fluid Mass between Reference Frames

This section describes how the momentum and fluid mass components developed in one reference frame can be related to those in another using the various transformations previously described. Linear momentum vector $\mathbf{P}$ of an infinitesimal volume of fluid is defined as the mass m of the infinitesimal fluid volume times the fluid perturbation velocity vector $\mathbf{U}$, such that $\mathbf{P} = m\,\mathbf{U}$.

3.7.1. Vectors Defined for Incompressible Flow Systems

Define vector set $\{\mathrm{P}_{ic1}, \mathrm{P}_{ic2}, \mathrm{P}_{ic3}\}$ to represent the momentum components when evaluated in a *FIS* reference frame for incompressible flow conditions, such that $\mathbf{P}_{ic} = m_{i\mathrm{s}}\,\mathbf{U}_{ic}$. Term $m_{i\mathrm{s}}$ represents the fluid mass of an infinitesimal volume of fluid when evaluated with a *FIS* reference frame for incompressible flow conditions. Define vector set $\{\mathrm{p}_{ic1}, \mathrm{p}_{ic2}, \mathrm{p}_{ic3}\}$ to represent the momentum components evaluated with a *FTV* reference frame for incompressible flow conditions, such that $\mathbf{p}_{ic} = m_{i\mathrm{v}}\,\mathbf{u}_{ic}$. Term $m_{i\mathrm{v}}$ represents the fluid mass of an infinitesimal volume of fluid when evaluated with a *FTV* reference frame for incompressible flow conditions.

3.7.2. Vectors Defined for Compressible Flow Systems

In a similar manner, define vector set $\{\mathrm{P}_{c1}, \mathrm{P}_{c2}, \mathrm{P}_{c3}\}$ to represent the momentum components when evaluated with a *FIS* reference frame for compressible flow conditions, such that $\mathbf{P}_{c} = m_{c\mathrm{s}}\,\mathbf{U}_{c}$. Term $m_{c\mathrm{s}}$ represents the fluid mass of an infinitesimal volume of fluid when evaluated with a *FIS* reference frame for compressible flow conditions. Let vector set $\{\mathrm{p}_{c1}, \mathrm{p}_{c2}, \mathrm{p}_{c3}\}$ represent the momentum components evaluated with a *FTV* reference frame for compressible flow conditions, such that $\mathbf{p}_{c} = m_{c\mathrm{v}}\,\mathbf{u}_{c}$. Term $m_{c\mathrm{v}}$ represents the fluid mass of an infinitesimal volume of fluid when evaluated with a *FTV* reference frame for compressible flow conditions.

3.7.3. Coordinate System and Reference Frame Transforms

Assume the momentum terms for incompressible flow in the *FIS* reference frame are related by a transformation to the incompressible flow system in a *FTV* reference frame in the following manner when the free-stream flow (with speed $V_{1\infty}$) is parallel to the X-axis:

$$\left.\begin{aligned} \mathrm{p}_{ic1} &= E_{ic}\,\mathrm{P}_{ic1} + F_{ic}\,m_{i\mathrm{s}}V_{1\infty} \\ \mathrm{p}_{ic2} &= \mathrm{P}_{ic2} \\ \mathrm{p}_{ic3} &= \mathrm{P}_{ic3} \\ m_{i\mathrm{v}}V_{1\infty} &= G_{ic}\,\mathrm{P}_{ic1} + H_{ic}\,m_{i\mathrm{s}}V_{1\infty} \end{aligned}\right\} \tag{71}$$

The four coefficients $\{E_{ic}, F_{ic}, G_{ic}, H_{ic}\}$ in (71) are unknown constants. The Jacobian determinant of the matrix in (71) must equal a value of plus one, such that:

$$Det\begin{bmatrix} E_{ic} & F_{ic} \\ G_{ic} & H_{ic} \end{bmatrix} \equiv +1 \tag{72}$$

Upon examination of the momentum terms in (71) and (72), it is clear that they resemble the same expressions used in deriving coordinate reference frame transformations. A reasonable guess for the unknown coefficients based on an incompressible flow system with *FTV* and *FIS* reference frames is given by the matrix on *line a* in Table 3 for just the X and T coordinates:

$$\begin{pmatrix} \mathrm{p}_{ic1} \\ \mathrm{p}_{ic1\infty} \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1 & 1 \\ M_\infty^2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{P}_{ic1} \\ \mathrm{P}_{ic1\infty} \end{pmatrix} \tag{73}$$

The above guess will be shown in (81) that the matrix used in (73) is in fact the correct one.

Replace the momentum and Mach terms in the second line of (73) with their mass and velocity components, such that:

$$m_{i\mathrm{v}}V_{1\infty} = \frac{1}{\beta}\frac{V_{1\infty}^2}{c_\infty^2} m_{i\mathrm{s}}U_{ic} + \frac{1}{\beta} m_{i\mathrm{s}}V_{1\infty} \tag{74}$$

Divide (74) by the free-stream speed $V_{1\infty}$ and rearrange terms to solve for the fluid mass $m_{i\mathrm{v}}$ in an incompressible flow system and a *FTV* reference frame, such that:

$$m_{i\mathrm{v}} = \frac{1}{\beta} m_{i\mathrm{s}}\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right) \tag{75}$$

Substitute term $\left(1 + U_{ic}V_{1\infty}/c_\infty^2\right)/\beta$ from (53) into (75) and rearrange terms, such that:

$$m_{i\mathrm{v}}\sqrt{1 - \frac{\left|\mathbf{u}_{i\mathrm{v}}\right|^2}{c_\infty^2}} = m_{i\mathrm{s}}\sqrt{1 - \frac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}} = m_0 \tag{76}$$

Term m_0 is called the *rest mass*. It represents the fluid's mass when the fluid at a particular point is brought to zero-speed conditions in the incompressible flow system. Hence, a relationship can be developed for the fluid mass in either the *FTV* or *FIS* reference frames of incompressible flow by rearranging (76), such that:

$$m_{i\mathrm{v}} = m_0 \Bigg/ \sqrt{1 - \frac{\left|\mathbf{u}_{ic}\right|^2}{c_\infty^2}} \tag{77}$$

$$m_{i\mathrm{s}} = m_0 \Bigg/ \sqrt{1 - \frac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}} \tag{78}$$

The above two equations (77) and (78) for fluid mass for incompressible flow systems restricted to subsonic flow conditions are identical in form to the expression for relativistic mass [24, 25]. The denominator terms are called the *Lorentz factor*. It should be obvious that formulas (77) and (78) predict fluid mass in an incompressible flow system to approach infinity as the magnitude of either of the *FTV* velocity $\left|\mathbf{u}_{ic}\right|$ or *FIS* velocity $\left|\mathbf{U}_{ic}\right|$ perturbation velocity vectors approach the characteristic speed c_∞. Clearly predicting an infinite increase in mass as the free-stream speed increases is not physically meaningful since the linearized expressions used in the derivation for potential flow are no longer valid. Modern day jet airplanes are quite capable of flying at both transonic and supersonic speeds without gaining an infinite mass.

The momentum vector $\mathbf{p}_{ic}$ in the *FTV* reference frame for an incompressible flow system can now be expressed in terms of the rest mass m_0 by substituting (77) for mass $m_{i\text{v}}$ back into the definition (69), such that:

$$\mathbf{p}_{ic} \quad = \quad \mathbf{u}_{ic}\, m_0 \Big/ \sqrt{1 - \frac{\left|\mathbf{u}_{ic}\right|^2}{c_\infty^2}} \tag{79}$$

In a similar manner, the momentum vector $\mathbf{P}_{ic}$ in the *FIS* reference frame for an incompressible flow system can now be expressed in terms of the rest mass m_0 by substituting (78) for mass $m_{i\text{s}}$ back into the definition (70), such that:

$$\mathbf{P}_{ic} \quad = \quad \mathbf{U}_{ic}\, m_0 \Big/ \sqrt{1 - \frac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}} \tag{80}$$

Equation (80) is in an identical form to that presented by [26].

3.7.4. Verifying Fluid Mass Formulation

The approach presented in the previous section that derived the fluid mass for an incompressible flow system will now be verified. Write out the first line of either (71) or (73) for momentum, such that:

$$m_{i\text{v}}\, u_{ic} \quad = \quad \frac{1}{\beta} m_{i\text{s}}\, U_{ic} \; + \; \frac{1}{\beta} m_{i\text{s}}\, V_{1\infty} \tag{81}$$

Replace the mass term $m_{i\text{v}}$ in (81) with (77) and mass term $m_{i\text{s}}$ with (78), such that:

$$\frac{u_{ic}\, m_0}{\sqrt{1 - \dfrac{\left|\mathbf{u}_{ic}\right|^2}{c_\infty^2}}} \quad = \quad \frac{1}{\beta} \frac{(U_{ic} + V_{1\infty})\, m_0}{\sqrt{1 - \dfrac{\left|\mathbf{U}_{ic}\right|^2}{c_\infty^2}}} \tag{82}$$

Multiply (82) by the square root term in the denominator on the left side of the equal sign in (82), replace the resultant ratio of square roots using (53), and rearrange terms, such that:

$$u_{ic} \quad = \quad \left(U_{ic} + V_{1\infty}\right)\left(1 + U_{ic}\frac{V_{1\infty}}{c_\infty^2}\right)^{-1} \tag{83}$$

Formula (83) for the u_{ic} velocity component for an incompressible flow system with a *FTV* coordinate frame matches exactly that given by the $IS \rightarrow IV$ transformation matrix listed on *line a* of Table 4.

3.7.5. Summary of Subsonic Fluid Mass Transformations

A summary for the twelve reference frame formulas used for fluid mass transformations is given in Table 6.

4. Summary

The paper starts by reviewing the wide spread use of Prandtl-Glauert correction factors by aeronautical engineers before World War II. Correction factors were needed to account for the effects of compressibility as a function of subsonic air speeds when using theoretical formulas based on an incompressible flow theory. This approach fell out of favor with the introduction of jet engines that could propel aircraft from subsonic to supersonic speeds. Engineers now use computational fluid dynamic algorithms to solve the complete Navier-Stokes equations that explicitly include all of the compressibility effects.

A more detailed examination of the mathematics behind the origin of the correction factors reveals a profound analogy to the equations of special relativity. To see this connection, a derivation of the classical three dimensional, convected wave equation is given in Cartesian coordinates representing the disturbance produced by the subsonic movement of a slender, solid object through still air with velocity $\mathbf{V}_\infty$. The velocity potential of the disturbed air is solved under the assumptions of compressible flow conditions and a fixed-to-vehicle (*FTV*) coordinate reference frame, abbreviated as reference frame *CV*. Atmospheric air is assumed to have the properties of irrotational, inviscid, barotropic, isentropic flow conditions; all external forces such as gravity are negligible; and the characteristic speed c_∞ of free-stream air is constant. In order to simplify the expressions, the vehicle velocity $\mathbf{V}_\infty$ is aligned along the X-axis of the coordinate system. Four partial differential equations for the transient wave equation of perturbed velocity potential are presented to represent alternative reference frames labeled as *CV* (compressible-*FTV*), *CS* (compressible-*FIS*), *IV* (incompressible-*FTV*), and *IS* (incompressible-*FIS*). Only the X space and T time coordinate components of the partial differential equations vary between the different reference frames. A brute force algorithm is described that searches for the coefficients of the 2x2 linear transformation matrices with a unit Jacobian determinant to convert the partial differential equation in reference frame *CV* to those of frames *CS*, *IV*, and *IS*. There are a total of twelve matrices needed to describe both forward and reverse transformations between the four reference frames. After trying approximately one million combination of terms, only the coefficients for three unique transformation matrices are found: the unit, inverse Galilean, and Miles matrices. The Lorentz matrix is obtained by multiplying the Galilean and Miles matrices together. Incremental differences in all of the space and time terms in the coordinate transformation matrices are taken. Expressions for the twelve 3x3 velocity transformation matrices are found by dividing the dX coordinate equation in each matrix by the corresponding dT equation. Similar steps are used to derive expressions for twelve 3x3 acceleration transformation matrices, and finally twelve 1x1 fluid mass transformation matrices.

Every transformation matrix shown in Tables 4 to 6 has an inverse form for itself. If $\mathrm{N_i}$ represents the i$^{\mathrm{th}}$ matrix in one of these three tables, then the dot product of the matrices $\mathrm{N_1 \cdot N_7}$, $\mathrm{N_2 \cdot N_{12}}$, $\mathrm{N_3 \cdot N_6}$, $\mathrm{N_4 \cdot N_{10}}$, $\mathrm{N_5 \cdot N_9}$, and $\mathrm{N_8 \cdot N_{11}}$ equals the identity matrix. In addition, linking the dot products of all twelve matrices from a given table into a certain order, such as the sequence $\mathrm{N_8 \cdot N_{10} \cdot N_6 \cdot N_3 \cdot N_5 \cdot N_7 \cdot N_2 \cdot N_{11} \cdot N_9 \cdot N_4 \cdot N_{12} \cdot N_1}$ will produce a circular chain returning to the same coordinates that it started with.

Table 6. Summary of subsonic fluid mass transformations as a function of using *FTV* versus *FIS* reference frames and compressible versus incompressible flow systems

Left Side of the Equal Sign		Formula	Right Side of the Equal Sign
a	*IV*	$m_{i\mathrm{v}} = \dfrac{1}{\beta} m_{i\mathrm{s}} \left(1 + U_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*IS*
b	*IV*	$m_{i\mathrm{v}} = \dfrac{1}{\beta} m_{c\mathrm{s}} \left(1 + U_{c} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*CS*
c	*IV*	$m_{i\mathrm{v}} = \dfrac{1}{\beta} m_{c\mathrm{v}} \left(\beta^2 + u_{c} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*CV*
d	*CV*	$m_{c\mathrm{v}} = m_{c\mathrm{s}}$	*CS*
e	*CV*	$m_{c\mathrm{v}} = m_{i\mathrm{s}}$	*IS*
f	*CV*	$m_{c\mathrm{v}} = \dfrac{1}{\beta} m_{i\mathrm{v}} \left(1 - u_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*IV*
g	*IS*	$m_{i\mathrm{s}} = \dfrac{1}{\beta} m_{i\mathrm{v}} \left(1 - u_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*IV*
h	*IS*	$m_{i\mathrm{s}} = m_{c\mathrm{s}}$	*CS*
i	*IS*	$m_{i\mathrm{s}} = m_{c\mathrm{v}}$	*CV*
j	*CS*	$m_{c\mathrm{s}} = m_{c\mathrm{v}}$	*CV*
k	*CS*	$m_{c\mathrm{s}} = m_{i\mathrm{s}}$	*IS*
l	*CS*	$m_{c\mathrm{s}} = \dfrac{1}{\beta} m_{i\mathrm{v}} \left(1 - u_{ic} \dfrac{V_{1\infty}}{c_\infty^2} \right)$	*IV*

5. Discussion

It is worth the effort to compare the form of the convected wave equation describing compressible flow in (84) against the form of the wave equation for incompressible flow in (85) (i.e., the analogous equation used in electromagnetic theory):

$$\frac{\partial^2\varphi_c}{\partial x_c^2}(1-M_\infty^2)+\frac{\partial^2\varphi_c}{\partial y_c^2}+\frac{\partial^2\varphi_c}{\partial z_c^2}= M_\infty^2\frac{\partial^2\varphi_c}{\partial \tau_c^2}+2M_\infty^2\frac{\partial^2\varphi_c}{\partial x_c\partial \tau_c} \quad (84)$$

$$\frac{\partial^2\varphi_{ic}}{\partial x_{ic}^2}+\frac{\partial^2\varphi_{ic}}{\partial y_{ic}^2}+\frac{\partial^2\varphi_{ic}}{\partial z_{ic}^2}= M_\infty^2\frac{\partial^2\varphi_{ic}}{\partial \tau_{ic}^2} \quad (85)$$

Just based on visual aesthetics, virtually anyone would find (85) more appealing than (84). The extra coefficient $1-M_\infty^2$ and XT cross-derivative in (84) resemble a type of rococo styled mathematics. Surely (85) is the true form of the equation representing the simplest expression for electromagnetic wave propagation. Why would it ever need to be more complicated? However, only (84) accounts for the effect of fluid compressibility in an absolute coordinate system of space and time. Removing the extra coefficient and cross-derivative by means of a coordinate transformation renders (85) an expression for a fictitious fluid and for space and time coordinates that change in magnitude as a function of speed.

It took many years for the majority of the public and scientific community to accept both heavier-than-air flight and faster-than-sound flight before WWII. Acceptance was stymied by arguments put forward to society through the popular press that were often based on political or religious ideology and parsimonious logic (i.e., if God intended man to fly he would have given us wings).

As another example, the use and interpretation of coordinate transformations can be confounded by using distance and time measuring devices that are also affected by the compressibility of the media itself. If one used an acoustic timing device in an inflight vehicle connected to outside conditions, its time delay measurements would also be affected by changes in air compressibility as speed varied. It then follows that if one rejects the hypotheses of speed affecting air compressibility, then discrepancies in the time delay measurements from the acoustic device would falsely be interpreted as space and time coordinates being bent.

The derivations given herein demonstrate that the Lorentz factor is not *special* in the normal sense of the word. Rather, it is a compressibility correction factor arising from the consequence of ignoring compressibility in the formulation of the convected wave equation. When compressibility is ignored, then correction factors are needed to compensate for the mathematical artefacts arising from spatial contraction and temporal dilation. In addition, velocity, acceleration, and mass vary with speed and no longer obey simple addition rules. The story of subsonic compressible aerodynamics is presented here because some of the same theoretical and mathematical developments occurring in manned flight were occurring almost simultaneously in the field of physics. The difference between the evolution of theory in aerodynamics and physics is that the majority of aerodynamists have accepted the concept of air compressibility but the majority of physicists have rejected the concept of vacuo compressibility.

6. Conclusions

Twelve coordinate transformations, plus formulas for subsonic velocity, acceleration, and mass are derived entirely from a brute force iteration routine using the mathematical characteristics of the partial differential equations from classical fluid dynamics for compressible flow and its transformation to incompressible flow. The mathematical derivation and interpretation for the twelve sets of transforms are apparently new. In addition, it was shown that there was no need to introduce additional assumptions concerning the positioning and timing between clocks, chord lengths, and speeding planes.

Examination of the formulas for coordinate transformation, velocity addition, and mass equations for subsonic conditions shows the equations are mathematically identical to those used in special relative to describe the motion of an electromagnetic wave or a particle, with the speed of light in vacuum used in place of the speed of sound for air. The convected wave equation for compressible flow and the special relativity wave equation only match exactly if the special relativity equations are assumed to be based on vacuum conditions, where the vacuum is represented by an incompressible flow system with a fixed-in-space reference frame (IS).

The proper starting form for using the convected wave equation is the form which includes cross derivatives in time and space. Any mathematical transform that removes these cross derivatives will convert the resultant equation into a non-physical form that represents a fictitious fluid.

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