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Drawing Movements as an Outcome of the Principle of Least Action

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According to the two-thirds power law the cube of the speed of a drawing movement is proportional to the radius of curvature of the trajectory, and the coefficient of proportionality has the meaning of mechanical *power*. We derive this empirical law from the variational principle known in physics as the *principle of least action*. It states that if a movement between two points of a given path obeys the two-thirds law, then the amount of work required to execute a trajectory in a fixed time is minimal. In this strict sense one may say that among infinitely many ways to execute a given path, the central nervous system chooses the most economical. We show that the kinematic equations for all drawing movements are solutions of a certain differential equation with a single (time-variable) coefficient. We consider several special cases of drawing movements corresponding to simplest forms of this coefficient.

Drawing and handwriting involve complex movements executed as a chain of individually planned simple movements. Figure 1 provides a good example of such planning: a great artist in a signature-like movement drew a silhouette of a bird (Clark, 1993). It seems reasonable to think that an internal representation of this drawing in a form of an image or silhouette was available to the implementation stage prior to the inception of the movement.[1] The segmentation into units seems to occur at the cusps, points of inflection, or contour discontinuities, and within each segment the curvature preserves its sign.

It has been empirically established (Lacquaniti, Terzuolo, & Viviani, 1983; Viviani & Cenzato, 1985) that within each segment of execution a power-law

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[1] Here is how Picasso understands the process: "Drawing is no joke. There is something very serious and mysterious about the fact that one can represent a living human being with line alone and create not only his likeness but, in addition, an image of how he really is. That is the marvel!" (Clark, 1993).

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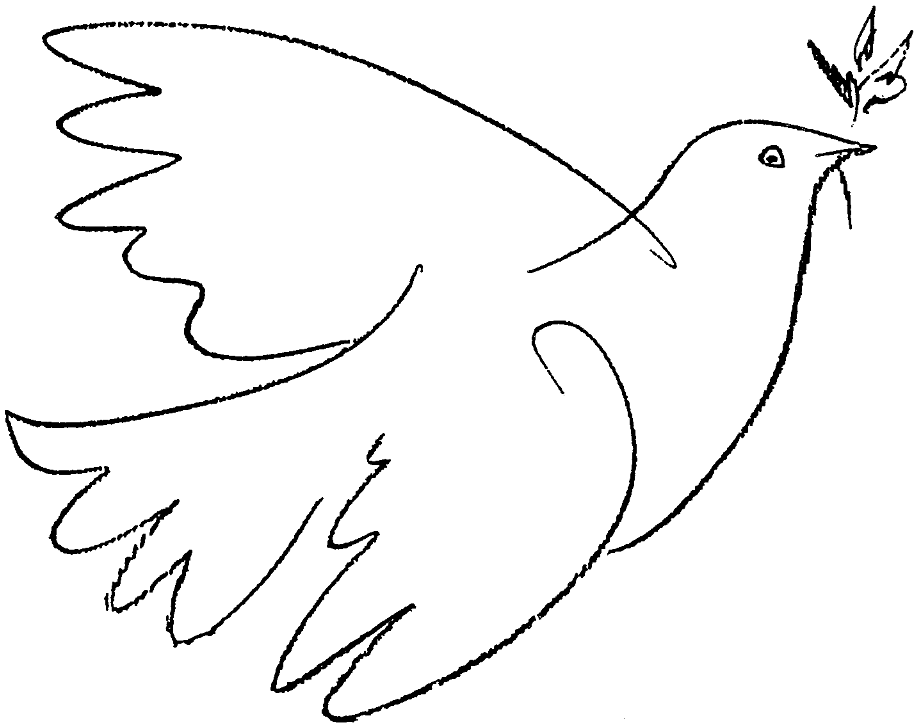

FIG. 1. Pablo Picasso, *Dove*, 1961, © 1999 Estate of Pablo Picasso/Artists Rights Society (ARS), New York. This drawing suggests that the entire silhouette can be decomposed into about 20 segments, and the segmentation seems to occur at the cuspids and the points of inflection.

relation exists between the angular velocity of a movement and the curvature of the trajectory

$$\Omega(t) = K \cdot C(t)^{2/3}, \tag{1}$$

where t denotes an instant in time, $\Omega(t)$ is the angular velocity, and $C(t)$ is the curvature. The coefficient of proportionality K is called the *gain factor* and remains approximately constant within each segment of execution (Viviani & Cenzato, 1985). This dependence has been known as the "two-thirds power law." This law is also known in a different but mathematically equivalent form (Viviani & Cenzato, 1985),

$$V(t) = K \cdot R(t)^{1/3}, \tag{2}$$

where $V(t)$ and $R(t)$ denote instantaneous values of the tangential velocity of the movement and the radius of curvature of the trajectory, correspondingly.

Although a given drawing trajectory can be traced in infinitely many different ways, the nervous system effectively reduces this excess of degrees of freedom (Viviani & Flash, 1995; Viviani & Schneider, 1991). In this article we derive the two-thirds power law from the principle of least action and show the constraints imposed on the parametric equations of drawing movements. These constraints are sufficient to characterize the control system that governs the execution of drawing movements and the reduction in the degrees of freedom that are available before the inception of such a movement.

1. TWO-THIRDS POWER LAW AND THE STRATEGY OF LEAST ACTION

Equation (2) can be rewritten in the form

$$\frac{V^3(t)}{R(t)} = P, \tag{3}$$

where $P = K^3 = \text{const}$. The left-hand side of (3) can be decomposed into

$$\frac{V^2(t)}{R(t)} \cdot V(t) = A_n(t) \cdot V(t), \tag{4}$$

where $A_n(t)$ is the magnitude of the *normal* or *centripetal acceleration*, that is, an acceleration directed toward the center of the curvature of the path. The centripetal acceleration characterizes the rate of change in the direction of the velocity vector (Yavorsky & Detlaf, 1982). Then (3) takes the form

$$A_n(t) \cdot V(t) = P, \tag{5}$$

which states that in drawing movements the product of the instantaneous values of centripetal acceleration and the tangential velocity remains constant during segment execution. This product is known in physics as mechanical *power*, and it characterizes the amount of mechanical work per unit of time necessary to keep movement along the trajectory.

The substitution into (3)

$$\begin{aligned} R(t) &= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}, \\ V &= \sqrt{\dot{x}^2 + \dot{y}^2}, \end{aligned} \tag{6}$$

where $x = x(t)$, $y = y(t)$ are the kinematic equations of movement, and $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$ denote their first and second derivatives with respect to time, gives

$$P = \dot{x}\ddot{y} - \dot{y}\ddot{x}, \tag{7}$$

which is the two-thirds power law in a differential form (see also Viviani & Cenzato (1985)). It is known that along a given path $y = y(x)$, $\dot{x}\ddot{y} - \dot{y}\ddot{x} = \dot{x}^3 y''_x$ (Bronshtein & Semendyayev, 1985). Thus (7) can be rewritten in the form

$$P = \dot{x}^3 y''_x, \tag{8}$$

where y''_x denotes the second derivative of y with respect to x.

The very fact that the mechanical power remains constant throughout the movement suggests that there exists a certain criterion from which the two-thirds power law can be derived. To find this criterion, consider the problem: along a given path $y=f(x)$, find a movement whose kinematic equations require the smallest amount of mechanical work necessary to accomplish the movement between two given points $M(x_0, y_0)$ and $N(x_1, y_1)$ of a curve in a fixed time $T=T_1-T_0$. Mathematically, this means that the kinematic equations $x=x(t)$ and $y=y(t)$ must minimize the integral

$$\int_{T_0}^{T_1} \dot{x}^3 y''_x \, dt. \tag{9}$$

In Appendix A we show that the necessary conditions for an extremum of (9) are relations (8), that is, $\dot{x}^3 y''_x = \text{const}$.

In the field of theoretical physics the time integral of the form (9) is known as the *action* of the system, and the assertion that the action on an actual path is minimal is called the *principle of least action* (Cohen, 1981; Kompaneyets, 1961; Landau & Lifshitz, 1969). In this strict sense we may say that in drawing movements the central nervous system follows the strategy of least action: among infinitely many ways to execute a given path, the system chooses the most economical. In Section 4 we illustrate this statement by considering three different ways of drawing a segment of a Lissajous figure, the movement with a constant velocity, the case when the x and y coordinates oscillate independently with different angular frequencies (Lissajous motion), and the movement governed by the strategy of least action.

2. DIFFERENTIAL FORM OF THE TWO-THIRDS POWER LAW AND ITS PROPERTIES

By the two-thirds power law, the mechanical power of a drawing movement remains constant during trajectory execution:

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = P = \text{const}. \tag{10}$$

To see what kind of constraints it imposes on the parametric equations of movement we differentiate it with respect to time,

$$\dot{x}\dddot{y} - \dot{y}\dddot{x} = 0 \tag{11}$$

or

$$\frac{\dddot{x}}{\dot{x}} = \frac{\dddot{y}}{\dot{y}}, \tag{12}$$

and this is another useful form of the two-thirds power law. It states that in drawing movements the ratios of the third to the first derivatives of the two kinematic equations must be equal. It follows immediately from (12) that the kinematic equations of movement, $x(t)$, $y(t)$, must simultaneously satisfy the equation

$$\dddot{u} + q(t)\,\dot{u} = 0, \tag{13}$$

that is, $\dddot{x} + q(t)\dot{x} = 0$ and $\dddot{y} + q(t)\,\dot{y} = 0$.

Equation (13) is a third-order linear ordinary differential equation. It is homogeneous (right-hand term equals zero), and its solutions depend solely on the coefficient $q(t)$. Thus, all drawing movements can be classified with respect to the form of this coefficient. In the next section we consider three simple special cases of $q(t)$.

It can be seen from (13) that $u = \text{const}$ is one of its solutions. Then its general solution is of the form

$$u = C_0 + C_1 w_1(t) + C_2 w_2(t), \tag{14}$$

where $w_1(t)$, $w_2(t)$ are any linearly independent particular solutions of (13), and C_0, C_1, C_2 are arbitrary constants. Without loss of generality we assume $C_0 = 0$. Then

$$x = C_{x1} w_1(t) + C_{x2} w_2(t)$$

$$y = C_{y1} w_1(t) + C_{y2} w_2(t).$$

3. CONSTANT-COEFFICIENT CASE: ELLIPSES, HYPERBOLAS, AND PARABOLAS

The simplest case of (13) is obtained by putting $q(t) = \text{const}$. If this constant is nonzero, then we have (see, e.g., Korn & Korn, 1968)

$$\text{Class 1:} \quad q = \omega^2 > 0$$

$$x = A_1 \sin(\omega t + \varphi_1) \tag{15}$$

$$y = A_2 \sin(\omega t + \varphi_2),$$

$$\text{Class 2:} \quad q = -\omega^2 < 0$$

$$x = A_1 e^{\omega t} + A_2 e^{-\omega t} \tag{16}$$

$$y = B_1 e^{\omega t} + B_2 e^{-\omega t}.$$

In the case when $q = 0$, direct integration leads to

$$\text{Class 3:} \quad x = A_0 t A_1 t + A_2 t^2$$

$$y = B_0 t B_1 t + B_2 t^2.$$

or, by assigning the initial positions to zero,

$$x = A_1 t + A_2 t^2$$
$$y = B_1 t + B_2 t^2. \tag{17}$$

The motion described by (15) is called an *elliptically polarized oscillation*; the term is due to the fact that the path-equation in the Cartesian coordinate system is an ellipse (Yavorsky & Detlaf, 1982). The path-equation of (16) is a *hyperbola*, and the path-equation of (17) is a *parabolic* path. Thus, whenever the coefficient q in (13) is a constant, the two-thirds law allows the motor planning only in terms of the second-order curves: ellipses, hyperbolas, parabolas, or their degenerated cases, e.g., straight lines.

The vast majority of experimental studies exploit the elliptic trajectory. In the study of drawing movements in children, Viviani & Schneider (1991) considered movements along an elliptic path with tangential velocity of the form $V(t) = K \cdot R(t)^{\beta}$, $K, \beta > 0$. They define this class of movements as *generalized Lissajous elliptic motion* (GLEM model). When $\beta = 1/3$ we have the two-thirds power law movement along an elliptic trajectory, that is, *mutually perpendicular harmonic oscillations of equal frequency*, or our Class 1. The two other classes have never been used.

4. LISSAJOUS MOTIONS DO NOT OBEY THE TWO-THIRDS POWER LAW

The class of *Lissajous motions* is defined as harmonic oscillations of the form

$$x = A_1 \sin(\omega_1 t + \varphi_1)$$
$$y = A_2 \sin(\omega_2 t + \varphi_2), \tag{18}$$

with commensurable angular frequencies

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2},$$

where n_1, n_2 are some integers (e.g., Symon, 1953; Yavorsky & Detlaf, 1982). The closed paths of such motions are called Lissajous figures. Figure 2 presents one such path with $\omega_2 = 2\omega_1$.

By taking first and third derivatives of Eq. (18) one can see that whenever $\omega_1 \neq \omega_2$, Eq. (12) does not hold. Thus, with the exception of simple harmonic oscillations of equal angular frequencies, Lissajous motions do not obey the two-thirds power law. In Appendix B we derive equations necessary to draw Lissajous figures obeying the two-thirds power law, and Fig. 2 provides an example.

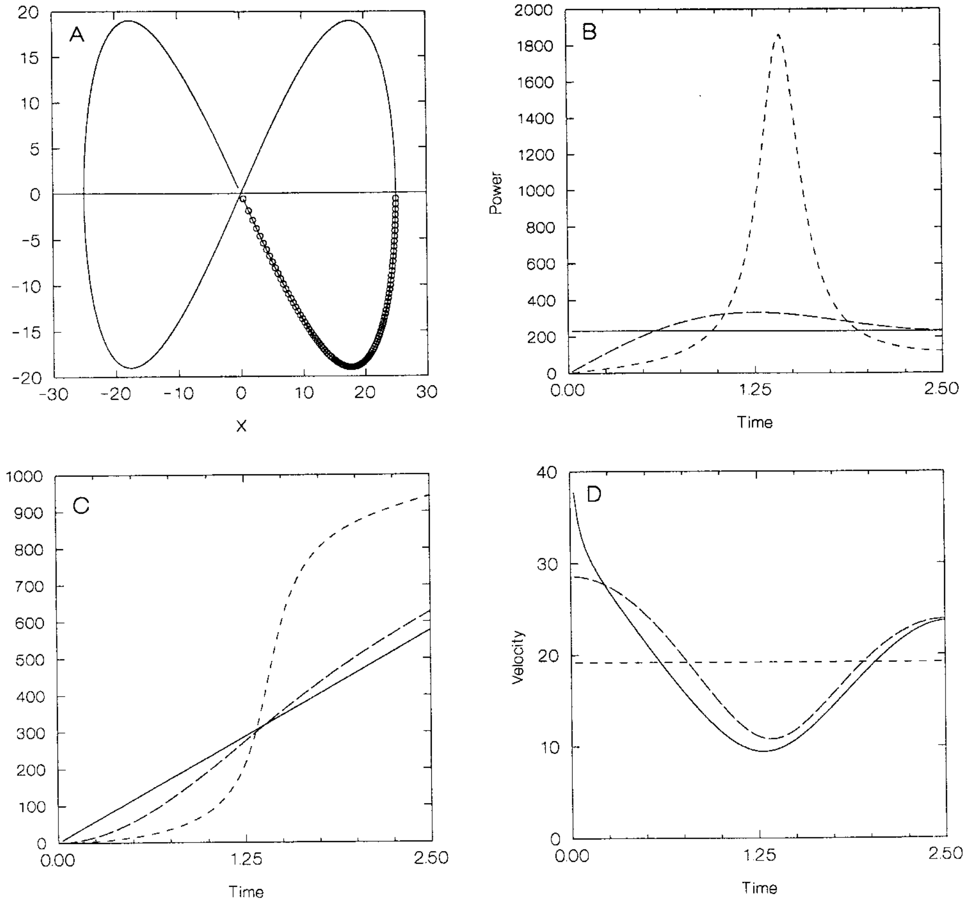

FIG. 2. (A) Lissajous figure: $x = 25\ \sin(0.2\pi t + 0)$, $y = 19\ \sin(0.4\pi t + 180)$. The figure consists of four identical, mirror-transformed segments. If a movement follows the above equations, it will have a period of 10 s. The data present counterclockwise movements along the lower right segment beginning from the center. All movements must be accomplished in 2.5 s. (B) The mechanical power of movements as a function of time for constant velocity movement (short-dashed line), Lissajous movement (long-dashed line), and the optimal, two-thirds power law movement (solid line). (C) Mechanical work as a function of time, for each of the movements. It can be seen that the integral assumes a minimum value for the two-thirds power law, and that the constant velocity strategy requires the greatest amount of mechanical work. (D) Velocities of the respective movements.

5. DISCUSSION

It is generally accepted in the fields of drawing motions, handwriting, and similar movements, that motor planning takes place prior to the inception of movement: a complex trajectory is planned as a chain of concatenated segments, and each segment is executed by the two-thirds power law (Massey, Lurito, Pellitzer, & Georgopoulos, 1992; Schwartz, 1994; Viviani & Flash, 1995; Wolpert, Ghahramany, & Jordan, 1995). One of the articles (Massey *et al.*, 1992) ends with this statement: "However, the neural basis which would explain why this relation follows a power law of relatively constant exponent remains to be explored."[2] The principle of least action is the most general principle to describe the behavior of any mechanical system (Landau & Lifshitz, 1969). In drawing movements we deal with a mechanical system of a special nature: it is planned and governed by the central nervous system (CNS). From the principle of least action it follows that the CNS does not impose the power law directly, but follows the strategy of accomplishing the desired goal in a preset time with the minimum mechanical work required.

In drawing elliptic templates, both accuracy and consistency of drawing increase with age (Viviani & Schneider, 1991). Rhesus monkeys must be trained properly, with rewards, before they are able to draw a spiral, but after successful training they perform the task as humans do, in compliance with the two-thirds power law (Schwartz, 1994). These facts suggest that the two-thirds power law is the result of the principle of least action acquired through a training process: among infinitely many possible ways to move along a given path, the CNS chooses the one which minimizes the amount of mechanical work needed to perform the desired task.

APPENDIX A: DERIVATION OF THE TWO-THIRDS LAW FROM THE PRINCIPLE OF LEAST ACTION

We show here that if the functional

$$I = \int_{T_0}^{T_1} \dot{x}^3 y''_x \, dt \tag{A1.1}$$

is minimal, then the *Lagrangian function* must be a constant, or

$$\dot{x}^3 y''_x = \text{const.} \tag{A1.2}$$

The *Euler–Lagrange equation* is

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0, \tag{A1.3}$$

where F_x and $F_{\dot{x}}$ are partial derivatives of the Lagrangian function in (A1.1) with respect to x and $\dot{x}$.

For the function $\dot{x}^3 y''_x$ we have

$$F_x = \dot{x}^3 y'''_x, \qquad F_{\dot{x}} = 3\dot{x}^2 y''_x, \qquad \text{and} \qquad \frac{d}{dt} F_{\dot{x}} = 6\dot{x}\ddot{x} y''_x + 3\dot{x}^3 y'''_x;$$

[2] A similar question could have been applied to Kepler's famous two-thirds power law (known as Kepler's third law of planetary motion), which states that the major semiaxes of elliptic orbits of the planets are proportional to the periods of revolution of the planets around the sun to a power of two-thirds. Nobody understood why the planets follow this law until the Universal Law of Gravitation was discovered, from which all Kepler's Laws follow with necessity (Motz & Duveen, 1977).

hence, (A1.3) is

$$3\dot{x}\ddot{x}y''_x + \dot{x}^3 y'''_x = 0, \tag{A1.4}$$

By multiplying both sides of (A1.4) by $\dot{x}$ one gets

$$3\dot{x}^2\ddot{x}y''_x + \dot{x}^4 y'''_x = 0, \tag{A1.5}$$

which is equivalent to

$$\frac{d}{dt}(\dot{x}^3 y''_x) = 0 \tag{A1.6}$$

or

$$\dot{x}^3 y''_x = \text{const}, \tag{A1.7}$$

which is the two-thirds power law.

APPENDIX B: DRAWING LISSAJOUS FIGURES WITH RESPECT TO THE TWO-THIRDS POWER LAW

Here we derive the differential equation for drawing the Lissajous contour in Fig. 2 in accordance with the two-thirds power law (solid lines), and we explain how these lines have been obtained. By the two-thirds power law we have

$$\dot{x}^3 y''_x = P = \text{const}. \tag{A2.1}$$

To obtain the expression for y''_x, we first eliminate time from the kinematic equations of Lissajous motion (Eq. (18)),

$$x = A_1 \sin(\omega_1 t + \varphi_1)$$
$$y = A_2 \sin(\omega_2 t + \varphi_2).$$

We get

$$y = A_2 \sin\left(\delta \arcsin \frac{x}{A_1} + \phi\right), \tag{A2.2}$$

where

$$\delta = \frac{n_2}{n_1}, \qquad \phi = \varphi_2 - \delta\varphi_1. \tag{A2.3}$$

Then, after differentiating (A2.2) twice with respect to x we get

$$y''_x = \frac{A_2 \delta x}{(A_1^2 - x^2)^{3/2}} \cos\left(\delta \arcsin \frac{x}{A_1} + \phi\right) - \frac{A_2 \delta^2}{A_1^2 - x^2} \sin\left(\delta \arcsin \frac{x}{A_1} + \phi\right). \tag{A2.4}$$

Equation (A2.1) can be rewritten in the form:

$$\dot{x} = \sqrt[3]{\frac{P}{y''_x}}. \tag{A2.5}$$

To produce Fig. 2 (solid lines) we integrated (A2.5) by the Runge–Kutta fourth-order method (Jeffrey, 1995).

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