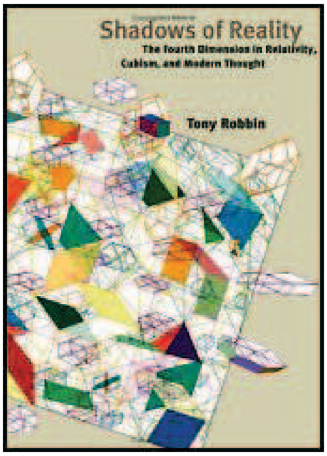
Book Review



Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought

Reviewed by Tony Phillips

Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought
Tony Robbin
Yale University Press, 2006
$40.00
160 pages, ISBN 978-0300110395

The fourth dimension? As La Rochefoucauld observed of true love, many people talk about it, but few have seen it. H. S. M. Coxeter, whose *Regular Polytopes* is the primary reference on four-dimensional constructions, says [Cox, ix]: "we can never fully comprehend them by direct observation." But he goes on: "In attempting to do so, however, we seem to peek through a chink in the wall of our physical limitations, into a new world of dazzling beauty." This is how the fourth dimension appeals to a geometer; for the popular imagination, fired by late nineteenth and early twentieth century science-fiction, theosophy and charlatanism, it became a fad with transcendental overtones. Just as UFOs are interpreted today as manifestations of a superior, alien civilization, so the fourth dimension gave room, lots of room, for the spirit world.

Tony Robbin tells this story in *Shadows of Reality: The Fourth Dimension in Relativity, Cubism, and Modern Thought*, along with an analysis of the impact of four-dimensional thought on modern art, and its importance in modern science.

An additional and more idiosyncratic theme is sounded by the invocation, in the book's title, of Plato's myth of the cave. The prisoners in the cave can see only shadows projected on the wall before them; and as Socrates explains to Glaucon: "to them the truth would be literally nothing but the shadows of the images." Thus our own truth may be only the projection of a more meaningful four-dimensional reality. But there is more. The concept itself of *projection* has enormous importance for Robbin; he systematically opposes, and prefers, projection (from four dimensions to three dimensions, and then usually to two to get things on a page) to the dual perspective on four-dimensional reality, three-dimensional *slicing*, which he characterizes as "the Flatland model". He is referring to Edwin A. Abbott's book *Flatland*, where four dimensions are explained by analogy with the Flatlanders' conception of the three-dimensional space we humans inhabit. "Consider this book a modest proposal to rid our thinking of the slicing model of four-dimensional figures and spacetime in favor of the projection model." I will come back to this point later.

The jacket blurb for *Shadows of Reality* describes the work as "a revisionist math history as well as a revisionist art history."

"Revisionist Art History"

Chapter 3, "The Fourth Dimension in Painting", contains an analysis of three well-known paintings by Picasso: *Les Demoiselles d'Avignon* (1907), the *Portrait of Ambroise Vollard*, and the *Portrait of Henry Kahnweiler* (both 1910). Robbin argues that "at one propitious moment a more serious and sophisticated engagement with the fourth dimension pushed [Picasso] and his collaborators into the discovery of cubism," and that that moment occurred between the painting of *Les Demoiselles* and that of the two portraits. The source of the geometry is pinned down: the *Traité élémentaire de géométrie à quatre dimensions* by Esprit Jouffret

Tony Phillips is professor of mathematics at Stony Brook University. His email address is tony@math.sunysb.edu.

(1903) [Jouf], and the bearer of this knowledge to Picasso is identified as Maurice Princet, a person with some mathematical training who was a member of the Picasso group.

This is indeed revisionist, because it claims a hitherto unrecognized mathematical influence on the most famous artist of the twentieth century.

The interaction between geometry and art at the beginning of the twentieth century has already been studied in encyclopedic detail by Linda Dalrymple Henderson [Hen]. When it comes to the fourth dimension and Picasso, Henderson is much more cautious: "Although Picasso has denied ever discussing mathematics or the fourth dimension with Princet, Princet was a member of the group around Picasso by at least the middle of 1907, and probably earlier. It thus seems highly unlikely that during the several years which followed, Picasso did not hear some talk of the fourth dimension. . . " But she presents eye-witness and contemporary testimony from Jean Metzinger, an artist member of the group, which states that the influence ran the other way: "Cézanne showed us forms living in the reality of light, Picasso brings us a material account of their real life in the mind—he lays out a free, mobile perspective, from which that ingenious mathematician Maurice Princet has deduced a whole geometry." [Metz]

Henderson gives solid documentation of the interest in mathematical ideas among the artists of "Picasso's circle", especially after they drifted away from him in 1911. And if images like Jouffret's figures were part of the visual ambience in the 1907–1910 period, they may well have caught Picasso's omnivorous eye. But for a specific mathematical influence on Picasso there is no evidence, and Robbin presents no reasonable argument beyond his own gut feeling: "Indeed, as a painter looking at the visual evidence I find that Picasso. . . clearly adopted Jouffret's methods in 1910." So we have to take his statement in the preface—"Pablo Picasso not only looked at the projections of four-dimensional cubes in a mathematics book when he invented cubism, he also read the text, embracing not just the images but the ideas"—as pure, unadvertised invention. Let us leave the last word on this topic to the artist himself: "Mathematics, trigonometry, chemistry, psychoanalysis, music and whatnot, have been related to cubism to give it an easier interpretation. All this has been pure literature, not to say nonsense, which brought bad results, blinding people with theories." [Pic]

"Revisionist Math History"

The agenda is stated in the preface. "Contrary to popular exposition, it is the projection model that revolutionized thought at the beginning of the twentieth century. The ideas developed as part of this projection metaphor continue to be the basis for the most advanced contemporary thought in mathematics and physics." The rest of the paragraph amplifies this assertion, naming projective geometry, Picasso (as cited above), Minkowski ("had the projection model in the back of his head when he used four-dimensional geometry to codify special relativity"), de Bruijn (his "projection algorithms for generating quasicrystals revolutionized the way mathematicians think about patterns and lattices"), Penrose ("showed that a light ray is more like a projected line than a regular line in space," with this insight leading to "the most provocative and profound restructuring of physics since the discoveries of Albert Einstein"), quantum information theory, and quantum foam. "Such new projection models present us with an understanding that cannot be reduced to a Flatland model without introducing hopeless paradox."

These items are fleshed out in the main text of the book, roughly one per chapter. I will look in detail at Chapter 4 ("The Truth"), which covers the geometry of relativity, and at Chapter 6, "Patterns, Crystals and Projections".

Relativity, and Time as the Fourth Dimension

For mathematicians, dimension is often just one of the parameters of the object under investigation. Their work may be set in arbitrary dimension ("in dimension *n*") or, say, in dimension 24, as in the work of Henry Cohn and Abhinav Kumar, who caused a stir in 2004 [Cohn] when they nailed down the closest regular packing of 24-dimensional balls in 24-dimensional space. So we really should be talking about "a fourth dimension", and not "the fourth dimension". For the general public, on the other hand, four dimensions are esoteric enough. They are usually thought of as the familiar three plus one more; hence "the fourth dimension". So where is this fourth dimension? Nineteenth-century work on 4-dimensional polytopes (reviewed in [Cox]) clearly posited an extra spatial dimension. But at the same time a tradition, going back to Laplace, considered *time* as "the fourth dimension". In fact partial differential equations relating space derivatives to time derivatives leave us no choice: their natural domain of definition is four-dimensional space-time.

Here there is some point to Robbin's anti-slicing strictures. Special relativity implies that space-time cannot be described as a stack of three-dimensional constant-time slices. More precisely, the most general Lorentz transformation, relating two overlapping (x, y, z, t) coordinate systems of which one may be traveling at constant speed with respect to the other, mixes the (x, y, z)s and the ts in such a way that the $\{t = \text{constant}\}$ slices are not preserved. But projections do not help. Here is Minkowski as quoted by Robbin: "We are compelled to admit that it is only in four dimensions that the relations here taken under consideration reveal their inner being in full simplicity, and that on a three-dimensional space forced on us *a priori* they cast only a very complicated projection." Minkowski is saying that projections from four dimensions to three are *not* very useful in understanding relativistic reality. Robbin gets the word "projection" back into play by describing Lorentz transformations as being "revealed when one geometric description of space is projected onto another." But the projections here are isomorphisms from one four-dimensional coordinate system to another; they are not the kind of projections, onto lower-dimensional spaces, that Robbin is contrasting to slicing.

Patterns, Crystals and Projections

Chapter 6 begins with a description of nonperiodic tilings and how they may be generated by "matching rules" and by repeated dissection and inflation. It then goes on to discuss the fascinating and illuminating relationship, discovered by Nicolaas de Bruijn [de B], between nonperiodic tilings in dimension n and projections of approximations of irrational slices of periodic tilings: cubical lattices in dimension $2n$ or $2n + 1$. The simplest example involves dimensions 1 and 2: take the (x, y)-plane with its usual tiling by unit squares; the vertices are at points (n, m) with integer coordinates. Draw the straight line $y = \varphi x$ with irrational slope $\varphi = 1.6180339\ldots$ (the "golden mean" number). Starting at $(0, 0)$, cover the line with the smallest possible subset of the tiling. The squares are shown here shaded.

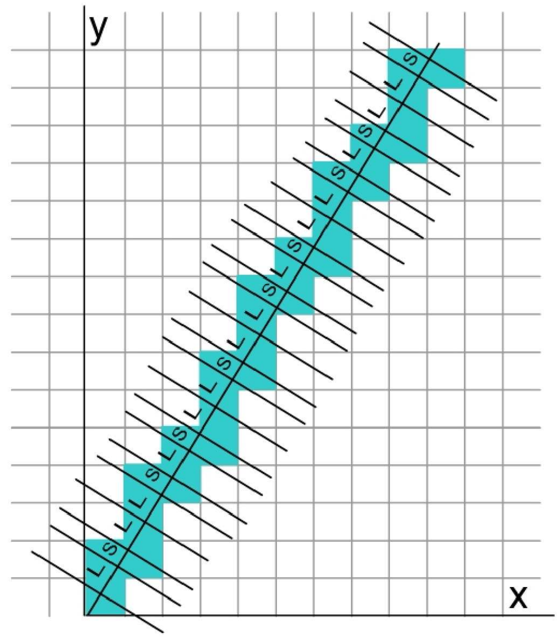


Figure 1.

Because the slope is irrational, the line does not pass through any lattice vertex other than $(0, 0)$, so the choice of covering tile is always unambiguous. Now project the center of each covering tile perpendicularly onto the irrational line. The projected points will define a tiling of the line by intervals: a long interval (L) if the two consecutive tiles were adjacent vertically, and a short one (S) if they were adjacent horizontally. A trigonometric calculation shows that the length ratio of S to L is φ. Our tiling by intervals is non-periodic because φ is irrational, but it has other amazing properties because φ is such a special number. For example, if we represent the short tile by S, and the long one by L, the tiling of the positive x-axis corresponds to a sequence of Ls and Ss. It turns out (the argument requires some linear algebra; see [web]) that the sequence may also be generated as follows: start with S, and copy the string over and over, each time rewriting every S as L, and every L as LS:

S
L
LS
LSL
LSLLS
LSLLSLSL
LSLLSLSLLSLLS
LSLLSLSLLSLLSLSLLSLSL
LSLLSLSLLSLLSLSLLSLSLLSLLSLSLLSLLS
...

Figure 2.

The sequence may also be generated like the Fibonacci numbers, starting with S and L and making each succeeding line the concatenation of its two predecessors. And there is more.

Instead of drawing a line with golden slope through the 2-dimensional lattice of unit squares, let us slice a plane (a golden plane—see below) through the analogous 5-dimensional lattice, cover the plane with the smallest possible subset of the 5-cubes of that lattice, and project the center points of those cubes orthogonally onto the plane. The projected points turn out to be the vertices of Penrose's non-periodic tiling [de B]. In this tiling, the plane is covered by copies of two rhombs (equilateral parallelograms), a fat one, with lesser angle $2\pi/5$, and a skinny one $\pi/5$, whose areas, just like the lengths of L and S, are in the golden ratio.

(We use the plane in 5-space spanned by the vectors

$$(1, -\cos(\pi/5), \cos(2\pi/5), \\ -\cos(3\pi/5), \cos(4\pi/5))$$

and

$$(0, -\sin(\pi/5), \sin(2\pi/5), \\ -\sin(3\pi/5), \sin(4\pi/5))$$

[web2]; equivalently, in terms of the golden mean, we may scale those vectors to

$$(2, -\varphi, \varphi-1, \varphi-1, -\varphi)$$

and

$$(0, -1, \varphi, -\varphi, 1).$$

A straightforward linear algebra computation shows that the 10 different unit squares in the 5-dimensional lattice project to the 10 possible orientations (5 each) of the 2 rhombs in the Penrose tiling.)

De Bruijn's construction of the Penrose tiling is intellectually satisfying. Instead of the arbitrary-sounding matching rules, which add one tile at a time, or the stepwise generation of tilings of finite portions of the plane by dissection and inflation, we have a single, intelligible definition tiling the entire plane. Furthermore the magical-seeming recurrence properties of the tiling are anchored back to the well-understood behavior of the fractional part of consecutive integral multiples of an irrational number, which has been studied since Kronecker. As an additional bonus, the construction also yields a new insight into the psychological phenomenon noted by Martin Kemp [Kem] in describing how a rhombic Penrose tiling affects the eye: "We can, for instance, play Necker cube-type games with apparent octagons, and facet the surface into a kind of cubist medley of receding and advancing planes." The surface in 5-space, which projects to the tiling, is made up of square facets lined up with the coordinate planes. Three by three these facets determine a cube, which projects correctly; but these cubes live in 5 dimensions, and they fit together in a manner inconceivable in 3-space.

Robbin puts the impact of de Bruijn's discoveries eloquently but inaccurately: "Once one accepts the counterintuitive notion that quasicrystals are projections of regular, periodic, cubic lattices from higher-dimensional space, all the other counterintuitive properties soon become clear in a wave of lucid understanding." The inaccuracy, which is fundamental, stems from Robbin's failure to recognize that these tilings are projections of *sections* of cubic lattices. At the end of the chapter the language becomes even more vivid, but the thesis is still fundamentally wrong-headed: "Quasicrystals show us that objects and systems described by projective geometry cannot be made Euclidean without introducing the most mystical and anthropomorphic properties into the system. Making what is generated or accurately modeled by projective geometry into a traditional Euclidean static model takes us away from science and moves us towards fetishism."

There are smaller-scale problems also, in Robbin's portrayal of Penrose's tilings and de Bruijn's construction. The main ideas are there, with correct references, but a naïve and attentive reader can only be bewildered by the presentation. We read things that are obviously just plain wrong ("The mechanics of the loom enforce a periodic repeat"), things that turn out to be wrong ("Originally, the Amman bars were equidistant"), and things that are misleading ("It was long thought to be impossible to have a tiling made up of pentagons") or unnecessarily obscure ("The dual net is a mathematical device that is often used to analyze pattern"—no further description). The figures only add to the confusion. Figure 6.7 is mislabelled, and here is Figure 6.8:

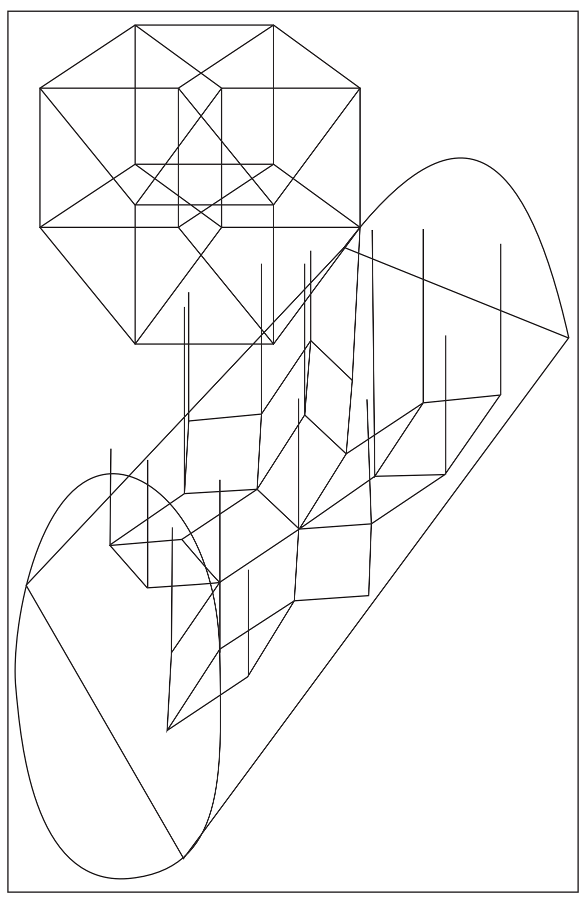


Figure 3. Robbin's Figure 6.8.

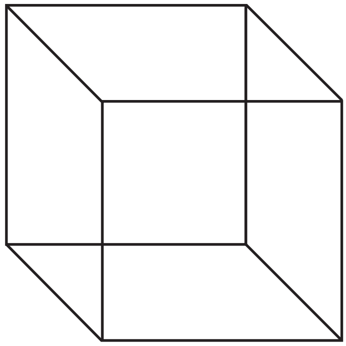


Figure 4. "Necker" cube: the eye imposes one of two three-dimensional interpretations of this figure as a cube, and oscillates between them.

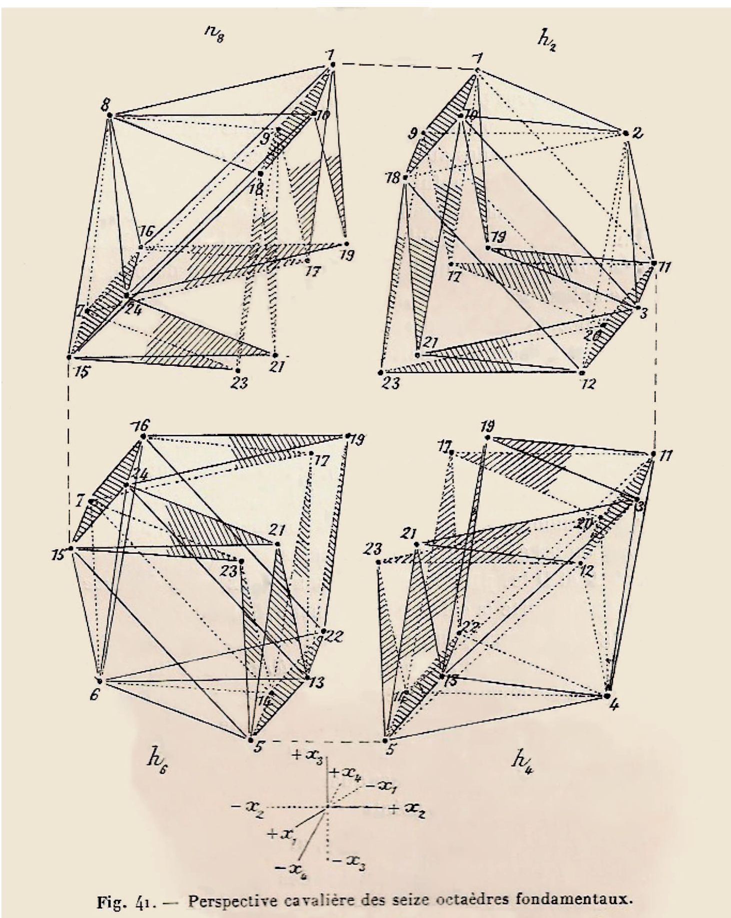


Figure 5. Jouffret's Figure 41, *perspective cavalière* of the 16 fundamental octahedra.

The caption ("The de Bruijn projection method is applied to a grid of hypercubes to produce a Penrose tiling. Only one hypercube is shown here.") does not mention the slicing plane, and the "projection method" is represented by a *four*-dimensional hypercube floating in the upper left-hand corner, along with vertical lines attached to the vertices of the Penrose tiling drawn in perspective in the center. I do not believe this figure is meant to be judged as a work of art; but I know that as a mathematical diagram, while it contains some suggestive truth, it is too impressionistic to be useful.

Conclusion

In sum, this is an attractive but very disappointing book. The topic has great potential, and Robbin's earlier work [Rob], both entertaining and enlightening, promised an idiosyncratic, artist's-eye look, with perhaps new insights, at these fascinating phenomena. In fact I am grateful to Robbin for introducing me to de Bruijn's work and for bringing to my attention Esprit Jouffret, a charming mathematical representative of the Belle Époque. But the book is seriously marred by the inadequacy and inaccuracy of its mathematical component: the author has been allowed to venture, alone, too far above the plane of his expertise. In his acknowledgments he singles out several "readers of the manuscript" and several Yale University Press staffers. He alone of course is responsible for the text and illustrations, but I am sure that some of those individuals wish now that they had done more to give this beautiful idea a more accurate and intelligible realization.

Appendix: Section, Projection, and *Perspective Cavalière*

Section and projection are mathematically dual operations. Any finite n-dimensional object can be completely reconstructed either from a 1-dimensional set of parallel $(n-1)$-dimensional slices or from a 1-dimensional set of $(n-1)$-dimensional projections, by stacking in one case or by a tomographic-style calculation in the other. The total information is the same, although packaged quite differently. When it comes to geometric objects, like regular polyhedra, the (innate or trained) ability of the human eye, which can resolve perspective problems almost automatically, means that a 2-dimensional projection can often be counted on to be "read" to give a mental image of its 3-dimensional antecedent. The "Necker cube" phenomenon referred to by Kemp is a familiar example of this human propensity.

But in fact neither of the natural interpretations is even plausible unless we *assume* that we are looking at the projection of a connected polyhedron. The configuration we see might be like one of the constellations in the night sky: our mind joining several completely unrelated objects.

Projection of more general objects is usually less informative. Knot theorists work comfortably with two-dimensional "projections" of three-dimensional knots, but these projections have been enhanced with clues: at each intersection a graphical convention gives information (from "the third dimension") telling which strand goes over and which goes under.

To explore the limitations of projection as a means for studying four-dimensional polyhedra, let us look at an image from Jouffret, reproduced both in Henderson's book and in Robbin's (Figure 5).

Jouffret's Figure 41 is part of his effort to explain the structure of the polytope C_{24}, which is a 3-dimensional object, without boundary (as such it can only exist in 4-dimensional space), made up of 24 octahedra. This polytope has 24 vertices, 96 edges, and 96 triangular faces. It is the most interesting of the 3-dimensional regular polytopes because it is the only one of the 6 that has no analogue among our familiar 2-dimensional polyhedra (in fact its group of symmetries is the Weyl group of the exceptional Lie group F_4 [Cox]). Jouffret works rigorously, in the style of the descriptive geometry *épures* required of every French candidate to the École Polytechnique or the École Normale Supérieure between 1858 and 1960 [Asa]: he constructs a C_{24} of edge-length a in 4-dimensional space as a polytope with vertices at the 24 points $(\pm\alpha, \pm\alpha, 0, 0)$, $(\pm\alpha, 0, \pm\alpha, 0)$, $(\pm\alpha, 0, 0, \pm\alpha)$, $(0, \pm\alpha, \pm\alpha, 0)$, $(0, \pm\alpha, 0, \pm\alpha,)$, $(0, 0, \pm\alpha, \pm\alpha)$, where $\alpha = a\frac{\sqrt{3}}{2}$. Since each vertex is connected to eight others, when you put in all the edges and project the polytope onto a page of the book you get something like one of the two pictures in Figure 6.

Which one you get depends on the projection you choose from 4-space to 2-space. These pictures are attractive, but they do not convey much information about how 24 octahedra fit together to form an object with no boundary. Putting in any indication of where the 96 2-dimensional faces fit in would certainly not improve legibility. Jouffret credits Schoute with the idea of a *perspective cavalière*: a cavalier bending of the rules in the interest of intelligibility. In this case there are two violations. First, sixteen of the octahedra are *selected* to represent the surface. The sixteen are geometrically *projected* as before. That projected image is *dissected* into four pieces, with common edges and faces drawn twice, and common vertices two or four times. The shading, which, as Henderson notes, gives the image a "shimmering quality of iridescence", marks the faces that are duplicated. (Note here two small errors: the face <9, 18, 21> in quadrant h_8 should be shifted to become <10, 18, 21>, and the face <17, 13, 14> in quadrant h_6 should be moved to <17, 22, 14>).

Before trying to understand how these pieces all fit together, and then where the missing eight octahedra should be inserted, we should note how far this picture is, even when restricted to one of the quadrants, from a straightforward projection. When two lines meet in this figure, a heavy dot tells us whether their intersection is essential (they actually meet in four dimensions) or contingent (two of their points happen to project to the same place). The lines themselves come with two different values: full and dotted, according to a somewhat arbitrary convention of Jouffret's regarding which lines are "seen" and which are "hidden".

Finally, even with all the extra information, and despite the accuracy of the *épure*, the picture is hard to read. There is a clue for modern readers: the rectangular parallelipipeds in each quadrant become a torus when the quadrants are slid back together. The inside and the outside of this torus are partitioned into 24 half-octahedra each by coning the boundary of each inside parallelipiped from an "interior" vertex (one of 2, 4, 6, or 8) and by coning the boundary of each outside block from its appropriate middle vertex (one of 17, 19, 21, 23). Each solid torus contains a necklace of 4 complete octahedra. The remaining *seize octaèdres fondamentaux* are formed when the two solid tori are glued along their boundaries to give a topological 3-sphere. I write "modern readers" because I do not believe that in 1903 this decomposition of the 3-sphere, even though it is almost unavoidable when S^3 is written as $\{|z_1|^2 + |z_2|^2 = 1\}$ in complex coordinates, was anywhere nearly as well known as it is today.

Jouffret remarks that the 16 vertices that span our torus are the vertices of a hypercube (*octaédroïde*), but he does not use this information to clarify the combinatorial structure of C_{24}. He also remarks that the other 8 vertices define a 16-cell (*hexadécadroïde*) of side $a\sqrt{2}$, and that this partition into 16 + 8 can be done in three symmetric ways.

References

[Asa] BORIS ASANCHEYEV, Épures de géométrie descriptive, *Concours d'entrée à l'Ecole Normale Supérieure*, Hermann, Paris, 2002,

[Cohn] HENRY COHN and ABHINAV KUMAR, The densest lattice in twenty-four dimensions, *Electronic Research Announcements of the AMS*, **10** (2004), 58–67, arXiv:math.MG/0408174.

[Cox] H. S. M. COXETER, *Regular Polytopes*, second edition, McMillan, New York, 1963.

[de B] NIKOLAAS G. DE BRUIJN, Algebraic theory of Penrose's non-periodic tilings of the plane, i, ii, *Nederl. Akad. Wetensch. Proc. Ser. A*, **84**, 38–52, 53–66, 1981.

[Hen] LINDA DALRYMPLE HENDERSON, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, Princeton, 1983.

[Jouf] ESPRIT JOUFFRET, *Traité élémentaire de géométrie à quatre dimensions*, Hermann, Paris, 1903.

[Kem] MARTIN KEMP, A trick of the tiles, *Nature* **436** (2005), 332–332.

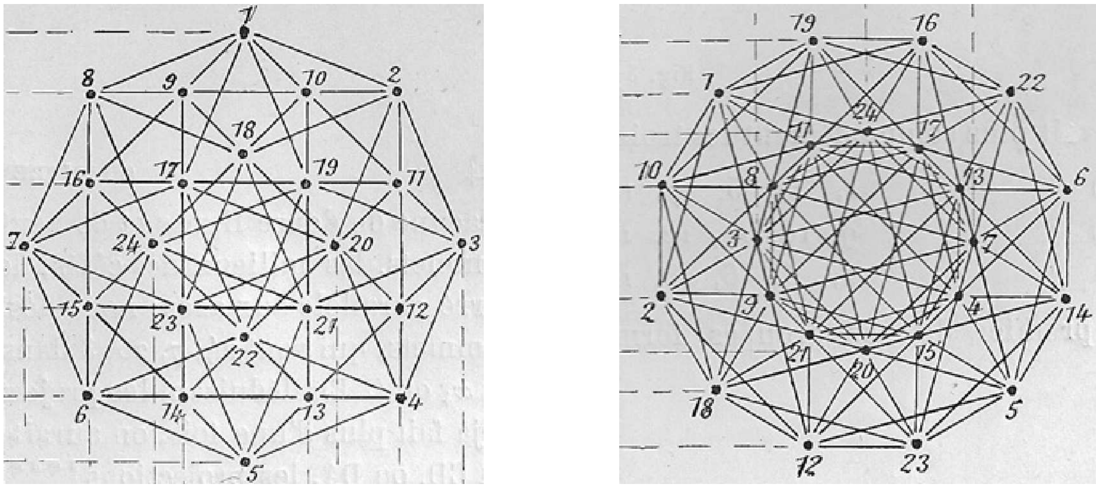


Figure 6. From Jouffret's Figure 36, two projections of the 24-cell.

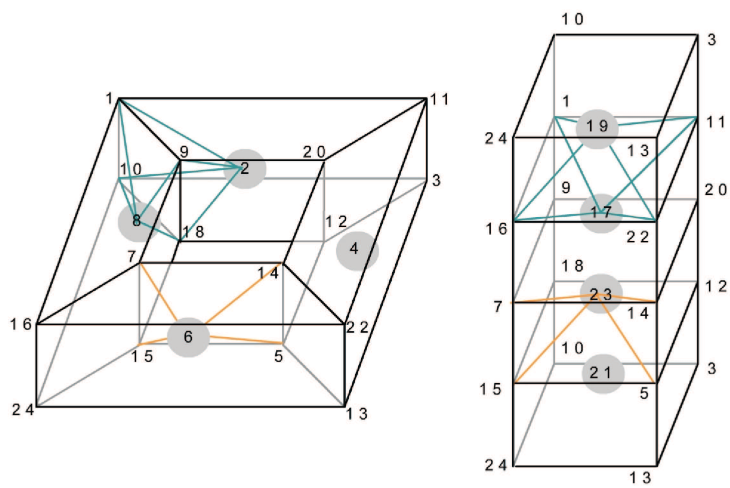


Figure 7. The two solid tori implicit in Jouffret's diagram, with their interior vertices. The extra edges for two of the "necklace" octahedra are shown in green; those for one of the *seize octaèdres fondamentaux* are shown in orange.

[Metz] Jean Metzinger, Notes sur la peinture, *Pan*, Oct.-Nov. 1910, translated and partially reprinted in Edward Fry, *Cubism*, McGraw-Hill, New York 1966. Quoted in [Hen] page 64. Here is the whole paragraph: "Picasso ne nie pas l'objet, il l'illumine avec son intelligence et son sentiment. Aux perceptions visuelles, il joint les perceptions tactiles. Il éprouve, comprend, organise: le tableau ne sera transposition ni schema, nous y contemplerons l'équivalent sensible et vivant d'une idée, l'image totale. Thèse, antithèse, synthèse, la vieille formule subit une énergique interversion dans la substance des deux premiers termes: Picasso s'avoue réaliste. Cézanne nous montra les formes vivre dans la réalité de la lumière, Picasso nous apporte un compte-rendu matériel de leur vie réelle dans l'esprit, il fonde une perspective libre, mobile, telle que le sagace mathématicien Maurice Princet en déduit toute une géométrie."

[Pic] Picasso speaks, *The Arts*, New York, May 1923. Reproduced in Alfred H. Barr Jr., *Picasso: Fifty Years of his Art*, Museum of Modern Art, New York 1946, reprint edition published for the museum by Arno Press, 1966. Also reprinted in Edward Fry, *Cubism*, McGraw-Hill, New York 1966.

[Rob] Tony Robbin, *Fourfield: Computers, Art & the 4th Dimension*, Little, Brown & Co., Boston, 1992.

[web] `http://www.math.okstate.edu/mathdept/dynamics/lecnotes/node29.html`, David J. Wright, Oklahoma State University.

[web2] `http://gregegan.customer.netspace.net.au/APPLETS/12/12.html`, Greg Egan—refers to Quasitiler documentation.