An Expert System Prototype for Inventory Capacity Planning: An Approximate Reasoning Approach

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ABSTRACT

An approximate reasoning framework is suggested for the development of an expert system prototype to aid management in planning inventory capacities. The development is considered to be a stage that comes after the analysis of a stochastic model. Such a model would provide the requisite insight and knowledge about the inventory system under specific assumptions. As a consequence, the model builder(s) would act as expert(s). The restructuring process from the stochastic model into the approximate reasoning framework is described in a case study analysis for a Markovian production model. The stochastic model considers a relatively simplified production process: one machine, constant production rate, a compound Poisson demand process for the product together with the reliability feature comprising the machine failure process and the ensuing repair action. In this context, the authors propose an approximate reasoning framework and describe (1) the identification of the managerial decision-making rules, which usually contain uncertain (vague, ambiguous, fuzzy) linguistic terms; and (2) the specification of membership functions that represent the meaning of such linguistic terms within context-dependent domains of concern. They then define a new

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universal logic incorporating these rules and functions and apply it to inventory capacity planning. Two case examples and a simulation experiment consisting of 21 cases are summarized with a discussion of results.

KEYWORDS: ***expert system, knowledge acquisition, approximate reasoning, inventory capacity planning, simulation experiments***

1. INTRODUCTION

The purpose of this paper is to suggest a possible approach to bridging the communication gap between stochastic model builders and the managers of inventory capacity planners. We use an approximate reasoning framework based on fuzzy logic for the development of an expert system prototype that can serve as an aid to managerial decision making in manufacturing.

It is generally known that although stochastic models provide valuable insights into a system's behavior under specific assumptions and identify useful guidelines, more often than not they are not implemented, either because managers do not understand the basic assumptions of the model, or because they are too complex, or because the precise information required to determine the values of the model parameters cannot be obtained or are not available. Hence, such models are, by default, inadequate to help managers to cope with the natural behavior of many real-life systems.

However, stochastic model builders are often experts who can interpret the results of their models and can express their insightful expert knowledge about such system behavior in a natural language that usually contains vague, ambiguous, uncertain, fuzzy linguistic terms. Such linguistic terms provide (1) a flexible expression of the system behavior subject to various uncertainties and imprecisions and at the same time (2) help model builders communicate their complex results to management in an appropriate natural language context.

Such concerns lead us to suggest the development of expert system prototypes via an approximate reasoning framework based on fuzzy logic but relying on insights obtained from stochastic models with the model builders acting as the experts.

In order to explain in detail how such a development would and could take place, we first summarize briefly both the approximate reasoning framework and the stochastic model under consideration in this paper.

Approximate Reasoning

Informally, approximate reasoning is the process or processes by which a possible imprecise conclusion is deduced from a collection of imprecise premises (Zadeh [1]). A number of alternative approaches are available for reasoning in the design of knowledge-based systems (Turksen [2], Turksen and

Zhong [3]). Before we discuss the details of an alternative approach, let us first review the structure of a possible knowledge-based system that may be used in prototype studies. (Clearly, the following is just a skeleton; other relevant aspects of the development of such a system are omitted in this paper.)

1. *Factual knowledge (observed system state) ("facts")*:

$$X_1 \text{ is } A_1^* \text{ AND } X_2 \text{ is } A_2^* \text{ AND} \cdots \text{AND } X_n \text{ is } A_n^* \tag{1}$$

for short:

$$A^* = A_1^* \text{ AND } A_2^* \text{ AND} \cdots \text{AND } A_n^*$$

The user of an expert system inputs the factual knowledge, choosing the appropriate linguistic terms A_1^*, $A_2^* \ldots$, A_n^* that describe the observed states of the system. Users may be given the option to state the meaning of these linguistic terms by specifying membership values and/or functions. Alternatively, they may be given the option to rely on the definitions provided by the domain experts.

2. *Rule base (expert rules) ("rules")*:

$$\text{IF } X_1 \text{ is } A_1 \text{ AND } X_2 \text{ is } A_2 \text{ AND} \cdots \text{AND } X_n \text{ is } A_n, \quad \text{THEN } Y \text{ is } B \tag{2}$$

for short:

$$A \rightarrow B, \text{ where } A = A_1 \text{ AND } A_2 \text{ AND} \cdots \text{AND } A_n$$

Domain experts (for this discussion, stochastic model builders) provide such rules during the knowledge acquisition phase, where A_1, $A_2, \ldots, A_n$ and B are the appropriate set of linguistic terms specified by the domain experts. The domain experts must also provide the meaning of these linguistic terms by specifying their membership values and/or functions over the domain of discourse. Rules are then encoded into the knowledge base by a knowledge engineer during the design and development phase.

3. *Expert system advice ("response")*:

$$Y \text{ is (should be) } B^* \tag{3}$$

The expert system "response" is provided to a user after the inference subsystem operates on the "rules" and the given "facts" in accordance with an inference scheme.

In the current literature of fuzzy sets, it may be observed that the meanings of linguistic terms and their logical combinations are generally represented by "point-valued" membership assignments. However, it has been observed that membership values obtained from domain experts and their logical combinations usually turn out to be "interval-valued" in most measurement experiments (Turksen [4–6], Chameau and Santamarina [7]).

In this paper we start out with a knowledge representation scheme based on point-valued membership functions obtained from experts (the stochastic model builders). That is, $A_1, A_2, \ldots, A_n, B$; $A_1^*, A_2^*, \ldots, A_n^*$ will all be represented by point-valued membership functions. However, the representation of the logical combination of AND and the logical implication IF ... THEN will be computed with the disjunctive and conjunctive normal forms, DNF and CNF, respectively. These representations and computations will be further discussed in appropriate sections in the remainder of the paper.

FRAMEWORK The basic approximate reasoning framework used in this paper consists essentially of the following:

(i) An interval-valued fuzzy set representation of "rules" and "facts"
(ii) A search for the "rules" closest to the "facts" based on a similarity measure
(iii) An inference based on the implication of the "rules" found in (ii) and the "facts"
(iv) An advice based on the linguistic approximation of the result(s) found in (iii)

The details of approximate reasoning methodology and its framework may be found in Turksen [2, 4–6, 8, 9].

Stochastic Model

Production/inventory/reliability models have been at the focus of production modeling for a long time now. The rule of inventory is to accommodate fluctuations from the demand side, and the fundamental problem is to relate the production process to the demand process so that, on the one hand, shortages are kept at a desirable low level but, on the other hand, no excessive inventory is built up.

An important but frequently neglected element in such systems is the imperfection of production systems. Posner and Berg [10] studied a model that incorporates reliability (or, indeed, unreliability) factors into the analysis and obtained closed-form analytical results under certain assumptions on the nature of the randomness in the production inventory system and the reliability factors.

MODEL ASSUMPTIONS We consider a Markovian production model in which a single machine produces items at a constant production rate 1 (without loss of generality) up to a level N, the inventory capacity. The production is halted whenever the inventory level reaches N and is resumed at the next demand epoch.

The demand process is compound Poisson: Demands arrive according to a Poisson process at rate λ, and order sizes are i.i.d. random variables exponentially distributed with a mean μ^{-1}. There is no backlogging of demand, so excess demand beyond the available inventory is lost. The operating time of the machine is exponentially distributed with parameter θ. Thus θ is the failure rate of the operating machine. The repair time of the failed machine is exponentially distributed with mean σ^{-1}.

ANALYTICAL RESULTS Let $X(t)$ be the inventory level at time t, and set $W(t) = N - X(t)$ to be the slack storage capacity at time t. Clearly, $W(t) \in [O, N]$. We introduce the variable $W(t)$ because to deal with $X(t)$ directly is less convenient than to deal with $W(t)$, but at the same time these two variables are equivalent to our understanding of the behavior of this system. The limiting density and distribution function of $W = \lim_{t\to\infty} W(t)$ are $f(\cdot)$ and $F(\cdot)$, respectively.

Since the knowledge of W alone does not indicate whether or not the machine is in repair, a supplementary variable approach is implemented through the generalized technique of "system point" in level-crossing analysis (Brill and Posner [11, 12]). W should be partitioned into two parts, W_0 and W_1, where W_0 indicates the portion of W while the machine is not under repair, and W_1 is the portion of W while the machine is under repair. Correspondingly, both $f(\cdot)$ and $F(\cdot)$ should be partitioned into two parts, denoted by $f_0(\cdot)$, $f_1(\cdot)$ and $F_0(\cdot)$, $F_1(\cdot)$, respectively. Naturally we have the following two equations:

$$f(\cdot) = f_0(\cdot) + f_1(\cdot)$$

$$F(\cdot) = F_0(\cdot) + F_1(\cdot)$$

It is to be observed that aside from the partial densities with respect to the state of the machine, there is also a probability mass $f_0 \equiv \Pr(W_0 = 0)$ associated with a full inventory accompanied by production shutoff, and $f_1^N \equiv \Pr(W_1 = N)$ associated with an empty inventory and no active production.

The result of the mathematical analysis due to Posner and Berg [10] is summarized in the Appendix.

The inventory capacity level N is the decision variable through which a desirable level of "the fraction of satisfied customer demand" ρ can be achieved. For computational convenience we define a surrogate decision variable.

$$V = \frac{N - N_0}{N_0} \tag{4}$$

where N_0 is the existing inventory capacity. V is thus the fraction of increase in the existing inventory capacity needed to achieve ρ_0, the desirable level of ρ.

For specificity, we shall set $\rho_0 = 0.95$ and let $N_0 = 1.5$. Setting $N_0 = 1.5$ means an inventory capacity of 1.5 "lots," where a lot corresponds to a prespecified number of items. This unit is also used in the definition of μ^{-1}, the mean size of demand. Another basic unit here is the unit of time that is used in the definition of the parameters λ, σ^{-1}, and θ. The production rate is based on both units and their standardized form, which corresponds to producing 1 lot per unit of time.

KNOWLEDGE ACQUISITION

The analytical result provides us with a thorough understanding of this production/inventory/reliability system. It is, however, not very practical from the point of view of management. The exactness of the system parameters is unnecessary to the managers, and furthermore it does not provide managers with insights into the operation of this system, which is of vital importance to the effective management of such a complex system. Many analytical models bear the same language barriers. Even though these models clearly depict the characteristics of a system for the well-trained model builders and their results could be effectively interpreted by their builders, they are usually inappropriate for managerial implementation and use.

An important, and at times critical, step in the design and development of an expert system prototype is knowledge acquisition in the form of rules and their components.

Preliminary Considerations

In accordance with the analytical model, we have decided to use the fraction of satisfied customer demand ρ as our performance criterion and the inventory capacity N as the decision variable, with other parameters tacitly assumed to be outside our control. Furthermore, in agreement with the approximate reasoning rule form (2), we opt for the following rule structure:

IF λ is A_1 AND μ^{-1} is A_2 AND θ is A_3
AND σ^{-1} is A_4 AND ρ is high
THEN take action B (with respect to N or, equivalently, V)

The A_i, $i = 1, \ldots, 4$, and B stand for sets of relevant linguistic descriptors. A basic set of descriptors for each of the A_i in this case is *low*, *medium*, *high* (Turksen [2]). (If necessary, this categorization can be made finer by adding linguistic descriptors such as *very low* or *moderately high*, as shown in the simulation experiments.)

Thus, an example of the rules we aim to construct is:

If the demand arrival rate λ is *low*, and the mean demand size μ^{-1} is *medium*, and the failure rate of the machine θ is *low* and its repair rate σ^{-1} is *high*, then in order to have a high fraction of satisfied demand ρ, increase N *moderately* (or, equivalently, set a *moderate* value for V).

The linguistic descriptors such as *low*, *medium*, and *high* are imprecise, but these are terms that a manager not only understands but is also generally willing to give a meaning representation to by specifying a membership function (Zysno [13], Turksen [4–6], Norwich and Turksen [14], Zimmerman and Zysno [15]). In contrast, to use the exact model (i.e., the stochastic model) as is, the manager would be required to substitute exact figures for the parameters—a responsibility he or she may be reluctant to assume because of the ever-acute shortage of data and the often encountered statistical inference difficulties, and so on.

We thus sacrifice some of the exactness of the original model in order to turn it into an implementable decision-making tool whenever we are confronted with either a lack of data or a limited amount of data. Indeed, the exact analysis can be viewed in this framework as a building block in the construction of this manageral aid (the expert system) where the other building blocks relate to the qualitative/quantitative interpretations of the linguistic descriptors as represented by the corresponding membership functions and inferencing methods, and so on. The detailed development of this procedure for the production/inventory/reliability case under consideration is described in the next section.

INTERPRETATION Before we turn to the actual construction of the rules, it is useful to review some of our interpretations regarding the regions of their applicability. The parameter N, the inventory capacity, is the sole decision variable by means of which we want to achieve a desirable level ρ_0 for the fraction of satisfied customer demand. While $\rho = \rho(N)$ is an increasing function of N, it is not at all clear, or indeed true, that any desirable ρ_0 can be achieved by merely increasing N or V [because we may very well have $\rho(\infty) < \rho_0$]. This is likely to happen when the effective production rate is small compared to the demand rate $\lambda\mu^{-1}$, because in this case inventory does not build up fast enough to allow changes on N relevant to our goal of reaching ρ_0. (By effective production rate we mean the theoretical production rate 1 minus the lost production rate due to machine unreliability.)

At the other extreme, we have the cases where even a much smaller level of

N than the present N_0 is good enough to achieve ρ_0. Generally speaking, a very low demand rate relative to the effective production rate makes a relatively small N sufficient to generate a high enough ρ.

When the demand rate is close to the production rate 1, the objective V and the action on N are very sensitive to the failure rate, particularly if the mean repair time σ^{-1} is high, because a greater or smaller failure rate can make all the difference between adequate supply and the occurrence of a shortage. Note that this statement refers not only to the expectation of the shortage but also to its variability. Thus, in the Poisson process, which in our theoretical model characterizes the failure process, the mean number of failure events per unit of time is proportional to the variance of this random quantity, and therefore a high failure rate can significantly increase the range of required changes on N, thereby detrimentally affecting the stability of the expert system's responses. This undesirable effect can be avoided by limiting the set of rules to low failure rates. Such a restriction is justified by the fact that machines with medium or high failure rates are, with a high possibility, not likely to be retained by the management. Consequently, rules involving medium or high failure rates will be eliminated from the system development.

Linguistic Descriptors

In order to design and develop an expert system that operates with the principles of approximate reasoning and performs a task heuristically equivalent to the analytical model considered earlier, we require that the model builders not only provide us with the linguistic descriptors that identify the aggregate patterns of behavior but also provide us with "meaning representation" for these linguistic descriptors. As suggested in the previous section these linguistic terms were identified as

L, low; M, medium; H, high

for each of the system (independent) parameters; that is,

Demand rate λ	A_1
Demand size μ^{-1}	A_2
Failure rate θ	A_3
Repair rate σ^{-1}	A_4

Turning to the set of actions B on N (or, alternatively, on V) that will yield the desirable performance level ρ, we consider the following four possible actions:

Low (L)	Increase N a bit (which includes the case of no increase at all) or, equivalently, set a low value for V.
Medium (M)	Increase N moderately or, equivalently, set a moderate value for V.
High (H)	Increase N a lot or, equivalently, set a high value for V.
Very high (VH)	Increase N quite a lot or, equivalently, set a very high value for N.

(The term "increase" is interpreted in these actions in percentage terms.)

In practice, the meaning representations of these membership functions are specified either by an expert of the production system under consideration or by its manager, in accordance with universally accepted forms and on the basis of relevant considerations (as delineated above for the failure rate). For this exercise, we have determined the membership functions in a cooperative effort between the operations research (OR) specialist and the knowledge engineer. Due to the context-dependent nature of the curves, these functions need to be adjusted and modified for a given production system. For all parameters, we have kept the convention of scaling the parameter's ranges into the [0, 1] interval by normalizing each base variable with its maximum in the following manner:

Normalized demand rate: $$DR = \lambda/\lambda_{max}, \tag{5a}$$

Normalized demand size: $$DS = \mu^{-1}/\mu_{max}^{-1}, \tag{5b}$$

Normalized failure rate: $$FR = \theta/\theta_{max} \tag{5c}$$

Normalized repair rate: $$RR = \sigma^{-1}/\sigma_{max}^{-1} \tag{5d}$$

Normalized decision variable: $$DV = V/V_{max} \tag{5e}$$

where the maximum values are set to

$$\lambda_{max} = 1.0, \qquad \mu_{max}^{-1} = 1.0, \qquad \theta_{max} = 0.05,$$
$$\sigma_{max}^{-1} = 5.0, \qquad V_{max} = 7.0$$

MEMBERSHIP FUNCTIONS The notion of crossover point (Zadeh [16]) with the membership value of 1/2 plays an important role in determining the membership functions. For each variable, we must first identify the points that are significant enough to divide the interval of the universal set [0, 1] into corresponding linguistic subintervals of low (L), medium (M), and high (H) or very high (VH), varying according to different system states, in order to

Table 1. Subintervals of Linguistic Terms

Variable	Low	Medium	High	Very high
DR	[0, 0.3]	(0.3, 0.7]	(0.7, 1.0]	
DS	[0, 0.3]	(0.3, 0.7]	(0.7, 1.0]	
RR	[0, 0.3]	(0.3, 0.8]	(0.8, 1.0]	
FR	[0, 0.1]	(0.1, 0.6]	(0.6, 1.0]	
DV	[0, 0.15]	(0.15, 0.35]	(0.35, 0.54]	(0.54, 1.0]

acquire meaningful membership functions for every linguistic term. Again these subintervals are identified by experts as shown in Table 1.

It should be noted that the separation point between regions is the crossover point of the two curves representing the membershp functions of the two corresponding linguistic terms, and the membership grade (MG) of this point has the largest uncertainty (Kosko [17]) for both of these fuzzy sets. For example, consider DR. The point separating the low and medium subintervals is 0.3:

$$MG_{\text{Low}}(0.3) = MG_{\text{Medium}}(0.3) = 0.5$$

An element of a fuzzy set with a membership grade greater than 0.5 is more likely to belong to this set than not, while a membership grade less than 0.5 indicates that an element is less likely to belong to this set. For example, the interval [0, 0.3] of DR is intended to be regarded as a low-level demand rate rather than a medium-level demand rate by experts. Therefore 0.3 is chosen as the point separating the low-level and medium-level intervals of DR, and accordingly a membership grade of 0.5 is assigned for the fuzzy sets of both low DR and medium DR. The membership curves are shown in Figures 1–5.

A different case is the fuzzy set of Very High for the service level, as shown in Figure 5. Even though the VH curve does not intersect the H curve except at the extreme right side of the figure, it still has a membership grade of 0.5 at

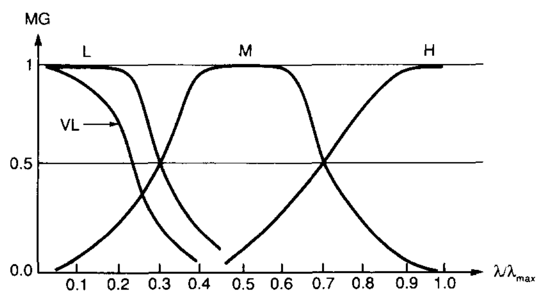

Figure 1. Four membership functions of VL, L, M, H for DR.

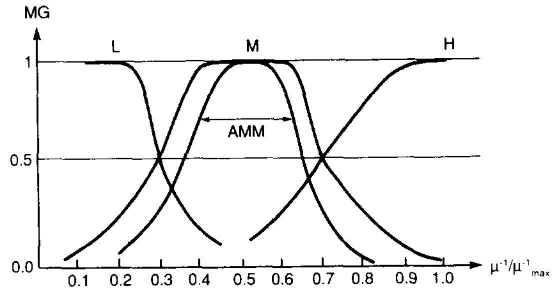

Figure 2. Four membership functions of L, M, AAM, H for DS.

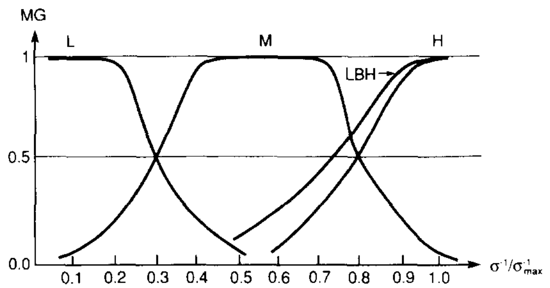

Figure 3. Four membership functions of L, M, LBH, H for RR.

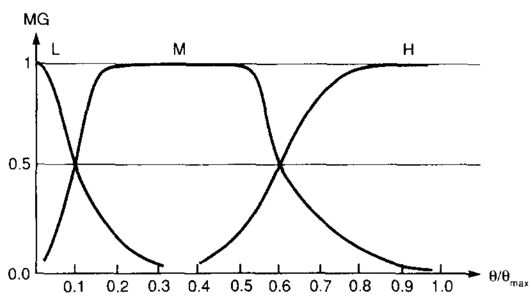

Figure 4. Three membership functions of L, M, H for FR.

point 0.54, which separates intervals of H and VH, following the same philosophy as we discussed above.

Each of these meaning representations of the membership curves needs to be further justified by the expert in an appropriate manner. For example, note the steep slope near FR = 0. This is due to the sensitivity of the notion of ""low failure rate" to even moderate (absolute) increases; thus, whereas a 0.05 failure rate may look to the manager to be low with a degree of determination

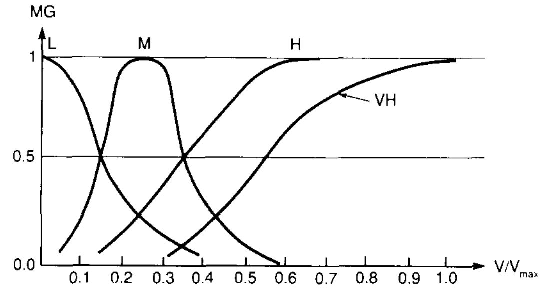

Figure 5. Four membership functions of L, M, H, VH for DV.

0.95, a 0.1 failure rate is assigned a degree of determination 0.5, indicating the highest level of entropy in the assessment (Kosko [17]). However, with a parameter like the demand rate DR, such sensitivity is unlikely, because demand volatility is anticipated anyway because of its dependence on human choices (as opposed to machine performance, for which a relatively high degree of precision is expected). Similar considerations help determine the shapes of all the membership curves (Figures 1-5). Some figures include the membership curves for additional linguistic terms that will be needed for the case study examples discussed later (after we present the inference procedure).

Structure of the Rule Base

Let us now proceed to the actual construction of the basic set of rules of the expert system. In principle, we should have 81 basic rules, since each of the A_i is assigned three linguistic terms. However, as mentioned earlier, rules involving "above-low" failure rates are impractical because the machine is not likely to be used. Hence, fixing the failure rate input state to "low" reduces the number of input state vectors to the manageable size of 27.

Suppose a manager assesses all parameter values to be low. This will correspond to an input state vector

$$\mathbf{I} = (\mathrm{L}, \mathrm{L}, \mathrm{L}, \mathrm{L})$$

where the first, second, third, and fourth components of **I** correspond to the normalized parameters DR, DS, FR, and RR, respectively. The question is what (minimal) action on N, or equivalently on V—L, M, H, or VH—is required to achieve the desired performance level ρ_0 (0.95 in the illustration here).

To answer this question we again require expertise. The expertise needed now is of a more general nature. Such expertise should be valid for an entire class of production/inventory systems (with machine imperfection incorpo-

Table 2. Midpoints of Linguistic Subintervals (Normalized)

Variable	Low	Medium	High
DR	0.15	0.5	0.85
DS	0.15	0.5	0.85
RR	0.15	0.55	0.9
FR	0.05		

rated), and it could be obtained, for example, from production/inventory/reliability system managers. In our case, however, the source of this expertise is again the OR experts who derived the mathematical results and interpreted them with the following simulated analysis.

For each independent variable DR, DS, RR, FR, choose the midpoint of the three linguistic subintervals (L, M, H) as a representation of this linguistic term. These points are shown in Table 2. It should be noted that a decision to consider only the Low failure rates means that FR is set at 0.05 for L.

First, the expert chooses these midpoints as a representation of the linguistic terms in deciding the formation of the left-hand side of the rules. Next, the corresponding values of these midpoints are determined by (5a)-(5d) and substituted into the analytical solution procedure that is shown in the Appendix to compute the corresponding inventory level and hence by Eq. (4) the service level. The service level in turn is converted to the corresponding subinterval by (5e), which identifies the value of the base variable in Figure 5 and determines the linguistic term for the response. Thus, the linguistic term is determined for the right-hand side of the rule. All 27 rules are determined in this manner; the results are presented in Table 3.

REPRESENTATION AND INFERENCE

Observe from Table 3 that each rule is a fuzzy relation, with the left-hand side of the rule being an AND combination among three variables, connected with the right-hand side of the rule by an IF . . . THEN (implication) relation.

In order to carry out the operation of AND combination and IF . . . THEN relation, a number of finite support points for each linguistic term must be identified in Figures 1-5. The values of the support points for each term are a compromise between the efficiency of computation and the accuracy of the inference results. In this study we choose seven points for each term as shown in Table 4.

Recall from the Introduction that in this paper the linguistic combinations are

Table 3. The Rule Base for Three Independent Variables[a]

Rule Variable No.	Independent Variables			Decision
	DR	DS	RR	DV
1	L	L	L	L
2	L	L	M	L
3	L	L	H	L
4	L	M	L	M
5	L	M	M	M
6	L	M	H	M
7	L	H	L	M
8	L	H	M	M
9	L	H	H	M
10	M	L	L	L
11	M	L	M	L
12	M	L	H	L
13	M	M	L	M
14	M	M	M	M
15	M	M	H	M
16	M	H	L	H
17	M	H	M	H
18	M	H	H	H
19	H	L	L	L
20	H	L	M	L
21	H	L	H	L
22	H	M	L	M
23	H	M	M	M
24	H	M	H	M
25	H	H	L	VH
26	H	H	M	VH
27	H	H	H	VH

[a]Recall that the failure rate FR is set to L only.

based on the disjunctive and conjunctive normal forms, DNF and CNF, respectively, which are extensions of the canonical forms in Boolean logic.

AND Composition

The DNF and CNF for AND combination are

$$\mathrm{DNF}(A \text{ AND } B) = A \cap B \tag{6}$$

$$\mathrm{CNF}(A \text{ AND } B) = (A \cup B) \cap (A \cup B^c) \cap (A^c \cup B) \tag{7}$$

Table 4. Typical Linguistic Descriptors and Their Membership Values for Each Fuzzy Set

DR:

VL = (1.0/0.0, 0.4/0.25, 0.32/0.3, 0.0/0.5, 0.0/0.7, 0.0/0.8, 0.0/1.0)
SL = (1.0/0.0, 0.57/0.25, 0.25/0.3, 0.0064/0.5, 0.0/0.7, 0.0/0.8, 0.0/1.0)
LBL = (1.0/0.0, 0.87/0.25, 0.71/0.3, 0.28/0.5, 0.0/0.7, 0.0/0.8, 0.0/1.0)
L = (1.0/0.0, 0.76/0.25, 0.5/0.3, 0.08/0.5, 0.0/0.7, 0.0/0.8, 0.0/1.0)
M = (0.0/0.0, 0.24/0.25, 0.5/0.3, 1.0/0.5, 0.5/0.7, 0.2/0.8, 0.0/1.0)
QBM = (0.0/0.0, 0.057/0.25, 0.25/0.3, 1.0/0.5, 0.25/0.7, 0.04/0.8, 0.0/1.0)
H = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.08/0.5, 0.5/0.7, 0.8/0.8, 1.0/1.0)

DS:

L = (1.0/0.0, 0.76/0.25, 0.5/0.3, 0.08/0.5, 0.0/0.7, 0./0.8, 0.0/1.0)
M = (0.0/0.0, 0.24/0.25, 0.5/0.3, 1.0/0.5, 0.5/0.7, 0.2/0.8, 0.0/1.0)
QBM = (0.0/0.0, 0.1/0.25, 0.24/0.3, 1.0/0.5, 0.3/0.7, 0.15/0.8, 0.0/1.0)
H = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.08/0.5, 0.5/0.7, 0.8/0.8, 1.0/1.0)
VH = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.0/0.5, 0.25/0.7, 0.64/0.8, 1.0/1.0)

RR:

VL = (1.0/0.0, 0.64/0.25, 0.25/0.3, 0.0/0.55, 0.0/0.8, 0.0/0.86, 0.0/1.0)
L = (1.0/0.0, 0.8/0.25, 0.5/0.3, 0.01/0.55, 0.0/0.8, 0.0/0.86, 0.0/1.0)
NM = (1.0/0.0, 0.73/0.25, 0.5/0.3, 0.0/0.55, 0.5/0.8, 0.74/0.86, 1.0/1.0)
M = (0.0/0.0, 0.27/0.25, 0.5/0.3, 1.0/0.55, 0.5/0.8, 0.26/0.86, 0.0/1.0)
H = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.01/0.55, 0.5/0.8, 0.8/0.86, 1.0/1.0)
LBH = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.17/0.55, 0.8/0.8, 1.0/0.86, 1.0/1.0)
VH = (0.0/0.0, 0.0/0.25, 0.0/0.3, 0.0/0.55, 0.25/0.8, 0.64/0.86, 1.0/1.0)

DV:

L = (1.0/0.0, 0.5/0.15, 0.2/0.25, 0.09/0.35, 0.0/0.5, 0.0/0.54, 0.0/1.0)
M = (0.0/0.0, 0.5/0.15, 1.0/0.25, 0.5/0.35, 0.13/0.5, 0.04/0.54, 0.0/1.0)
H = (0.0/0.0, 0.06/0.15, 0.18/0.25, 0.5/0.35, 0.95/0.5, 1.0/0.54, 1.0/1.0)
VH = (0.0/0.0, 0.0/0.15, 0.02/0.25, 0.1/0.35, 0.32/0.5, 0.5/0.54, 1.0/1.0)

where $\cap$, $\cup$, and superscript c are used in the set notation to correspond to the intersection, union, and complementation operators, respectively.

All the fuzzy sets we have defined so far are point-valued. However, by applying Eqs. (6) and (7) for AND combination, interval-valued fuzzy sets are generated, with CNF(.) and DNF(.) being the upper and lower bounds, respectively. This is certainly not a surprise, as it is argued in Turksen [4–6] that most of the linguistic combinations would be interval-valued when imprecise knowledge is extracted from experts. That means that our four independent variables and one decision variable could most probably have been interval-valued fuzzy sets. For ease of computation we adopt a point-valued approach and choose the midpoint of an interval for fuzzy relations whenever such an interval is given or is generated by DNF and CNF operations.

Let us choose rule 6 from Table 3 as an example to explain these calculations. The left-hand side of this rule is

$$R_{\mathrm{LMH}} = \mathrm{DR_L} \text{ AND } \mathrm{DS_M} \text{ AND } \mathrm{RR_H} \tag{8}$$

A finite support membership value is chosen from each of these three fuzzy sets, say $a_i \in \mathrm{DR_L}$, $b_j \in \mathrm{DS_M}$, $c_k \in \mathrm{RR_H}$, $1 \le i, j, k \le 7$. Let us also choose Max and Min operations, denoted by $\vee$ and $\wedge$, to correspond to the $\cup$ and $\cap$ in the set notation of CNF and DNF in Eqs. (6) and (7) and throughout this paper. This choice is needed because there are a large collection of operators that correspond to $\cup$ and $\cap$ (Turksen [4–6]).

Hence the membership grades (MGs) for the first two variables are computed as

$$\mathrm{MG}(\mathrm{DNF}(\mathrm{DR_L} \text{ AND } \mathrm{DS_M})) = a_i \wedge b_j \tag{9}$$

$$\mathrm{MG}(\mathrm{CNF}(\mathrm{DR_L} \text{ AND } \mathrm{DS_M})) = (a_i \vee b_j) \wedge (a_i \vee b_j^c) \wedge (a_i^c \vee b_j) \tag{10}$$

where the complementation is chosen to be the pseudocomplement:

$$a_i^c = 1 - a_i, \qquad b_j^c = 1 - b_j, \qquad c_k^c = 1 - c_k$$

The midpoint r_{ij} of the interval-valued fuzzy set

$$\mathrm{R_{LM}} = \mathrm{DR_L} \text{ AND } \mathrm{DS_M}$$

is computed as

$$r_{ij} = (1/2)[\mathrm{MG}(\mathrm{DNF}(\cdot)) + \mathrm{MG}(\mathrm{CNF}(\cdot))] \tag{11}$$

Next r_{ij} is used to compute the CNF and DNF of AND with c_k similarly determining r_{ijk}, which is the midpoint of the interval-valued fuzzy set of R_{LMH} in Eq. (8). R_{LM} is a 7×7 matrix, and R_{LMH} is a $7 \times 7 \times 7$ matrix. Altogether we have 27 such matrices, since we have 27 rules in our rule base.

An α-cut for each matrix or each rule needs to be computed by

$$\alpha = V(r_{ijk} \wedge r_{ijk}^c), \qquad 1 \le i, j, k \le 7, \tag{12}$$

to satisfy the sufficiency condition required for the generalized modus ponens (Turksen [2]). Thus, we have 27 α's in all, that is, α_n, $1 \le n \le 27$, which are then used to truncate the decision space as discussed next.

IF . . . THEN Composition

The 27 three-dimensional matrices are then combined with the right-hand side of each rule, the decision space, by IF . . . THEN composition to construct the fuzzy relation of each rule in Table 3. But we must first truncate the decision space by α-cuts, where the α's are given by Eq. (12). For example, suppose we have $\alpha_6 = 0.5$. Rule 6 shows that DV should be M, with the following finite support:

$$\mathrm{DV_M} = (0.5, 1, 0.5, 0.13, 0.04)$$

After truncation, we get

$$\mathrm{DV_M^{(T)}} = (0.5, 1, 0.5)$$

Those elements that are less than α_6 are discarded, because they are insignificant in carrying the information through this inference procedure (Turksen [2–6]).

Now we are ready to do DNF and CNF evaluation of the IF . . . THEN relation between R and DV. The DNF and CNF expressions of IF . . . THEN composition are

$$\mathrm{DNF}(A \to B) = (A \cap B) \cup (A^c \cap B) \cup (A^c \cap B^c) \tag{13}$$

$$\mathrm{CNF}(A \to B) = A^c \cup B \tag{14}$$

In the membership domain, again with the application of Max and Min operators, we get

$$r_{ijkl} = (1/2)\left[\mathrm{MG}(\mathrm{DNF}(R_{\mathrm{LMH}} \to \mathrm{DV_M^{(T)}})) + \mathrm{MG}(\mathrm{CNF}(\mathrm{R_{LMH}} \to \mathrm{DV_M^{(T)}}))\right] \tag{15}$$

where r_{ijkl} is the midpoint of the interval-valued fuzzy sets in matrix R_{LMHM}, that is,

$$R_{\mathrm{LMHM}} = R_{\mathrm{LMH}} \to \mathrm{DV_M^{(T)}}$$

Equation (15) will reduce interval-valued membership grades to point-valued ones.

The 27 matrices generated in this manner are represented in four-dimensional matrices with membership values. So far we have constructed a set of rules and the database that is the set of these matrices, which are required for the inferences in approximate reasoning.

Inference Procedure

A number of alternative procedures are available for the approximate reasoning approach (Turksen [2, 9], Turksen and Zhong [3]). In this paper, we discuss Zadeh's composition rule of inference [16]. In addition, from among the many approximate reasoning modes, we choose the modus ponens known as the generalized modus ponens (GMP) in our discussion.

For GMP, compositional inference is written as

$$R^* \circ (R \to \mathrm{DV}) = \mathrm{DV}^* \tag{16}$$

where $\circ$ is the compositional rule of inference; R^* is an AND combination of observed states of the system, DR*, DS*, RR*; and $R \to$ DV is a rule to be selected from the rule base with a suitable distance or similarity measure. DV* is the expert system advice (response) to be provided to a user for the combined observed system state R^*. Such expert system responses are determined in the following manner.

Suppose a user inputs DR*, DS*, and RR* (the observed states); then the inference engine first computes

$$R^*_{\mathrm{LOWER}} = \mathrm{DNF}(\mathrm{DR}^* \text{ AND } \mathrm{DS}^* \text{ AND } \mathrm{RR}^*)$$

$$R^*_{\mathrm{UPPER}} = \mathrm{CNF}(\mathrm{DR}^* \text{ AND } \mathrm{DS}^* \text{ AND } \mathrm{RR}^*)$$

$$\mathrm{MG}(R^*) = (1/2)\left[\mathrm{MG}(R^*_{\mathrm{LOWER}}) + \mathrm{MG}(R^*_{\mathrm{UPPER}})\right]$$

Next the closeness of R^* is checked against R, the left-hand side of each of the 27 rules. Among the many distance measures (Zwick et al. [18]), we choose the Hamming distance as the closeness measure for this paper, which is

$$d_{\mathrm{H}}(R - R^*) = \sum_{ijk} | r_{ijk} - r^*_{ijk} | \tag{17}$$

The rule whose left-hand side R has the minimum distance among the 27 rules to the observed system state R^* is chosen for GMP computations.

The compositional rule of inference is applied for the rule so chosen. For example, suppose the rule so chosen has left-hand side and right-hand side denoted by R^*_0 and DV^*_0, respectively; then

$$R^* \circ (R^*_0 \to \mathrm{DV}^*_0) = \mathrm{DV}^* \tag{18}$$

By substituting DNF and CNF operations of $R^*_0 \to \mathrm{DV}^*_0$ into Eq. (18), we can obtain the DNF and CNF of DV* and the midpoint of this interval as

$$\mathrm{MG}(\mathrm{DV}^*) = (1/2)\left[\mathrm{MG}(\mathrm{DNF}(\mathrm{DV}^*)) + \mathrm{MG}(\mathrm{CNF}(\mathrm{DV}^*)\right] \tag{19}$$

Since DV^*_0 in Eq. (18) is already truncated in the implication composition by α_0 given by Eq. (12), DV* has only those elements corresponding to the truncated DV^*_0.

Finally, DV* is compared to all the linguistic terms in the decision space, that is, $D \in \{\mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}\}$ to find the closest linguistic term based on the distance measure, in this study the Hamming distance d_{H} given by Eq. (17), that is,

$$d_{\mathrm{H}}(\mathrm{DV}^*, \mathrm{DV}) = \sum |\mathrm{MG}(\mathrm{DV}^*) - \mathrm{MG}(\mathrm{DV})|$$

Thus, the closest linguistic term is taken as the linguistic approximation to DV*. The output of the fuzzy inference engine is this linguistic term so chosen.

Please note that when we make the distance comparison between DV* and the linguistic terms in decision space, DV_0^* is truncated by the α_0-cut [Eq. (12)]. Thus, we should truncate the linguistic terms in the decision space accordingly in order to calculate the Hamming distance properly.

To indicate the closeness between DV* and DV we may choose another measure, besides the distance measure, which is somewhat more general in the sense that it is in the interval [0, 1] and is independent of any particular situation. This measure is called the *similarity measure* (Turksen and Zhong [3]) and is defined as

$$S = \frac{1}{1 + d}$$

where d is a distance measure. In our case, d_H is the Hamming distance; therefore,

$$S = \frac{1}{1 + d_H} \tag{20}$$

When $d_H = 0$, $S = 1$, indicating that the two linguistic terms are exactly the same; and when $d_H = \infty$, $S = 0$, meaning that the two terms are very far apart. It should be noted that d_H and S are equivalent measures of closeness.

Simulation Experiment

The proposed approximate reasoning approach was programmed in LISP and was run on an Apollo workstation in a UNIX environment for two simulation experiments with 21 hypothetical case data shown in Table 6. The membership values of the linguistic variables are shown in Table 4. The first experiment was run with the application of max-min operators, the second with the application of bold union/intersection operators. The results of these experiments are shown in Tables 7 and 8.

Before we discuss the outcome of these experiments, let us illustrate the inference procedure described in the previous subsection with two examples.

EXAMPLE 1 If the state of the system we have observed matches exactly to one of the left-hand sides of the 27 rules—for example; DR* is L, DS* is M, and RR* is M, that is, rule 5 in Table 3 and/or case 11 in Table 6—then $R^* = R_{LMM}$ and $d_H(R^*, R_{LMM}) = 0$.

If we input R^* into the fuzzy inference engine, we get $DV^* = DV_0^*$, which is M. Thus, M is the linguistic term we should choose according to this observed state of the system. This process is illustrated in Table 5. The output is M with a similarity measure of 1. The result is what we should have expected, because the observed state is exactly the left-hand side of rule 5. The

Table 5. Inference Procedure

Observed states

$$\mathrm{DR}^* = (1.0, 0.76, 0.50, 0.08, 0.0, 0.0, 0.0)$$

$$\mathrm{DS}^* = (0.0, 0.24, 0.50, 1.0, 0.50, 0.20, 0.0)$$

$$\mathrm{RR}^* = (0.0, 0.27, 0.50, 1.0, 0.50, 0.26, 0.0)$$

$$R^* = \mathrm{DR}^* \text{ AND } \mathrm{DS}^* \text{ AND } \mathrm{RR}^*$$

$\min | R^* - R | = 0$ identifies R_0^* of rule 5, with $\alpha_5 = 0.5$. Hence, DV_0^* is M.

$$\mathrm{DV}^* = R^* \circ (R_0^* \to \mathrm{DV}_0^*)$$

$$\mathrm{DV}^* = (0.5, 1., 0.5)$$

and

$$d_{\mathrm{H}}(\mathrm{DV}_0^*, \mathrm{DV}^*) = 0$$

$$S = \frac{1}{1 + d_{\mathrm{H}}} = 1$$

fuzzy inference engine does certainly produce, as it should, an output that is the right-hand side of rule 5.

This example illustrates the fact that an expert system based on our inference procedure produces the expected result when there is an exact match between the observed system state and the left-hand side of a rule in the rule base. However, the real power of our approximate reasoning has a bit more intelligence built into it in the following sense. Suppose there is no exact match between the observed system state and the left-hand side of any rule in the rule base. Then the question is, What can our approximate reasoning procedure provide for the user? We show in the next example that even when there is not an exact match, our approximate reasoning has the power to generate appropriate advice for the users.

EXAMPLE 2 If the state of the system we have observed is an arbitrary one—say, DR* is *very low*, DS* is *quite a bit medium*, and RR* is a *little bit high*—then the inference procedure is as follows.

If they are not available in the system already, the expert (the model builder) should give us meaning representations of all the allowable linguistic terms

Table 6. Simulation Inputs: Observed System States

Case No.	Linguistic Descriptors of Input Parameters		
	DR*	DS*	RR*
1	L	M	H
2	VL	QBM	LBH
3	VL	M	VH
4	VL	QBM	VL
5	SL	QBM	VL
6	SL	QBM	VH
7	VL	M	NM
8	L	L	L
9	M	M	M
10	H	H	H
11	L	M	M
12	VL	L	VH
13	VL	L	VL
14	H	L	VH
15	H	L	LBH
16	LBL	L	H
17	M	H	VL
18	M	H	LBH
19	M	VH	VL
20	QBM	H	VH
21	QBM	H	VL

such as very low (VL), quite a bit medium (QBM), a little bit high (LBH), specifying their membership functions. (For the case study, our expert identified these membership functions as shown in Figures 1–3 and as defined in Table 4.)

Thus, DR* is VL, DS* is QBM, RR* is LBH, that is, case 2 in Table 6. Observe that this does not match the left hand-side of any of the rules in Table 3. First $R^* = \text{DR}^* \text{ AND } \text{DS}^* \text{ AND } \text{RR}^*$ is computed by DNF and CNF expressions of AND combination. Then the nearest R, denoted by R_0^*, is chosen from among the 27 left-hand sides of the rules in the rule base. In this case, rule 6 is selected; hence, DV_0^* is M. With the GMP and an α-cut of 0.5, we get

$$\text{DV}^* = R^* \circ (R_0^* \to \text{DV}_0^*)$$

The result is $\text{DV}^* = (0.5/0.15, 1.0/0.25, 0.5/0.35)$.

The comparison of distance between DV* and other linguistic terms in decision space D can be summarized as follows:

Decision Space D Measure	Hamming Distance $\lvert DV_0^* - DV^* \rvert$	Similarity
L	1.21	0.45
M	0.0	1.0
H	1.26	0.44
VH	3.14	0.24

Therefore, the linguistic term M is the response of the expert system for the observed system state. That is, when the demand rate is very low, the demand size is quite a bit medium, and the repair rate is a little bit high, we should expect the service level to be medium.

SIMULATION EXPERIMENTS Let us now look at the two simulation experi-

Table 7. Simulation Experiments Results—(a) Simulation with Max-Min Operators; (b) Simulation with Bold Union/Intersection Operators

Case No.	Selected Rule Number		Linguistic Descriptor of System Response	
	(a)	(b)	(a)	(b)
1	6	6	M	M
2	6	6	M	M
3	6	6	M	M
4	4	4	M	M
5	4	4	M	M
6	6	6	M	M
7	5	6	M	M
8	1	1	L	L
9	14	14	M	M
10	27	27	VH	VH
11	5	5	M	M
12	3	3	L	L
13	1	1	L	L
14	12	12	L	L
15	12	12	L	L
16	3	3	L	L
17	16	16	H	H
18	18	18	H	H
19	16	16	H	H
20	18	18	H	H
21	16	16	H	H

Table 8. Simulation Experiments—Comparison of DV* for Max-Min and Bold Operator with α-Cut of 0.5

Case No.	DV*, Max-Min	DV*, Bold
2	(0.50, 1.00, 0.50)	(0.65, 1.00, 0.65)
3	(0.50, 1.00, 0.50)	(0.50, 1.00, 0.50)
4	(0.50, 1.00, 0.50)	(0.506, 1.00, 0.506)
5	(0.50, 1.00, 0.50)	(0.646, 1.00, 0.646)
6	(0.50, 1.00, 0.50)	(0.646, 1.00, 0.646)
7	(0.75, 1.00, 0.75)	(1.00, 1.00, 1.00)
12	(1.00, 0.50, 0.00)	(1.00, 1.00, 0.00)
13	(1.00, 0.50, 0.00)	(1.00, 1.00, 0.00)
14	(1.00, 0.75, 0.00)	(1.00, 1.00, 0.00)
15	(1.00, 0.75, 0.00)	(1.00, 1.00, 0.00)
16	(1.00, 0.50, 0.00)	(1.00, 1.00, 0.00)
17	(0.95, 1.00, 1.00)	(1.00, 1.00, 1.00)
18	(0.95, 1.00, 1.00)	(1.00, 1.00, 1.00)
19	(0.95, 1.00, 1.00)	(1.00, 1.00, 1.00)
20	(0.95, 1.00, 1.00)	(1.00, 1.00, 1.00)
21	(0.95, 1.00, 1.00)	(1.00, 1.00, 1.00)

ments based on the 21 cases shown in Table 6. Both of these experiments were run using the Hamming distance measure and the associated similarity measure as defined by Eq. (20) in identifying the rule to be selected and in identifying the linguistic descriptor to be displayed as the system response. It is observed in Table 7 that, with the exception of case 7, there were no differences in the rule selection. In particular, for case 7, we observe that max-min operators selected rule 5 but bold union/intersection selected rule 6. Furthermore, there were no differences in the system response in terms of the linguistic descriptors on the surface. However, if we look at the internal value of DV* vectors before it is approximated to a linguistic descriptor via the use of Hamming distance and similarity measures, we observe some differences (see Table 8). Hence, we realize that the similarity measures have a smoothing effect on the system response in terms of the linguistic descriptors. Whether such a smoothing effect is desirable or not may be context- and domain-dependent. Therefore, a system designer in cooperation with the users can decide whether to output the system response in terms of just the linguistic descriptors or just DV* or both. It should be noted that in Table 8 we list only cases 2–7 and 12–21, where the observed system states do not match the left-hand side of any rule in the rule base. Since cases 1, 8, 9, 10, and 11 have an exact match, we have no reason to list those cases. As explained in Example 1, the system response corresponds to the right-hand side of the rule as expected.

Let us now reconsider cases 2–7 and 12–21. They were selected randomly, and case 2 is explained in detail in Example 2. It is rather interesting to note that cases 2–7 describe an observed system behavior around "medium" and the system response is "medium," corresponding to our expectations. Similarly, when the system behavior is around "low," as in cases 12–16, the system response is "low," and when it is around "high" as in cases 17–21, the system response is "high," again corresponding to expectations. This is an indication of robust system behavior. Clearly these results support the hypothesis that approximate-reasoning-based expert systems would better serve the operations managers of such robust systems.

Finally, these simulation experiments appear to suggest that once the rule is identified with the help of a similarity measure we could directly fire the rule, that is, give the right-hand side of the rule as a response linguistic variable. This needs to be validated theoretically and experimentally in the future. If this finding is true, then there would be no need for a compositional rule of inference. This would be analogous to modus ponens in two-valued logic. Could this be true for the case of robust systems?

CONCLUSIONS

In this paper, we have described our approximate reasoning approach for an expert system design and development as an aid to management confronted with a production/inventory capacity problem. We have attempted to show how operations research and approximate reasoning can be synthesized for the solution of real-life problems in the era of knowledge-based systems. Operations research methodologies can generate valuable insights into the understanding of problem domains. Approximate reasoning provides a framework where the insight gained from OR models could be restructured for real-life problems either where there is insufficient information to identify parameters of the system at hand or where such data are not available owing to various factors and uncertain environmental conditions. Hence, the best we can do is rely on OR experts' assessments and the interpretation of real system behavior via model analysis. Since such assessments can best be expressed in a natural language setting with linguistic terms providing flexibility of expression in human judgments, and since approximate reasoning based on fuzzy logic can handle such linguistic uncertainties and imprecision, it appears that the marriage of operations research and approximate reasoning is inevitable with the advance of expert systems.

Approximate reasoning with linguistic variables and their terms provides aggregation of human knowledge as well as user friendliness. Thus, approximate reasoning is a reasonably good analog to human reasoning. This was illustrated with the second example in the last section. Furthermore, the simulated cases 2–7 and 12–21 discussed in the previous section indicate and support the commonsense reasoning that managers in charge of robust systems would get the appropriate support without detailed precision with the aid of approximate-reasoning-based expert systems. The approximate reasoning shows an advantage over analytical models in that it allows a certain degree of freedom from accuracy in the information and knowledge acquisition. This freedom and flexibility combined with the power of the aggregation due to linguistic variables allows expert system designers and developers to summarize the available knowledge in terms of far fewer rules than if one were to design expert systems that require precise information leading to rule explosion.

On the other hand, the extra intelligence provided in the inference procedure gives us a way to cope with situations not explicitly included in the rule base and hence with unforeseen future conditions in the observed system behavior. This means that the use of an expert system is not confined to the set of rules provided in the knowledge base.

There are certain issues that need more elaboration. Some of these are:

1. Throughout our discussion, point-valued fuzzy sets are employed instead of interval-valued ones, with the exception of CNF and DNF expressions, which creates interval-valued results for the logical combinations. However, by taking the average of the upper and lower bounds, respectively, we reduce the intervals to points. However, several experimental results obtained so far suggest that interval-valued fuzzy sets represent experts' assessment more naturally. Even though we have shown that interval-valued inference is quite possible, computations required for interval-to-interval inference are a lot more complex and costly at this point in our research (Turksen [9], Turksen and Zhong [3]).
2. In the inference procedure, we have used GMP and the Hamming distance and a similarity measure based on this distance measure. This choice needs justification. It is known that there are other distance measures and other similarity measures (Zwick et al. [18], Turksen and Zhong [3]). However, the choice of a distance and a similarity measure is still an open question. Until we identify the context-dependent effects of these distance and similarity measures, we cannot know how to choose an appropriate measure.

These issues are left for future research.

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APPENDIX

$$f_0(\omega) = A(e^{-\beta_1 \omega} - re^{-\beta_2 \omega})$$

$$f_1(\omega) = \frac{A}{\lambda}\left[(\mu - \lambda - \beta_1)e^{-\beta_1 \omega} - \frac{r}{\lambda}(\mu - \lambda - \beta_2)e^{-\beta_2 \omega}\right]$$

$$f_0 = \frac{A(1 - r)}{\lambda}$$

and

$$f_1^N = \frac{A}{\sigma}\left[\frac{\mu - \lambda - \beta_1}{\beta_1 - \mu}(e^{-\mu N}e^{-\beta_1 N}) - \frac{r(\mu - \lambda - \beta_2)}{\beta_2 - \mu}(e^{-\mu N} - e^{-\beta_2 N})\right]$$

where

$$\lambda A^{-1} = \frac{\mu}{\beta_1}\left(1 - \frac{r\beta_1}{\beta_2}\right) + \left(1 - \frac{\mu}{\beta_1}\right)e^{\beta_1 N} - r\left(1 - \frac{\mu}{\beta_1}\right)e^{-\beta_2 N}$$
$$+ \frac{\lambda}{\sigma}(\mu - \lambda - \beta_1)\left[\frac{\beta_2 - \beta_1}{(\mu - \beta_1)(\mu - \beta_2)}e^{-\mu N}\right.$$
$$\left. + \frac{e^{-\beta_1 N}}{\mu - \beta_1} - \frac{e^{-\beta_2 N}}{\mu - \beta_2}\right]$$

$$B = -Ar$$

and

$$r = \frac{(\mu - \lambda - \beta_1)/(\mu - \beta_1)}{(\mu - \lambda - \beta_2)/(\mu - \beta_2)}$$

where A, B, β_1, β, and r are intermediate parameters used in the analytical solution procedure.

A main concern of management is the fraction of satisfied customer demands. This can be expressed in terms of the model parameters as

$$\rho = \frac{\mu}{\lambda} A \left[\frac{1 - e^{-\beta_1 N}}{\beta_1} - \frac{r}{\beta_2}\left(1 - e^{-\beta_2 N}\right) \right]$$

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