Kuznetsov M.P., Strijov V.V. Methods of expert estimations concordance for integral quality estimation // Expert Systems with Applications, 2014, 41(4-2) : 1988-1996.

Methods of expert estimations concordance for integral quality estimation

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Abstract

The paper presents new methods of alternatives ranking using expert estimations and measured data. The methods use expert estimations of objects quality and criteria weights. This expert estimations are changed during the computation. The expert estimation are supposed to be measured in linear and ordinal scales. Each object is described by the set of linear, ordinal or nominal criteria. The constructed object estimations must not contradict both the measured criteria and the expert estimations. The paper presents methods of expert estimations concordance. The expert can correct result of this concordance.

Keywords: expert estimations, integral quality, object ranking, preference learning

1. Introduction

To make decisions about managed objects, for example, about nature protected areas or state regions, one must rank this objects according to an integral quality estimations, or, equivalently, construct a binary preference function over the set of objects. To construct the integral quality estimation several steps must be performed [1].

1. A quality criterion must be chosen to compare the objects.
2. A set of objects must be selected according to the quality criterion.
3. The expert must select a set of features describing the objects.
4. A design matrix "objects-features" must be fulfilled.
5. The expert estimations of the objects quality and the criteria weights must be collected. Further we suppose that multicollinearity of the criteria is not significant.

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We consider various scales for the expert estimations [2]: linear, ordinal and nominal. Each scale defines a method of transformation to be applied: for instance, any monotonic transformation can be applied to the ordinal scale.

Decision making and preference learning propose several methods to estimate the integral quality of objects [3, 4]. Unsupervised methods construct the estimation using the objects description and the quality criterion. This paper introduces the principal component analysis as an example of the unsupervised methods [5, 6]. According to this method, an integral quality estimation is a projection of the objects to a first component of the design matrix. Also Pareto slicing and metric method can be regarded as the unsupervised methods.

The supervised methods use expert estimations of the objects quality or the criteria weights [7] besides the design matrix. This paper presents the linear regression method [1], where the target variable is a vector of the expert-given object estimations. We consider linear and ordinal scaled expert estimations.

The paper considers a linear model for objects quality estimation [8]. The expert estimations of the criteria weights and objects quality are measured in the linear or ordinal scale. In general, model-computed object estimations doesn't equal expert-given estimations. This means that expert estimations and model-computed estimations contradict each other [9, 10]. We propose the method of the expert estimations concordance. We consider three various cases corresponding to the different types of measurement scales.

The first case considers linear scaled both expert estimations and measured data. We propose the estimations concordance method as follows. A set of the admissible expert estimations is a segment restricted by the maximum and minimum value of the estimation. The model uses a structure parameter to find the solution as a point of this segment.

The second case considers expert estimations of objects and criteria weights to be ordinal scaled. Ordinal-scaled expert estimations define a convex polyhedral cone. The design matrix defines a linear mapping of this cone to object space. The mapped cone can intersect a cone defined by the expert estimations of the objects. In this case the expert estimations of criteria weights and objects supposed to be concordant. In the converse case, we present a method of ordinal concordance. The method minimizes a distance between the vectors in the cones.

The third case considers ordinal scaled criteria [11]. The proposed method of objects quality estimation is as follows. Each criterion corresponds to the convex polyhedral con in objects space. According to the linear model, an admissible set of the object estimations is a Minkowski sum of the corresponding cones [12, 13]. The computed estimation is a projection of the expert estimation to the admissible set of values.

The data: Nature Protected Areas' Annual Report. We use the report to illustrate the proposed methods. Table 1 shows a part of the data. The problem is to estimate efficiency of each NPA using the measured data and the expert estimations.

Table 1: Nature Protected Areas report with the expert object qualities and criteria weights estimations

Object number	National park name	**Expert estimations of object qualities**	Number of environmental expertises	Number of individual grants	Number of PhDs	Number of employees	Number of organizations at the NPA territory	Number of journal papers	Number of journal authors	Number of students
			1.00	*0.85*	*0.78*	*0.70*	*0.69*	*0.58*	*0.48*	*0.29*
$\mathbf{x}_1$	NPA 1	**1.00**	3	0	1	4.5	3	3	2.5	0
$\mathbf{x}_2$	NPA 2	**0.83**	1	7	1	8	2	8.5	5	40
$\mathbf{x}_3$	NPA 3	**0.67**	2	1	1.5	9	1.5	9.5	6.5	66
$\mathbf{x}_4$	NPA 4	**0.63**	1	3	3	5	4.5	18	11	7
$\mathbf{x}_5$	NPA 5	**0.58**	0	12	8	19	11	7.5	11	62
$\mathbf{x}_6$	NPA 6	**0.50**	0	0	2	5	2.5	2.5	3.5	11
$\mathbf{x}_7$	NPA 7	**0.44**	1	5	4	11	20	16.5	15.5	4
$\mathbf{x}_8$	NPA 8	**0.38**	0	0	3	7	1	4.5	2.5	0
$\mathbf{x}_9$	NPA 9	**0.33**	0	6	0	7	1.5	7	4.5	1
$\mathbf{x}_{10}$	NPA 10	**0.17**	0	0	1	4	2	2.5	3.5	14.5

2. The integral quality estimation problem

Denote by $\mathbf{X} = \{\mathbf{x}_{ij}\}_{i,j=1}^{m,n}$ a design matrix "objects-features". The object description is a vector $\mathbf{x}_i$, the i th string of the matrix $\mathbf{X}$.

The object ranking problem is to find a binary relation $\prec$ defined on the set of object pairs, $\mathbf{x}_i \prec \mathbf{x}_k$. To solve this problem we find a mapping $f : \mathbb{X} \to \mathbb{R}$, where $\mathbb{X}$ is a set of object values. A set of values $\mathbb{R}$ of the mapping f has a natural binary relation, that is finding a mapping f is sufficient to solve object ranking problem.

Denote an integral quality estimation of the object $\mathbf{x}_i$ by y_i. We consider a linear model, where the value of integral quality y_i is a linear combination of the criteria, elements of the vector $\mathbf{x}_i$:

$$y_i = f(\mathbf{w}, \mathbf{x}_i) = \sum_{j=1}^{n} w_j x_{ij}. \tag{1}$$

Denote by $\mathbf{f}$ a vector of the values of the function f over the set of objects,

$$\mathbf{f} = [f(\mathbf{w}, \mathbf{x}_1), ..., f(\mathbf{w}, \mathbf{x}_m)]^\mathsf{T} = \mathbf{X}\mathbf{w}, \tag{2}$$

where $\mathbf{w}$ is a vector of criteria weights.

Let each criteria be mapped to the scale $[0, 1]$:

$$x_{ij} \mapsto \frac{x_{ij} - \min\limits_i x_{ij}}{\max\limits_i x_{ij} - \min\limits_i x_{ij}}, \quad i \in \{1, ..., m\}, \quad j \in \{1, ..., n\}.$$

This paper considers the case of the full rank of $\mathbf{X}$, $\text{rank}(\mathbf{X}) = n$, with $m \geqslant n$.

3. Unsupervised integral quality estimation

We will consider the principal component analysis as an unsupervised method for integral quality estimation. This method finds the objects' projections to the principal component coordinates such that the sum of squared distances between the objects and their projections to the first component is minimum. Consider the orthogonal matrix $\mathbf{W}$ from the linear combination $\mathbf{Z}^\mathsf{T} = \mathbf{X}^\mathsf{T}\mathbf{W}$ where columns $\mathbf{z}_1, ..., \mathbf{z}_n$ of the matrix $\mathbf{Z}$ have the maximum sum of variances,

$$\sum_{j=1}^{n} \sigma^2(\mathbf{Z}_j) \to \max,$$

where

$$\sigma^2(\mathbf{Z}_j) = \frac{1}{m}\sum_{i=1}^{m}(z_i - \overline{z})^2, \quad \overline{z} = \frac{1}{m}\sum_{i=1}^{m} z_i.$$

Columns of the matrix $\mathbf{W}$ are the eigenvectors of the covariance matrix $\mathbf{\Sigma} = \mathbf{X}^\mathsf{T}\mathbf{X}$. The matrix $\mathbf{\Sigma} = \mathbf{X}^\mathsf{T}\mathbf{X}$ can be found using singular values decomposition of the matrix $\mathbf{X}^\mathsf{T}\mathbf{X}$. Since $\mathbf{\Sigma} = \mathbf{X}^\mathsf{T}\mathbf{X} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^\mathsf{T}$, it follows that $\mathbf{\Sigma}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}$ is an eigenvectors system of the matrix $\mathbf{\Sigma}\mathbf{W}$.

Therefore the vector of object estimations $\hat{\mathbf{y}}_{\text{PCA}}$ is the projection of the row-vectors of the matrix $\mathbf{X}$ to its first principal component, and $\mathbf{w}$ is the first row-vector of the matrix $\mathbf{W}$. This vector corresponds to the maximum eigenvalue of the matrix $\mathbf{\Sigma}$.

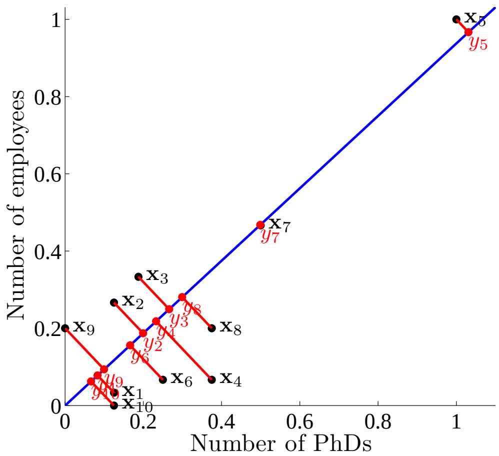


Figure 1: The principle component method illustration

Figure 1 illustrates principal component analysis for objects quality estimation. The black points are the NPAs from the table 1 describing by the criteria ¡¡Number of PhDs¿¿ and ¡¡Number of Employees¿¿. The PCA estimations y_i of the objects $\mathbf{x}_i$ are the points projections to the first principal component indicated by the blue line.

4. Supervised integral quality estimation

The supervised methods use the model (1), the design matrix $\mathbf{X}$, the expert estimations of the objects $\mathbf{y}_0$ or of the criteria weights $\mathbf{w}_0$.

4.1. The linear-scaled expert estimations

Weighted sum. Consider the linear-scaled expert estimations of the criteria weights $\mathbf{w}_0$. The vector of the object estimations is the linear combination,

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{w}_0.$$

This is the simplest method of objects estimation. The main drawback is the lack of robustness of the result estimations. The small changes of the expert estimations $\mathbf{w}_0$ may cause enormous changes in the result estimation.

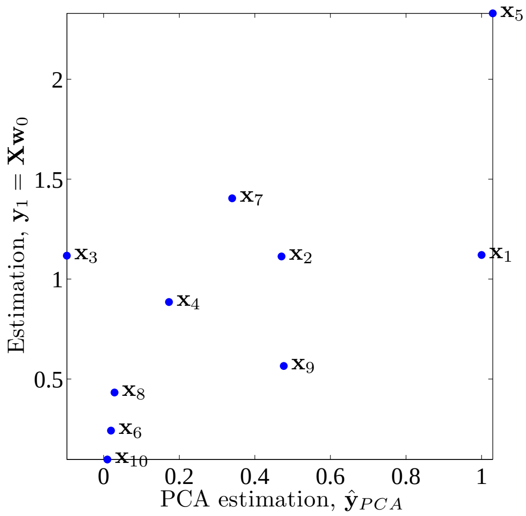


Figure 2: Comparison of the weighted sum method with the principal component method

Fig. 2 shows a comparison of the weighted sum estimation $\mathbf{y}_1 = \mathbf{X}\mathbf{w}_0$ with the principal component estimation $\hat{\mathbf{y}}_{\text{PCA}}$. Each point is an object, NPA. The x-axis shows the principal component estimation $\hat{\mathbf{y}}_{\text{PCA}}$, the y-axis — the weighted sum estimation $\mathbf{y}_1$.

Expert-statistical method. Consider the linear-scaled expert estimations $\mathbf{y}_0$. The method obtains the criteria weights $\hat{\mathbf{w}}$ as the argument of minimum of a distance between the expert estimations $\mathbf{y}_0$ and the computed estimations $\mathbf{y}_0' = \mathbf{X}\hat{\mathbf{w}}$,

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}\in\mathbb{R}^n} \|\mathbf{X}\mathbf{w} - \mathbf{y}_0\|^2.$$

The solution of this problem is given by the ordinal least squares method,

$$\hat{\mathbf{w}} = (\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}\mathbf{X}^\mathsf{T}\mathbf{y}_0. \tag{3}$$

The vector of the object estimations $\mathbf{y}_0' = \mathbf{X}\hat{\mathbf{w}}$. This vector is contained in space of the columns of the matrix $\mathbf{X}$ and is the nearest vector to the $\mathbf{y}_0$.

5. The expert estimations concordance

In this section we will consider both expert estimations of the objects $\mathbf{y}_0$ and of the criteria weights $\mathbf{w}_0$.

5.1. Linear concordance of the expert estimations

Consider the computed object estimations $\mathbf{y}_1 = \mathbf{X}\mathbf{w}_0$ using the vector $\mathbf{w}_0$ and the computed criteria weights $\mathbf{w}_1 = \mathbf{X}^+\mathbf{y}_0$ using the vector $\mathbf{y}_0$. Here the pseudo-inverse linear operator $\mathbf{X}^+ = (\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}\mathbf{X}^\mathsf{T}$. In other words, the linear operator $\mathbf{X}$ maps the expert estimations $\mathbf{w}_0$ to the vector $\mathbf{y}_1$, and pseudo-inverse linear operator $\mathbf{X}^+$ maps the expert estimations $\mathbf{y}_0$ to the vector $\mathbf{w}_1$. In general case the computed and the expert-given estimations are different, $\mathbf{y}_1 \neq \mathbf{y}_0, \mathbf{w}_1 \neq \mathbf{w}_0$.

Call the expert estimations $\mathbf{y}$ and $\mathbf{w}$ *concordant* if the following conditions hold:

$$\mathbf{y} = \mathbf{X}\mathbf{w}, \quad \mathbf{w} = \mathbf{X}^+\mathbf{y}. \tag{4}$$

Hereafter we find the expert estimations under the conditions of concordance (4). Denote by $\mathbf{y}_0' = \mathbf{X}\mathbf{X}^+\mathbf{y}_0$ the projection of the vector $\mathbf{y}_0$ to the space of the columns of the matrix $\mathbf{X}$.

α-concordance method of the expert estimations. To resolve the contradiction in expert estimations let us consider the estimations

$$\mathbf{w}_\alpha \in [\mathbf{w}_0, \mathbf{w}_1] \text{ and } \mathbf{y}_\alpha \in [\mathbf{y}_1, \mathbf{y}_0']. \tag{5}$$

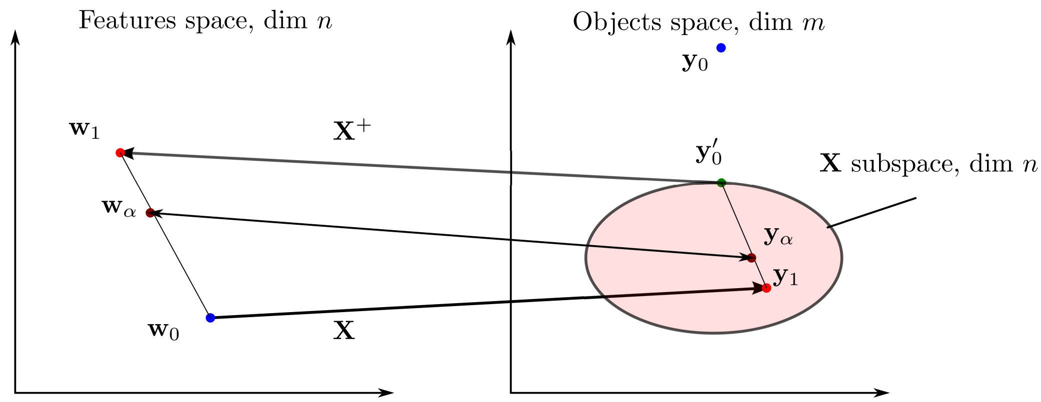


Figure 3: The α-concordance method illustration

The pair of vectors $\mathbf{w}_\alpha, \mathbf{y}_\alpha$ for the given α is defined by the following conditions,

$$\mathbf{w}_\alpha = \alpha\mathbf{w}_0 + (1-\alpha)\mathbf{X}^+\mathbf{y}_0', \quad \mathbf{y}_\alpha = (1-\alpha)\mathbf{y}_0' + \alpha\mathbf{X}\mathbf{w}_0.$$

Theorem 1. *The vectors $\mathbf{w}_\alpha, \mathbf{y}_\alpha$ are concordant (4).*

Proof. It is easily proved that $\mathbf{X}\mathbf{w}_\alpha = \mathbf{y}_\alpha$ and $\mathbf{X}^+\mathbf{y}_\alpha = \mathbf{w}_\alpha$:

$$\mathbf{X}\mathbf{w}_\alpha = \alpha\mathbf{X}\mathbf{w}_0 + (1-\alpha)\mathbf{X}\mathbf{X}^+\mathbf{y}_0' = \alpha\mathbf{X}\mathbf{w}_0 + (1-\alpha)\mathbf{y}_0' = \mathbf{y}_\alpha,$$

$$\mathbf{X}^+\mathbf{y}_\alpha = (1-\alpha)\mathbf{X}^+\mathbf{y}_0' + \alpha\mathbf{X}^+\mathbf{X}\mathbf{w}_0 = (1-\alpha)\mathbf{X}^+\mathbf{y}_0' + \alpha\mathbf{w}_0 = \mathbf{w}_\alpha.$$

Fig. 3 illustrates the $\alpha-$ concordance method. The vectors $\mathbf{y}_0$ $\mathbf{w}_0$ are the expert estimations of the objects and of the criteria weights. $\mathbf{y}'_0$ is the nearest point to the $\mathbf{y}_0$ in the n-dimensional subspace of the columns of the matrix $\mathbf{X}$. The pair of vectors $\mathbf{y}_\alpha$, $\mathbf{w}_\alpha$ is the concordant pair of the expert estimations.

The parameter α defines expert preferences to the expert estimations of the objects versus the expert estimations of the criteria weights. If α tends to zero the expert prefers the estimations of the objects; if α tends to zero the expert prefers the estimations of the criteria weights. One could allow expert to assign the parameter α according to his own preferences. Another way to define the parameter α is to compute it as the argument of minimum of the residuals sum Q,

$$Q = \frac{\|\mathbf{w}_0 - \mathbf{w}_\alpha\|}{n} + \frac{\|\mathbf{y}'_0 - \mathbf{y}_\alpha\|}{m} \to \min_\alpha . \tag{6}$$

γ-concordance method of the expert estimations. The γ-concordance method refuses from the restrictions (5). It finds concordant estimations in the neighborhoods of the vectors $\mathbf{w}_0, \mathbf{y}'_0$ as a solution of the optimization problem (6) with a regularization parameter $\gamma^2 \in [0, +\infty)$:

$$\mathbf{w}_\gamma = \arg \min_{\mathbf{w} \in \mathbb{R}^n} (\varepsilon^2 + \gamma^2 \delta^2),$$

where $\varepsilon^2 = \|\mathbf{w}_0 - \mathbf{w}_\gamma\|^2$ and $\delta^2 = \|\mathbf{y}'_0 - \mathbf{y}_\gamma\|^2$. The solution of this problem is the vector of criteria weights,

$$\mathbf{w}_\gamma = (\mathbf{X}^\mathsf{T}\mathbf{X} + \gamma^2 I_n)^{-1}(\mathbf{X}^\mathsf{T}\mathbf{y}'_0 + \gamma^2 \mathbf{w}_0), \tag{7}$$

and the concordant estimations of objects, $\mathbf{y}_\gamma = \mathbf{X}\mathbf{w}_\gamma$. The parameter γ^2 defines expert preferences to the expert estimations of the objects versus the expert estimations of the criteria weights, so as α.

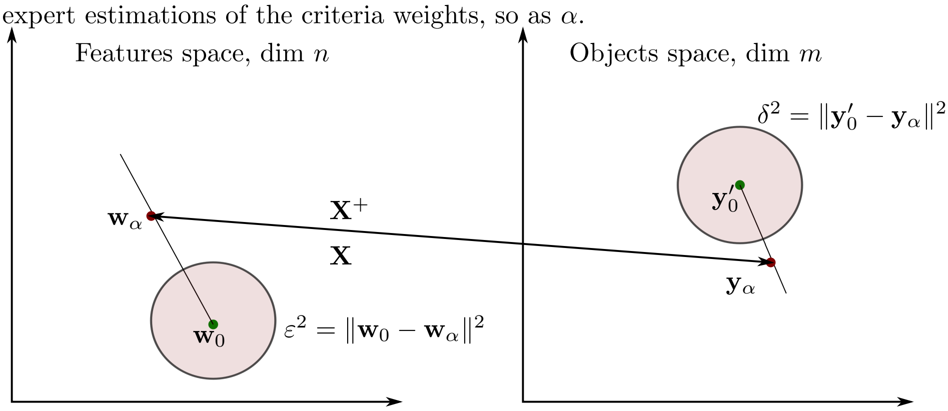
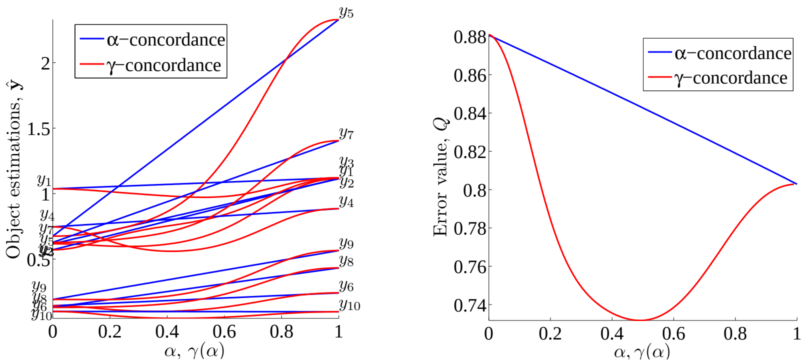


Figure 4: The γ-concordance method illustration

Fig. 4 illustrates the method of $\gamma-$concordance. The radiuses of the circumcircles of the points $\mathbf{w}_0$ and $\mathbf{y}'_0$ equal ε and δ, respectively.



(a) The change of the object estimations (b) The change of the error function

Figure 5: α- and γ-concordance illustration

Fig. 5 compares α- and γ-concordance methods. The x-axis shows the values of the parameter α changing from 0 to 1, whereas parameter γ is the function of α,

$$\gamma = \frac{\alpha}{1-\alpha},$$

so γ changes from 0 to ∞. The left figure shows object estimations changing. The blue lines indicate object estimations for α-concordance, the red lines indicate γ-concordance. The extreme cases correspond to the expert estimations $\mathbf{y}_0'$ and $\mathbf{y}_1 = \mathbf{X}\mathbf{w}_0$, respectively. The right figure shows changing of the error function (6). In the case of γ-concordance this function has a global minimum corresponding to the optimal value of the parameter γ.

Fig. 6 compares the expert-statistical method and the γ-concordance method for the optimal value of γ defined by minimal value of the error function (6). The x-axis shows the expert-statistical estimations $\hat{\mathbf{y}}_{\text{OLS}}$. The y-axis shows the γ-concordance estimations $\hat{\mathbf{y}}_\gamma$.

5.2. Ordinal concordance of the expert estimations

Let the expert estimations $\mathbf{y}_0, \mathbf{w}_0$ be measured in ordinal scales. It means that the set of the estimations is linearly ordered. Consider a cone in Euclidean space corresponding to this set. Without loss of generality suppose that the cone is described by the set of vectors $\mathbf{y} \in \mathbb{R}^m$ with the following restrictions on the components of $\mathbf{y}$:

$$y_1 \geqslant y_2 \geqslant \ldots \geqslant y_m \geqslant 0; \quad w_1 \geqslant w_2 \geqslant \ldots \geqslant w_n \geqslant 0.$$

The set of such vectors $\mathbf{y}$ is described by the system of linear inequalities,

$$\mathbf{J}^m \mathbf{y} \leqslant 0,$$

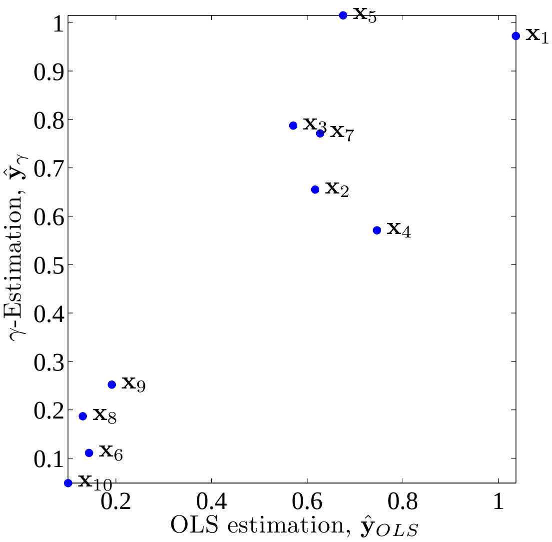


Figure 6: Comparison of the expert-statistical method with the γ-concordance method

where $\mathbf{J}^m$ is the $m \times m$ matrix,

$$\mathbf{J}^m = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

In the case of a random order $y_{i_1} \geqslant y_{i_2} \geqslant y_{i_m} \geqslant 0$, the matrix of the system will be constructed from the $\mathbf{J}^m$ by permutations of the corresponding columns.

In the same way, the cone of vectors $\mathbf{w}$ corresponding to the expert estimations of the criteria weights is described by the system of linear inequalities with the $n \times n$-matrix $\mathbf{J}^n$.

Therefore the expert estimations $\mathbf{y}_0$ and $\mathbf{w}_0$ correspond to the $m \times m$ and $n \times n$ matrices $\mathbf{J}^m$ and $\mathbf{J}^n$, respectively.

Correspondence of convex polyhedral cones to the expert estimations. Denote by $\mathcal{Y}_0$ and $\mathcal{W}_0$ the cones, corresponding to the expert estimations in the space of objects and in the space of features, respectively,

$$\mathcal{Y}_0 = \{\mathbf{y} | \mathbf{J}^m \mathbf{y} \leqslant 0\}, \quad \mathcal{W}_0 = \{\mathbf{y} | \mathbf{J}^n \mathbf{w} \leqslant 0\}.$$

The linear operator $\mathbf{X}$ maps the cone $\mathcal{W}_0$ of the expert estimations of the criteria weights $\mathbf{w}_0$ to the computed cone $\mathbf{X}\mathcal{W}_0$. The linear operator $\mathbf{X}\mathbf{X}^+$ maps the cone $\mathcal{Y}_0$ of the expert estimations of the objects $\mathbf{y}_0$ to the cone

$$\mathcal{Y}'_0 = \mathbf{X}\mathbf{X}^+\mathcal{Y}_0.$$

The cone $\mathcal{Y}'_0$ consists of the vectors from the subspace of columns of the matrix $\mathbf{X}$. Fig. 7 illustrated the defined cones.

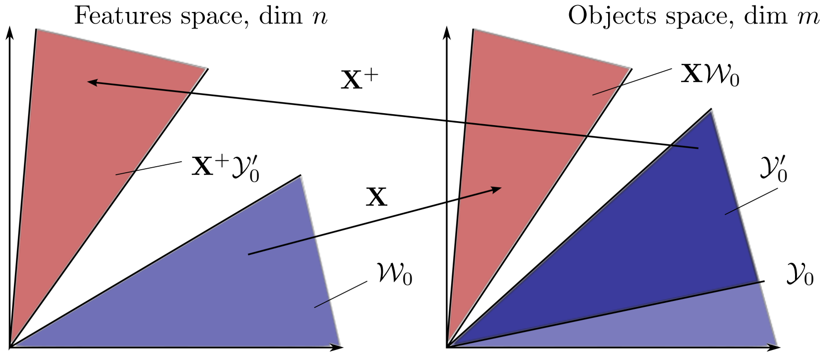


Figure 7: Cones corresponding to the expert estimations

The ordinal-scaled expert estimations $\mathbf{w}_0$ and $\mathbf{y}_0$ are called *concordant* if the cones $\mathbf{X}\mathcal{W}_0$ and $\mathcal{Y}'_0$ have a non-empty intersection besides the vector $\mathbf{0}$. In this case we can found vectors $\hat{\mathbf{y}} \in \mathcal{Y}'_0$ and $\hat{\mathbf{w}} \in \mathcal{W}_0$ satisfying the concordance conditions (4).

Properties of polyhedral cones. Now we introduce the following properties of the polyhedral cones corresponding to the expert estimations.

Lemma 2. *If two cones have vertices in the origin, their intersection is a polyhedral cone.*

PROOF. A polyhedral cone with the vertex in the origin is described by the system of linear inequalities. Let the first cone be described by the system $\mathbf{X}_1\mathbf{w} \geqslant \mathbf{0}$ and the second cone — by the system $\mathbf{X}_2\mathbf{w} \geqslant \mathbf{0}$. The intersection of this cones is described by the system with the matrix $\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}$. In other words, their intersection is the polyhedral cone with a vertex in the origin.

Lemma 3. *The locus* $\mathbf{X}\mathbf{w}$ *is a cone if* $\mathbf{X}$ *is a linear mapping.*

PROOF. For any vector $\mathbf{w} \in \mathcal{W}$ a vector $\lambda\mathbf{w} \in \mathcal{W}$. Therefore, if $\mathbf{y} \in \mathcal{Y}$, we get

$$\lambda\mathbf{y} = \lambda\mathbf{X}\mathbf{w} = \mathbf{X}(\lambda\mathbf{w}) \in \mathcal{Y} = \mathbf{X}\mathcal{W}.$$

This completes the proof.

It follows that if $\mathcal{W}$ is a polyhedral cone, the linear operator $\mathbf{X}$ maps it to the polyhedral cone $\mathbf{X}\mathcal{W}$. A corresponding pseudo-inverse operator $\mathbf{X}^+$ maps the cone $\mathcal{Y}$ to the cone $\mathbf{X}^+\mathcal{Y}$.

Lemma 4. $\mathcal{W}_0 \cap \mathbf{X}^+\mathcal{Y}_0' = \{\mathbf{0}\} \iff \mathbf{X}\mathcal{W}_0 \cap \mathcal{Y}_0' = \{\mathbf{0}\}$.

PROOF. Let us note that the cones $\mathcal{W}_0$ and $\mathcal{Y}_0' = \mathbf{X}\mathbf{X}^+\mathcal{Y}_0$ have the same dimension n since $rank(\mathbf{X}) = n$. This means that operators $\mathbf{X}, \mathbf{X}^+$ imply one-to-one correspondence from the features space to the space of the columns of the matrix $\mathbf{X}$.

Let the vector

$$\mathbf{w} \in \mathcal{W}_0 \cap \mathbf{X}^+\mathcal{Y}_0' \quad \mathbf{w} \neq \mathbf{0}.$$

Then the corresponding vector

$$\mathbf{X}\mathbf{w} \in \mathbf{X}\mathcal{W}_0$$

as well as

$$\mathbf{X}\mathbf{w} \in \mathbf{X}\mathbf{X}^+\mathcal{Y}_0' = \mathcal{Y}_0'.$$

That is the cones $\mathbf{X}\mathcal{W}_0$ $\mathcal{Y}_0'$ intersect at non-zero vector $\mathbf{X}\mathbf{w}$.

Let the vector

$$\mathbf{y} \in \mathcal{Y}_0' \cap \mathbf{X}\mathcal{W}_0 \quad \mathbf{y} \neq \mathbf{0}.$$

Then the corresponding vector

$$\mathbf{X}^+\mathbf{y} \in \mathbf{X}^+\mathcal{Y}_0'$$

as well as

$$\mathbf{X}^+\mathbf{y} \in \mathbf{X}^+\mathbf{X}\mathcal{W}_0 = \mathcal{W}_0.$$

That is the cones $\mathcal{W}_0$ and $\mathbf{X}^+\mathcal{Y}_0'$ intersect at non-zero vector $\mathbf{X}^+\mathbf{y}$.

From Lemma 4 it follows that for any vector $\mathbf{w}_p$ of the cone $\mathcal{W}_p = \mathcal{W}_0 \cap \mathbf{X}^+\mathcal{Y}_0'$ there exists a unique vector $\mathbf{y}_p' \in \mathcal{Y}_p' = \mathbf{X}\mathcal{W}_0 \cap \mathcal{Y}_0'$ such that the vectors $\mathbf{w}_p, \mathbf{y}_p'$ satisfy the conditions of concordance 4. Moreover, from the definition of the cone $\mathcal{Y}_0'$ it follows that for any vector $\mathbf{y}_p' \in \mathcal{Y}_p'$ there exists a unique vector $\mathbf{y}_p \in \mathcal{Y}_0$ such that $\mathbf{y}_p' = \mathbf{X}\mathbf{X}^+\mathbf{y}_p$. This implies that a concordant pair $\hat{\mathbf{w}}, \hat{\mathbf{y}}$ must satisfy following conditions,

$$\begin{cases} \mathbf{J}^n\hat{\mathbf{w}} \leqslant \mathbf{0}, \\ \hat{\mathbf{y}} = \mathbf{X}\mathbf{X}^+\mathbf{y}, \\ \mathbf{J}^m\mathbf{y} \leqslant \mathbf{0}. \end{cases}$$

Optimization problem for ordinal-scaled expert estimations concordance. Now we formulate an optimization problem for ordinal-scaled expert estimations concordance. We will find the nearest vectors $\hat{\mathbf{w}}$ and $\mathbf{y}_1$ in the cones $\mathcal{W}_0$ and $\mathcal{Y}_0$ as follows:

$$\begin{aligned} (\hat{\mathbf{w}}, \mathbf{y}_1) = \underset{\mathbf{w}\in\mathcal{W}_0, \mathbf{y}\in\mathcal{Y}_0}{\arg\min} \ \|\mathbf{X}^+\mathbf{y} - \mathbf{w}\|, \\ \text{subject to } \ \mathbf{y} \in \mathcal{Y}_0, \mathbf{w} \in \mathcal{W}_0, \|\mathbf{X}^+\mathbf{y}\| = 1, \|\mathbf{w}\| = 1, \end{aligned} \tag{8}$$

where $\|\cdot\|$ is the Euclidean metrics in the space $\mathbb{R}^m$.

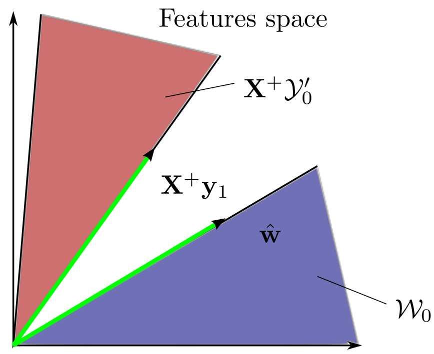


Figure 8: The nearest vectors of the cones

This means that the computed vector of criteria weights $\hat{\mathbf{w}}$ is a monotonic transformation of the vector $\mathbf{w}_0$. At the same time the objects estimation $\hat{\mathbf{y}} = \mathbf{X}\mathbf{X}^+\mathbf{y}_1$ is the nearest point to the expert estimation $\mathbf{y}_0$ from the subspace of the columns of the matrix $\mathbf{X}$. This method illustrated with fig. 8.

The problem of the nearest vectors can be solved by maximizing the rank correlation. That is, we will find the vectors $\hat{\mathbf{w}} \in \mathcal{W}_0$ and $\mathbf{y}_1 \in \mathcal{Y}_0$ such that Kendall correlation between $\hat{\mathbf{w}}$ and $\mathbf{y}_1$ is maximum:

$$(\hat{\mathbf{w}}, \mathbf{y}_1) = \arg \max_{\mathbf{w}\in\mathcal{W}_0, \mathbf{y}\in\mathcal{Y}_0} \rho(\mathbf{X}^+\mathbf{y}, \mathbf{w}) : \|\mathbf{X}^+\mathbf{y}\| = 1, \|\mathbf{w}\| = 1.$$

The algorithm of minimizing distance between vectors in cones. Rewrite the problem (8) as follows:

$$\begin{array}{lllll} \text{minimize} & \|\mathbf{X}^+\mathbf{y} - \mathbf{w}\| & & & \\ \text{subject to} & (\mathbf{X}^+\mathbf{y})^T\mathbf{X}^+\mathbf{y} = 1 & \text{and} & \mathbf{w}^T\mathbf{w} = 1, \\ & \mathbf{J}^n\mathbf{w} \leqslant \mathbf{0} & & \mathbf{J}^m\mathbf{y} \leqslant \mathbf{0}. \end{array}$$

To solve this problem we propose an iterative algorithm consequently finding approximations of vectors $\mathbf{y}^{(2k)}, \mathbf{w}^{(2k+1)}$ at every even and odd iteration. Define the vector $\mathbf{w}^{(0)} = \mathbf{w}_0$ at the iteration $k = 0$. Denote by $\mathbf{a} = \mathbf{y}^{(2k)}$ and $\mathbf{b} = \mathbf{w}^{(2k+1)}$ the solutions of two consequent optimization problems:

$2k$:		$2k+1$:	
minimize	$\|\mathbf{X}^+\mathbf{a} - \mathbf{w}^{(2k)}\|$	minimize	$\|\mathbf{X}^+\mathbf{y}^{(2k+1)} - \mathbf{b}\|$
subject to	$(\mathbf{X}^+\mathbf{a})^T\mathbf{X}^+\mathbf{a} = 1,$	subject to	$\mathbf{b}^T\mathbf{b} = 1 = 1,$
	$\mathbf{J}^m\mathbf{a} \leqslant 0.$		$\mathbf{J}^n\mathbf{b} \leqslant 0.$

While solving the optimization problem, define the constants

$$\mathbf{w}^{(2k)} = \mathbf{a}^{(2k-1)} \quad \text{and} \quad \mathbf{y}^{(2k+1)} = \mathbf{b}^{(2k)}.$$

Since the target function and inequality constraints are convex, the solution will be found for finite number of iterations. Methods of convex optimization to solve this problem are provided in [14]. To solve the problem of the rank correlation maximization we use a genetic algorithm.

In the case of a non-trivial intersection of the cones $\mathcal{W}_0$ and $\mathbf{X}^+\mathcal{Y}_0$ the solution of (8) is a vector $\hat{\mathbf{w}}$ from an intersection of this cones and a vector $\hat{\mathbf{y}}$ satisfying

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{X}^+\mathbf{y}_1 = \mathbf{X}\hat{\mathbf{w}}.$$

That is, vectors $\hat{\mathbf{y}}, \hat{\mathbf{w}}$ satisfy the concordance conditions (4). If an intersection of the cones is trivial, the proposed algorithm find the nearest non-concordant vectors. As in the case of linear scales, we present a method of the expert estimations concordance using a structure parameter α,

$$\mathbf{y}_\alpha = (1-\alpha)\hat{\mathbf{y}} + \alpha\mathbf{X}\hat{\mathbf{w}}.$$

Here the vector $\mathbf{y}_\alpha$ and the corresponding vector $\mathbf{w}_\alpha = \mathbf{X}^+\mathbf{y}_\alpha$ define the cones $\mathcal{Y}_\alpha$ and $\mathcal{W}_\alpha$, respectively. Furthermore, the intersection

$$\mathcal{Y}_\alpha \cap \mathbf{X}\mathcal{W}_\alpha \neq \emptyset.$$

As in the case of linear-scaled expert estimation concordance, the parameter α defines expert preferences to the expert estimations of the objects versus the expert estimations of the criteria weights. Below we present a method of constructing the linear-scaled estimations from the computed ordinal-scaled estimations.

Stable estimations with respect to the design matrix disturbance. Consider the computed cone $\mathcal{Y}_p = \mathcal{Y}_\alpha \cap \mathcal{W}_\alpha$ and the design matrix $\mathbf{X}$. Disturb the elements of this matrix,

$$\mathbf{X}_\Delta = \mathbf{X} + \Delta,$$

with a normal-distributed noise, $\Delta = \delta\mathbf{I}, \delta \sim \mathcal{N}(0, \sigma^2)$. An image of the linear mapping $\mathbf{y} = \mathbf{X}_\Delta\mathbf{w}$ is also normally distributed. According to the hypothesis call a stable solution $\mathbf{y}_p$ the central vector of the cone $\mathcal{Y}_p$ under the condition $\|\mathbf{y}_p\| = 1$. The so-called Chebyshev point $\mathbf{y}_p$ is a center of an incircle of the cone $\mathcal{Y}_p$.

Find the maximum distance from the target vector $\mathbf{y}_p$ to the cone faces as follows:

$$\hat{\mathbf{y}}_p = \arg\max_{\mathbf{y}_p \in \mathcal{Y}_p} \{\|\mathbf{y}_p - \mathbf{b}\|^2 : \mathbf{b} \in \mathbb{R}^m \setminus \mathcal{Y}_p, \|\mathbf{y}_p\|^2 \leqslant 1\}. \tag{9}$$

Fig. 9 illustrates the Chebyshev point $\mathbf{y}_p$ of the cone $\mathcal{Y}_p$.

Expert estimation concordance using isotonic regression. Consider the special case of the problem (8) such that the expert estimations of the objects $\mathbf{y}_0$ are linearly scaled and the expert estimations of the criteria weights are ordinal-scaled. In this case, the problem can be formulated in the terms of the well-known isotonic regression problem [15, 16].

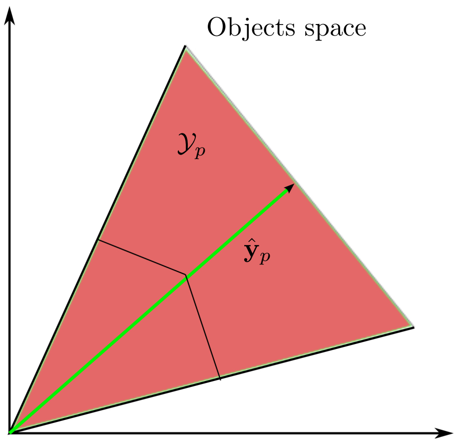


Figure 9: An stable solution: Chebyshev point $\mathbf{y}_p$

Let $\widetilde{\mathbf{w}} = X^{+}\mathbf{y}_0$. Find the monotonic sequence $w_1 \leqslant ... \leqslant w_n$ as the nearest to the vector $\widetilde{\mathbf{w}}$,

$$\begin{cases} \widehat{\mathbf{w}} = \arg\min_{w\in\mathbb{R}^n} \sum_{i=1}^{n} (\widetilde{w}_i - w_i)^2, \\ w_n \geqslant ... \geqslant w_1. \end{cases}$$

To make the expert estimation concordant, rewrite this problem with the structure parameter λ. If λ tends to zero the expert prefers the estimations of the objects; if λ tends to zero the expert prefers the estimations of the criteria weights. Find the vector $\widehat{\mathbf{w}}$ such that:

$$\widehat{\mathbf{w}} = \arg\min_{\mathbf{w}\in\mathbb{R}^n} \left(\frac{1}{2}\sum_{i=1}^{n}(\widetilde{w}_i - w_i)^2 + \lambda \sum_{i=1}^{n-1}(w_i - w_{i+1})_+ \right).$$

To solve this problem we use an algorithm proposed at [15].

6. Expert estimations concordance for the ordinal-scaled criteria

This section considers the case of the ordinal-scaled criteria. Write the matrix $\mathbf{X}$ as the concatenation of its columns, $\mathbf{X} = [\boldsymbol{\chi}_1, ..., \boldsymbol{\chi}_n]$. In the case of the ordinal criteria, the geometric shapes corresponding to the columns of $\mathbf{X}$ are cones $\mathcal{X}_1, ..., \mathcal{X}_n$. As described above, each cone is defined by the system of linear inequalities,

$$\mathcal{X}_j = \{\mathbf{x}_j | \mathbf{J}_j^m \mathbf{x}_j \leqslant 0\}, \quad j = 1, ..., m,$$

with the $m \times m$ matrices $\mathbf{J}_j^m$.

Consider a linear model of the objects quality estimation. In the terms of ordinal scales it means that the admissible set $\mathcal{X}$ of the object estimations $\hat{\mathbf{y}}$ is a set consisting of the all possible sums of vectors,

$$\mathcal{X} = \{\mathbf{x} | \quad \mathbf{x} = \mathbf{x}_1 + ... + \mathbf{x}_n, \quad \mathbf{x}_1 \in \mathcal{X}_1, ..., \mathbf{x}_n \in \mathcal{X}_n\}.$$

For the following consideration let us recall definition of the Minkowski sum. The Minkowski sum of two subsets L_1 and L_2 of the linear space is a set L' consisting of all possible sums of vectors from L_1 and L_2. Call an *admissible set for the linear model* a Minkowski sum,

$$\mathcal{X} = \mathcal{X}_1 + ... + \mathcal{X}_n.$$

To estimate vector of object qualities we construct an admissible set as the Minkowski sum of the convex polyhedra $\mathcal{X}_1, ..., \mathcal{X}_n$. To do this we use a method from [13]. The proposed method computes a matrix of a system of linear inequalities describing the sum of the polyhedra. The description of this method is given below.

Let two convex polyhedra $\mathcal{X}_1$ and $\mathcal{X}_2$ be described by the following system of inequalities:

$$\mathcal{X}_1 = \{\mathbf{x}_1 | \mathbf{J}_1\mathbf{x}_1 \leqslant \mathbf{b}_1\}, \quad \mathcal{X}_2 = \{\mathbf{x}_2 | \mathbf{J}_2\mathbf{x}_2 \leqslant \mathbf{b}_2\}.$$

The Minkowski sum of the polyhedra is the vector $\mathbf{x}$ satisfying the following conditions:

$$\begin{cases} \mathbf{x} - \mathbf{x}_1 - \mathbf{x}_2 = 0, \\ \mathbf{J}_1\mathbf{x}_1 \leqslant \mathbf{b}_1, \\ \mathbf{J}_2\mathbf{x}_2 \leqslant \mathbf{b}_2. \end{cases}$$

Transform the system replacing the variable $\mathbf{x}_1 = \mathbf{x} - \mathbf{x}_2$:

$$\begin{cases} \mathbf{J}_1\mathbf{x} - \mathbf{J}_2\mathbf{x}_2 \leqslant \mathbf{b}_1, \\ \mathbf{J}_2\mathbf{x}_2 \leqslant \mathbf{b}_2. \end{cases} \tag{10}$$

The following lemma describes the Minkowski sum of two polyhedra.

Lemma 5. $\mathbf{x} \in \mathcal{X}$ *if and only if it exists* $\mathbf{x}_2$ *satisfying (10).*

This means that to find a vector $\mathbf{x}$ one must solve the system of linear inequalities,

$$\mathbf{C}\mathbf{x}_2 \leqslant \mathbf{d}, \quad \mathbf{C} = \begin{pmatrix} -\mathbf{J}_1 \\ \mathbf{J}_2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{b}_1 - \mathbf{J}_1\mathbf{x} \\ \mathbf{b}_2 \end{pmatrix}.$$

To solve this system we use the following version of the Minkowski-Farkas lemma.

Lemma 6. *Let* $\mathbf{J}$ *and* $\mathbf{b}$ *be a matrix and a vector. The system of linear inequalities* $\mathbf{J}\mathbf{x} \leqslant \mathbf{b}$ *is solvable iff* $\mathbf{y}\mathbf{b} \geqslant 0$ *for any vector* $\mathbf{y}$ *satisfying the following conditions:*

$$\mathbf{y} \geqslant 0, \quad \mathbf{y}\mathbf{J} = 0.$$

In our case write the Minkowski-Farkas lemma as following,

$$\exists \mathbf{x}_2 : \quad \mathbf{C}\mathbf{x}_2 \leqslant \mathbf{d} \Leftrightarrow \forall \mathbf{z} : \mathbf{C}^T\mathbf{z} = 0, \quad \mathbf{z} \geqslant 0 \quad \to \quad (\mathbf{d}, \mathbf{z}) \geqslant 0.$$

Let $\mathbf{V}$ be a fundamental system of solutions (FSS) for this case. Therefore

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix},$$

where $\mathbf{V}_i$ is a FSS, corresponding to the matrix $\mathbf{J}_i$. It follows that the condition $(\mathbf{d}, \mathbf{z}) \geqslant 0$ must be rewritten as

$$\mathbf{V}_1^T(\mathbf{b}_1 - \mathbf{J}_1\mathbf{x}) + \mathbf{V}_2^T\mathbf{b}_2 \geqslant 0.$$

Denote

$$\mathbf{J} = \mathbf{V}_1^T\mathbf{J}_1, \qquad \mathbf{b} = \mathbf{V}_1^T\mathbf{b}_1 + \mathbf{V}_2^T\mathbf{b}_2,$$

and obtain the parameters $\mathbf{J}, \mathbf{b}$ of the system of inequalities describing the Minkowski sum $\mathcal{X}_1 + \mathcal{X}_2$.

To find the non-negative FSS of the system with the matrix $\mathbf{V}$ we use a method proposed in [13].

The solution of the concordance problem is the point $\hat{\mathbf{y}}$ nearest to the expert estimation $\mathbf{y}_0$ such that $\hat{\mathbf{y}} \in \mathcal{X}$. Having constructed the set $\mathcal{X}$, define the computed object estimations as the projection $P_{\mathcal{X}}(\mathbf{y}) \in \mathcal{X}$ satisfying the following conditions:

$$\hat{\mathbf{y}} = P_{\mathcal{X}}(\mathbf{y}) = \arg\min_{\mathbf{z}\in\mathcal{X}} \|\mathbf{y} - \mathbf{z}\|. \tag{11}$$

The projection is unique due to the convexity of the set $\mathcal{X}$. Fig. 10 illustrates a projection of the vector $\mathbf{y}_0$ to the admissible set $\mathcal{X}$.

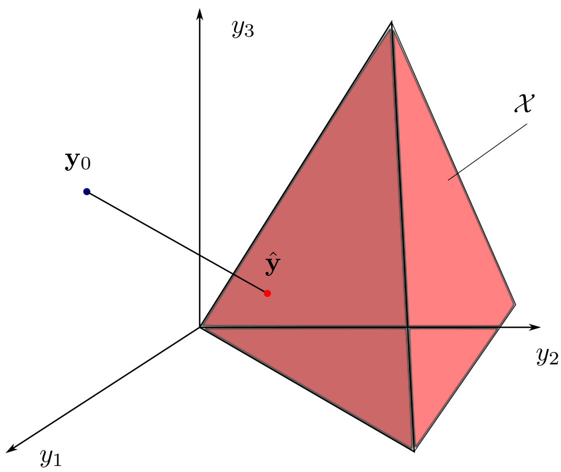


Figure 10: Projection of the point $\mathbf{y}_0$ to the cone $\mathcal{X}$

Fig. 11 compares the method of ordinal expert data (9) with the method of ordinal criteria (11). The x-axis shows the Chebyshev point estimation $\hat{\mathbf{y}}_{\text{Cheb}}$. The y-axis shows the projections $\hat{\mathbf{y}}_{\text{Cones}}$ to the admissible set $\mathcal{X}$.

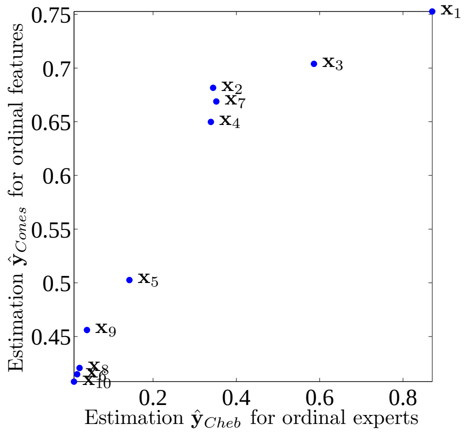


Figure 11: Comparison of the method of the ordinal expert data with the method of ordinal criteria

7. Computational experiment

Results for the real data. Table 2 shows estimations of the Nature Protected Areas obtained by the four proposed methods. The results are comparable. The specific method should be chosen according to some knowledge or assumptions about the data structure.

Analysis of the algorithms accuracy. The results of four proposed algorithms are demonstrated in the table 3. As the quality criterion we propose a correlation coefficient between expert estimation of objects $\mathbf{y}_0$ and the computed estimations $\hat{\mathbf{y}}$. We use Pearson correlation coefficient, $r(\mathbf{y}_0, \hat{\mathbf{y}})$, to measure the

Table 2: Integral quality estimations for the Nature Protected Areas

Object number	$\hat{\mathbf{y}}_{\text{OLS}}$	$\hat{\mathbf{y}}_{\gamma}$	$\hat{\mathbf{y}}_{\text{Cheb}}$	$\hat{\mathbf{y}}_{\text{Cones}}$
$\mathbf{x}_1$	1	2	1	1
$\mathbf{x}_2$	5	5	4	3
$\mathbf{x}_3$	6	3	2	2
$\mathbf{x}_4$	2	6	5	5
$\mathbf{x}_5$	3	1	6	6
$\mathbf{x}_6$	8	9	9	9
$\mathbf{x}_7$	4	4	3	4
$\mathbf{x}_8$	9	8	8	8
$\mathbf{x}_9$	7	7	7	7
$\mathbf{x}_{10}$	10	10	10	10

Table 3: Accuracy of the proposed algorithms

	$\hat{\mathbf{y}}_{\text{OLS}}$	$\hat{\mathbf{y}}_{\gamma}$	$\hat{\mathbf{y}}_{\text{Cheb}}$	$\hat{\mathbf{y}}_{\text{Cones}}$
Pearson, r	0.69	0.55	0.6	0.66
Kendall, τ	0.47	0.47	0.38	0.51

quality in the linear scales and we use Kendall correlation coefficient, $\tau(\mathbf{y}_0, \hat{\mathbf{y}})$, to measure quality in the ordinal scales. Note that the estimation $\hat{\mathbf{y}}$ was computed using Leave-One-Out method. Thus we can estimate a generalization ability of the proposed algorithms.

The results show the object estimations $\hat{\mathbf{y}}$ computed by the four proposed methods.

1. $\hat{\mathbf{y}}_{\text{OLS}}$ — estimations computed by the expert-statistical method (3),
2. $\hat{\mathbf{y}}_{\gamma}$ — estimations computed by the γ-concordance method (7),
3. $\hat{\mathbf{y}}_{\text{Cheb}}$ — estimations computed by the Chebyshev point finding (9),
4. $\hat{\mathbf{y}}_{\text{Cones}}$ — estimations computed by ordinal criteria method (11).

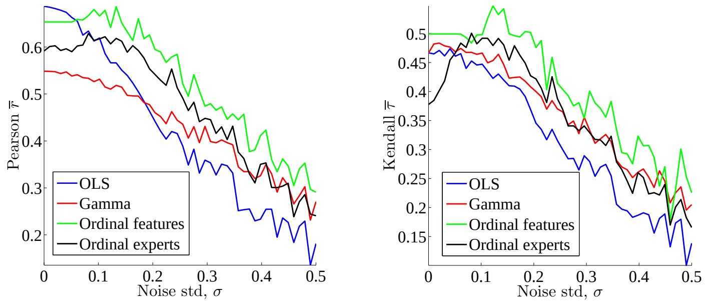
The results show that Pearson correlation has the maximum value for the OLS-estimation, whereas Kendall correlation is maximum for the ordinal criteria method.

Analysis of the algorithms stability. To analyse stability of the proposed algorithms we disturb the elements of the matrix $\mathbf{X}$. Consider the matrix $\mathbf{X}_\Delta = \mathbf{X}+\Delta$, where $\Delta(i,j) \sim \mathcal{N}(0,\sigma)$. We change the standard deviation σ of the disturbance from its minimum value $\sigma = 0$ to its maximum value $\sigma_{\max}$. Fig. (12) shows how changes quality criteria for the estimations $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{w}}, \mathbf{X}_\Delta)$. The left figure shows the changing of Pearson correlation. For the non-disturbed matrix $\mathbf{X}$ the expert-statistical method gives the best result but this result is less stable. The most stable results are indicated with the green line (ordinal criteria) and the black line (ordinal expert data).

Software implementation. The proposed methods were realized using MATLAB language. The open access code of algorithms and the computational experiment are located at [17].

The code consists of the two main modules. The module `comparison.m` performs pairwise algorithms comparison. The results of this module are illustrated at fig. (2), (11), (6). The module `test_noise.m` tests algorithms (9), (7), (11), (3) precision and stability. The results of this module are shown above in this section.

The input data structures `X, y0, w0` are the design matrix $\mathbf{X}$ and the expert estimations of the object qualities and of the criteria weights, respectively. This data correspond to the table 1.



(a) Pearson correlation, r (b) Kendall correlation, τ

Figure 12: Analysis of the algorithms stability for the disturbed matrix $\mathbf{X}$

8. Conclusion

The paper presents the methods of objects integral quality estimation based on expert estimations and measured data. Unsupervised and supervised methods are considered. The paper proposes the methods of the linear and ordinal expert estimations concordance. The methods use a structure parameter defining expert preferences to the expert estimations of the objects versus the expert estimations of the criteria weights. The paper presents the method of the stable object estimations construction and the method of the ordinal-scaled objects integral quality estimation. The error analysis is performed. The integral quality estimations are constructed for the Nature Protected Areas' annual reports.

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