Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 379 (2004) 439–455
www.elsevier.com/locate/laa

Student discussions on a Linear Algebra problem in a distance-education course

Asuman Oktaç

Departamento de Matemática Educativa, Centro de Investigación y de Estudios Avanzados (Cinvestav) del IPN, Av. IPN 2508, Col. San Pedro Zacatenco, México DF 07360, Mexico

Received 12 December 2002; accepted 26 August 2003

Submitted by F. Uhlig

Abstract

An introductory course on Linear Algebra was given at a distance, employing group work. We present one of the on-line discussions about a problem that involved eigenvalues and eigenvectors, commenting on the conceptual and logical difficulties of students, as well as the characteristics of the distance environment as far as they influence student interactions. We emphasize that our success in using new communicational technologies for instructional purposes will depend on our understanding of these new environments and this in turn will require change in points of view in interpreting teaching and learning phenomena.

Keywords: Distance education; Eigenvalues and eigenvectors; Chain of reasoning; E-mail discussion; Group discussion; Logical errors

1. Distance education

“Can we teach mathematics at a distance?” is a question that has generated a lot of discussion. With the developments in technology new ways of presenting and studying mathematics have been introduced. In some of these approaches technology is used as a communication tool whereas in others it serves the purpose of a pedagogical tool to enhance student understanding.

There is a considerable amount of general literature regarding the use of learning networks in education. Some of these studies refer to the principles of designing instruction and courseware for distance courses, while others compare on-line

E-mail address: oktac@mail.cinvestav.mx (A. Oktaç).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2003.08.021

and traditional approaches. There are also those that discuss the characteristics of on-line collaborative work and student attitudes towards this medium. Yet others choose possible influences of distance environments on the formation of learning and professional communities as their focus.

One study worth mentioning is the article by Blanton et al. [3] in which the authors review, using the perspective of social constructivism, several hundreds of articles regarding the application of computer-based telecommunications to teacher preparation. Their conclusion is that research in this area in general lacks theoretical and methodological rigor. They mention that many authors simply fit e-mail messages into categories and they present this as the result of their research, even though these categorizations seldom reflect a theoretical perspective. The authors also affirm that many studies make unsupported claims as to the participants' engagement in critical reflection, the democratization of relationships and environments, and the sharpening of analytic and verbal skills through writing as a result of telecommunications. The authors do, however, report that this technology seems to increase and improve communication among users.

Since in this article we will be analyzing an e-mail discussion, we think it is important to mention some of the general research results that involve asynchronous learning environments. Schahczenski [13] reports on an experiment in which students participated in on-line discussions about computer ethics. She concludes that since the students have time to reflect on what they write and structure it before sending their messages, the discussions in this environment become more like "the presentation of mini-essays". Furthermore, she contrasts the pace of the two discussion formats and states that on-line discussions develop at a slower pace whereas in-class discussions are short-lived, adding that student motivation is a key factor for on-line discussions to work.

Wang and Bonk [16] report that in a strategy that consists in "scaffolding" (Collins et al. [5] referred to in [16]) the teacher provides temporary support with difficult tasks, gradually fading this help until the students start working on their own. This study and several others acknowledge that e-mail and other computer conferencing tools have the power to do away with time and space constraints of traditional learning environments, offering an "anywhere, anytime" learning environment model. Of course this comment has to be evaluated taking into account issues of access and equity.

Anderson et al. [1] define "teaching presence" as having three categories—design and organization, facilitating discourse, and direct instruction. Their comments indicate how complex the teacher's already difficult role can become in a distance environment:

> Fulfilling the complex responsibilities of a teacher necessitates sustained and authentic communication between and among teachers and students. While control must be shared and choices provided, the discourse must also be guided toward higher levels of learning through reflective participation as well as by challenging assumptions and diagnosing misconceptions.

> This collaborative construction of knowledge is a challenge that all educators face. However, it is made extraordinarily difficult when it is the educator's responsibility to design, facilitate, and direct learning online.
> [. . .] Part of the challenge [. . .] is to develop compensatory behaviors for the relative lack of non-verbal and paralinguistic communication in a text-based medium such as computer conferencing. Another part of the challenge is to overcome the difficulty of conceiving the role of the teacher in online courses within the long established conceptual framework that we have built in the context of conventional, face-to-face teaching. [. . .] Especially in these "pioneering days" of online learning the thoughtful design of learning activities is critical to the attainment of educational outcomes. In the process of designing and using these tools, teachers are forced to be learners themselves and like all who experience learning, the learners themselves are changed [1].

With the idea that the instructor, taking the role of discussion leader, can impose an authoritarian presence, and that the students may become passive followers, Rourke and Anderson [12] report on the use peer teams where students took up the role of a facilitator in a graduate-level communications networks course. Their results indicate that the peer teams fulfilled each of the roles of the teacher presence mentioned above, and that the students preferred the peer teams to the instructor as discussion leaders. The authors state that although this method was "helpful in achieving higher order learning objectives", the discussions "could have been more challenging and critical" [12].

In the case of mathematics, there has not been much discussion concerning on-line collaborative work and interaction, or the kinds of tools that a distance-education environment can offer for studying students' conceptions, difficulties and mathematical reasoning. Among the few research studies that our search identified in this domain, Crowe and Zand [6] report on the results of a case study with distance students of mathematics, where the learning environment was improved through the use of electronic communication and enhanced by specialized software. The material that they used for their course contained diverse mathematical topics that involved geometry, functions, differentiation, recurrence relations and sequences. Students were encouraged to seek help electronically from their tutors when needed, and they were required to submit on-line assignments. One of the research questions that the authors posed is whether the students can work collaboratively on mathematics using electronic communication. In one phase of the study students were divided into small groups and were asked to work on an open-ended modelling project. Based on the results of this experiment, the authors conclude that electronic collaboration at a distance is possible.

Another paper in this category [9] mentions that their main objective was the exploration of the communication process that took place between a student studying systems of linear equations and his tutor. The student was an athlete who was not able to attend his regular classes and therefore was assigned a tutor with whom he could work at a distance. The author gives examples of the exchanges that occurred between the student and the tutor, without offering a detailed analysis. Due to the complexities of the communication, she calls for the formulation of new methodologies that take into account the written character of the interactions and their widely varying linguistic aspects, which would allow a more complete analysis.

In our previous work [10,11] we presented detailed information about a Linear Algebra course that was offered at a distance. This information included matters concerning instructional design, the team work that went into the preparation of the course, the didactical strategies employed and examples of on-line exchanges, as well as a discussion about the difficulties involved and the nature of the communication process. In [10], we report that on-line interactions, being asynchronous and writing-based, can help overcome conceptual difficulties in Linear Algebra experienced by the participants, resulting in correct solutions that have the consensus of all the group members.

2. This study

In this paper we will report on a particular on-line discussion that took place among a group of students that were trying to solve a problem involving eigenvalues and eigenvectors. That students face difficulties when learning Linear Algebra concepts can no longer be doubted. Several researchers have demonstrated that the abstract notions of Linear Algebra such as vector spaces and linear transformations cause considerable difficulties for novice students. The book on the teaching of Linear Algebra edited by Dorier [7] shows that these problems are general and not specific to one country or culture. Various studies have identified different causes for these problems and some have suggested ways to help students overcome them [8,15]. It is worth mentioning that except for the articles mentioned in the previous section, we found no study that specifically addresses the characteristics of distance teaching and learning, or that takes a critical look into the use of collaborative learning and on-line discussions in the case of Linear Algebra.

In the present study, the students who formed the discussion group were teachers at the "preparatory" level on different campuses of the Instituto Tecnológico y Estudios Superiores de Monterrey (ITESM) in Mexico and were enrolled in an introductory Linear Algebra course as part of a Master's in Education program. (After secondary school, at the preparatory level students are prepared to go on to the university.) In this section we describe the course and then we give a mathematical and conceptual analysis of the problem on which the students were working. In the next section we present the group discussion together with our interpretation of the interactions that took place, commenting on the effects of this virtual environment on the characteristics of the interaction.

2.1. The course

The instructor of the course was also the author of this paper. One of our didactical objectives in planning this course was to hand in the responsibility to the teachers who were taking it, in every single component of the course. The purpose of this was to give them the opportunity to progress towards becoming independent learners, to defend their points of view and to learn to collaborate with their colleagues using electronic means of communication. In this course the students had to read a lot of material on their own, try to make sense of the concepts involved, and share their ideas with the instructor as well as with their group members and the whole class. Most of the students who were enrolled in this course did not have any previous experience with the abstract concepts of Linear Algebra, although they were somewhat comfortable with solving systems of linear equations and performing matrix operations. Their contact with Linear Algebra had been mostly at an algorithmic level.

During satellite classes that were broadcast for 1.5 hours every two weeks, students saw the instructor on a television screen and were able to communicate with her by sending electronic messages that appeared on her computer screen, after first being filtered by the course assistant. The preparation of homework relied heavily on e-mail discussions. It is important to note that there was no lecturing by the instructor. During the satellite sessions selected homework questions were discussed, but only after they had been "handed-in". These sessions also served to clarify students' doubts and to emphasize important concepts and relationships between different concepts.

To do the homework students relied on their textbook [2] and "virtual" group discussions. This book was chosen due to the restriction that the textbook had to be in Spanish, and some others that we had considered for use were either out of print or were not available. We are using the word "virtual" in the sense here that in each group there were at most two people from the same campus, hence the students had to interact via e-mail or the internet with the others in their group. The discussion stage lasted less than two weeks from the time one satellite class ended up to about two days before the next one was held, so that there would be enough time to review the answers for the next session. It was one of the didactical contracts of this course that the initiative rested with the students. If there was enough effort to solve a particular problem, the instructor would help by asking further questions and giving hints, while trying to keep this to a minimum so that the students would rely on themselves for the solutions. The degree of this effort was not quantified, but it meant participating actively and on time in the group discussions towards the resolution of the homework problems.

2.2. The problem

Students were given the following chain of reasoning and were asked to find the flaw in it.

- "Let A and B be two $n \times n$ matrices, A with an eigenvalue λ and B with an eigenvalue μ.
- This implies that $Ax = \lambda x$ and $Bx = \mu x$ for some eigenvector x.
- We have

  $$ABx = A(\mu x) = \mu(Ax) = \mu(\lambda x)$$

  and

  $$BAx = B(\lambda x) = \lambda(Bx) = \lambda(\mu x)$$
- Since $\mu(\lambda x) = \lambda(\mu x)$, AB and BA have the same eigenvalues."

A second part asked whether the two matrix products AB and BA do in fact always have the same eigenvalues.

This is not a traditional problem in the sense that it does not ask the students to prove a mathematical statement. Rather, it demands that the students try to follow the presented chain of reasoning and find where it fails. Complicating the situation further is the fact that under the given condition that A and B are square matrices, the two products AB and BA do indeed have the same eigenvalues. One might expect that a student who has difficulties with logical and mathematical reasoning might group the given "proof", its steps and its result together, concluding that since the result is true, there is nothing wrong with its "proof". Others might pay more attention to the reasoning and discover that something is wrong with it and then answer—erroneously—the second part of the question in the negative, by simply thinking that "if there is a mistake in one line of the given proof, this makes the result wrong, too". On the other hand, a student who is at ease with mathematical theory and logic would conclude that the second part might be true or false independently of the chain of reasoning presented above.

The reason we chose to present on-line student exchanges on this specific problem is its explicit logical aspect and the fact that it caused difficulties for many students. One of the issues that has been identified as problematic in introductory courses is student difficulties with logic, but as we shall see, the nature of these difficulties could be quite different from what we as the instructors assume.

We would like to note that by our presentation of a step-by-step reasoning and by asking where the mistake lies, instead of presenting a statement and its "proof" separately, we have placed an emphasis on following the logic of an argument and on deciding on its coherence. As we point out when we discuss the students' contributions, some confusion might have resulted from this. However we preferred this version as it relates more closely to the kind of thinking that we wanted to develop in the students, in line with the objectives of the course.

2.3. Purpose of this study

Let us suppose that one group of students turn in the following argument as their final answer to this homework problem.

Group answer

That the eigenvector x is considered from the beginning as if it were the same in: $Ax = \lambda x$ and $Bx = \mu x$. In any case what the above outline [referring to the chain of reasoning] shows is that AB and BA for the same eigenvector have the same eigenvalues. That is, the mistake is in the way that the information is presented.

Do AB and BA have the same eigenvalues? If A and B are similar, then they have the same eigenvalues. If A and B are similar their multiplication will also be similar, therefore AB and BA will have the same eigenvalues.

What sense can we make of this answer as an instructor? Can we pinpoint the conceptual difficulties of students, concerning the notions of eigenvalues and eigenvectors? Can we guess where their logical errors lie?

In what follows, by presenting the solution process of the group who came up with this answer, we hope to show that the analysis of students' registered on-line discussions will on the one hand provide possible answers to these questions, and on the other hand they will guide us into directions for choosing further didactical strategies in dealing with student difficulties. Our comments will include observations about the students' conceptual difficulties with the linear algebra concepts, the problems with their logical reasoning, as well as the nature of the interactions that took place, as we believe that the cognitive and the interactional (social) aspects together provide us with clues as to how knowledge is constructed and the difficulties that arise. We also hope to shed some light into some of the new teaching/learning phenomena occurring as a result of new technological environments and warn against not noticing or misinterpreting them.

3. Group discussion

The group whose interactions we reproduce below is composed of four students, whose names have been changed to protect their identity. After every message that was posted by a student to his/her group we insert comments regarding his/her contribution to the solution. Our comments should be taken as possible interpretations of the situations, and not as claims to truth, as we are aware that other interpretations might be possible. As Sfard [14] points out, "the only viable possibility for the researcher is to provide a *convincing interpretation* of the observed phenomena, as opposed to their definitive explanation. The interpretation should try to be as compelling, cogent, and trustworthy as possible, but it will nevertheless always remain subject to questioning and modifications". In fact, the registered transcripts make it possible to share these episodes with other teachers and researchers with the purpose of entering into a dialogue about the nature of the teaching–learning process and in order to obtain insights into the observed phenomena.

In what follows we extract only those parts of messages that correspond to the particular problem presented above, although the group was discussing several homework questions at the same time. For each message we indicate the day and time when it was posted, counting as Day 1 the day after the previous satellite class was held. The transcripts are translated from Spanish.

> (Day 3, 11:21) *María*: Here I don't see the trick that they are doing. What I know, is that they assume that ABx and BAx are scalar multiples of x, which is false... Well, I will keep thinking about it.

Comments: One possibility for what is bothering María is the existence of two lines in the chain of reasoning affirming that $ABx = \mu(\lambda x)$ and $BAx = \lambda(\mu x)$. She might be seeing the right hand sides of these equations as a scalar multiple of a vector resulting in a vector, whereas the left hand side gives her the impression of an expression resulting in a matrix, therefore ignoring that the resulting matrix in both sides would have dimensions $n \times 1$. In this case what attracts her attention would be the "scalar" part, pointing out to the impossibility of multiplying x by a scalar and obtaining the left hand side. This might have to do with the visual effect of the expressions ABx and BAx being different from the visual effect that a scalar multiple of x produces, leading to the conclusion that they cannot be the same size. The other possibility could be that she sees BAx and ABx as vectors, but not as multiples of x. These two possibilities would imply difficulties with the structure and size of products involving matrices and vectors. Yet another interpretation is that she is concerned that the same x appears everywhere, although she does not say anything specific to that effect.

María points out to something which she says that she knows is false, however her answer indicates that she does not offer this as the solution to the problem. In other words she does not consider this as the failing piece in the argument. Furthermore she uses the term "assume" incorrectly, since in the argument this result follows from a previous error, and it is not one of the starting assumptions.

María was the one who opened the solution process to this problem on Day 3. Although the group members had exchanged opinions about the other homework questions during the first two days, no one had commented about this problem, which might be an indication that they were all having difficulty with it.

> (Day 5, 21:19) *John*: I am sending my ideas about the problem.
>
> (a) Here, they assume that $AB = BA$ which is false and you can show it for almost any $n \times n$ matrix.
> The idea that they have is that if
> $AB = BA$
> then
> $ABx = BAx$
> and then,

$A(\mu x) = B(\lambda x)$
$\mu(Ax) = \lambda(Bx)$
$\mu(\lambda x) = \lambda(\mu x)$
which is what is false because we cannot start with $AB = BA$ (multiplication of matrices is not commutative).

(b) Here I have tried some instances where I found the eigenvalues for AB and BA and they come out with the same eigenvalues. Until now nothing occurs to me as to why. I am going home.

Comments: John's message shows his confusion mainly of a logical nature, that is with the assumptions that are being made. Although nowhere in the chain of reasoning is the equality of AB and BA mentioned or used, John has the impression that it is being assumed. Our interpretation is that he starts with the conclusion obtained somewhere from the given argument that $ABx = BAx$, and thinks that one way to obtain this is by assuming that $AB = BA$. He does not think of the possibility that other assumptions might lead to this result, neither does he concentrate on what is being claimed erroneously in the chain of reasoning. This might point out to his confusion between equivalence and implication. We can compare his behavior to the attitude of a student who thinks that if we have a homogeneous system of linear equations, the solution set contains only the zero vector. However we note a difference in these two cases: in the case of the solution to a system of equations the student is choosing the trivial solution and ignoring the possibility of having other solutions. In the case of our problem, the student is imposing an assumption based on a result that is obtained. In a way, for him, this result itself is imposing the condition, as according to him the result cannot have been obtained in any other way. We also note that John changes the quantifier in the equation $ABx = BAx$ from a "there exists" to a "for all", consequence of assuming $AB = BA$. This might imply a difficulty with quantifiers on his part. In the second part of his answer, he seems surprised by the observed equality of the eigenvalues of AB and BA. He tries to make sense of it, but cannot. Note that John repeats the error in the given chain by making $Bx = \mu x$ and $Ax = \lambda x$ at the end of his argument.

From an interactional point of view, John presents his individual progress so far. He does not pay attention to the idea that María had suggested and does not comment anything about it.

(Day 6, 21:21) *Javier*: I have been studying the problem and I haven't found anything wrong. I developed a general equation to calculate eigenvalues of 2×2 matrices one for AB and another for BA, and they come out to be identical. Tomorrow I will send more details.

(Day 7, 13:10) *Javier*: With respect to what I wrote to you about this problem, here are the details.

> I have analyzed the argument, I couldn't find the mistake. I don't see that they are assuming that $AB = BA$ either, because they are handling the equation ABx separate and independent from BAx.
> Out of curiosity I developed a general equation for the calculation of the characteristic polynomial of AB and another for BA for 2×2 matrices, and I arrived at the same equation. I did it in the following manner [he writes his calculations of AB and BA in terms of their entries, which we do not repeat here].
> Here we can see that AB is not equal to BA. Applying the steps for the calculation of characteristic polynomials in both cases I arrived at the following general equation [he gives the equation of the characteristic polynomial in terms of the entries of the matrices A and B].
> I tried this equation with some examples and yes they came out to be equal doing it directly and with this formula.
> Of course this doesn't mean that the same thing happens for any $n \times n$ matrix. I did this only to see what the result was with 2×2 matrices. I wanted to do it with 3×3 matrices, but I didn't have time any more, and anyway this doesn't show the argument in this problem is correct or false. As I said before, I couldn't find anything wrong. And you? We also have to think about the possibility that there is nothing wrong with the proof… until the opposite is shown.
> I will be waiting for your comments.

Comments: Although Javier could not find what is wrong with the given chain of reasoning, he makes some important contributions to the solution process. He refutes John's claim that AB and BA are assumed equal and he gives a reason for it: that the equations involving ABx and BAx are handled independently, and therefore there was no relation established between them beforehand. This explanation also assumes that somehow the relationship between ABx and BAx might imply an assumption about a relationship between AB and BA. After showing algebraically the equality of the eigenvalues of AB and BA for the 2×2 case, he correctly argues that no matter what the result would be for the 3×3 case, this would have no implication on the correctness or falseness of the proof. He suggests that there may be nothing wrong with it. His behavior in this part of the solution displays his readiness to accept different possibilities until something is proven beyond doubt.

In his first message, Javier also follows the pattern by presenting what he had done so far individually. However he lets his group know that more is coming. With his second message, he starts interacting more directly with the others in the group. He does not simply reject John's argument, but he explains to the others in detail by a logical argument why John's solution was not correct . We can claim that the fact that the student contributions are written and not oral, facilitates the task of examining them and commenting about them later. Then, when Javier starts showing his calculations involving the characteristic polynomials and mentions the inequality of AB

and BA, he seems to be following on John's claim that "AB and BA are not equal and you can show it for any $n \times n$ matrix". His explanations contain mathematical elements as well as what he thinks about what constitutes justification of an argument and the nature of mathematical proof, and it seems that he is willing to communicate with the members of his group about his ideas. He asks them directly if they could find anything yet and he ends his message by asking for feedback and more comments from his group members. His messages show that he is not simply presenting individual progress, but that he takes the communication process seriously.

> (Day 8, 19:57) *Instructor*: I am sending you my comments about your solutions. María observes that in the argument it is assumed that ABx and BAx are scalar multiples of x and that is false. Good observation but actually they don't assume it. This follows from other assumptions that they make. Javier answers to John that they never assume the equality $AB = BA$. What is wrong then?

Comments: I as the instructor intervened in order to motivate more discussion by specifying what I could agree with so far. My answer reflects that María's message appears to take the problem to be that these two expressions are multiples of the same vector x. However as pointed out above, María might have been bothered by something else.

I acknowledge all the contributions made so far summarizing the main points, mentioning María's message alongside others, as it had not received an answer from the group. I repeat the question, calling for more reflection.

> (Day 10, 15:07) *Raúl*: In this problem the mistake that I observe in the argument is that it is only valid for the same eigenvector x, or could it be that it shows that AB and BA have the same eigenvalue when they have the same eigenvector????, anyway I agree with Javier that for 2×2 matrices AB and BA have the same eigenvalues, I would extend this to 3×3 matrices (I verified it with various examples) and they also have the same eigenvalues, the proof is algebraically tedious but relatively simple, I think that this extends to all matrices A and B but until now I haven't been able to carry out a general proof.

> (Day 10, 21:27) *Raúl*: I am sending you what I have so far of the homework. We have not reached an agreement on this problem, the comments that María and Javier sent seem to be refuted by the instructor. In summary we have to go over this problem.

In a draft that contains all the solutions to the homework, he suggests the following answer for this problem:

> That the eigenvector x is considered from the beginning as if it were the same in: $Ax = \lambda x$ and $Bx = \mu x$. In any case what the above outline [referring to the chain of reasoning] shows is that AB and BA for the same eigenvector have the same eigenvalue.

Comments: Raúl realizes that the same vector x is used, although he has difficulty expressing exactly how, and what the implications of this are. He seems to be bothered by the fact that the same x that appears in $Ax = \lambda x$ and $Bx = \mu x$ also appears in the expressions involving AB and BA. He then interprets this as if in the chain of reasoning the mistake lied in assuming that instead of A and B, it is AB and BA that have the same eigenvector x. He reinterprets the statement and the proof in the lines of "if AB and BA have the same eigenvector, then they have the same eigenvalue". The rest of Raúl's arguments are concerned with establishing that AB and BA have the same eigenvalues, apparently continuing along the lines of Javier's thinking. He looks for a general proof but faced with the lack of it (as he could not produce one), he refers to his conviction based on various examples and local arithmetic proofs in special cases.

Raúl enters late into the discussion and following the example of the others' in the group, first he presents his individual solution to the problem. Since the group members were taking turns for sending the results and this homework was his responsibility, he sends a draft calling attention to this problem, as they had not reached a solution yet. He reminds the fact that María's and Javier's arguments were refuted by the instructor, so he offers his solution as a possibility. He calls for more discussion, as there was no consensus yet.

> (Day 10, 22:27) *Javier*: I want to make some comments about the problems of the homework that we haven't been able to conclude yet. It looks like the proof works at least with 2×2 and 3×3 matrices. I personally still don't see the mistake. With respect to Raúl's comment, where the mistake is that the argument works only for the same eigenvector x, I don't see it as a mistake, because if one eigenvector for A is used and another different one for B, we wouldn't have a point of comparison for the proof.

Comments: Javier interprets what Raúl says in terms of the choice of the same x for A and B. It is as if Javier is suggesting that in order to be able to say anything, it is obvious that we have to work with the same vector x. Otherwise, we would not be able to write the rest of this proof. Therefore his reasoning takes the body of the proof as a starting point and justifies the rest in terms of it. His message is illustrative of his conception of what constitutes a chain of reasoning. In this message he lets us have a glimpse into the implicit assumptions that he is making about the nature of the problem.

Javier accepts Raúl's statement that the result holds with 3×3 matrices as well. He takes into account Raúl's suggestion for a possible flaw in the argument, he seems to understand what this suggestion means and presents his reasons for not agreeing with it. At this point the conflict seems to be based on the differences of interpretation as to what a chain of reasoning consists of, and what the problem is asking.

> (Day 11, 9:38) *Instructor*: Raúl has the correct idea. The mistake is in assuming that A and B have the same eigenvectors. Now, it is true that AB and BA have the same eigenvalues for square matrices. For the general proof, I suggest that you see the next exercise in the manual.

Comments: The next exercise of the manual asked the student to read the article "Gems of Exposition in Elementary Linear Algebra" [4] that suggests a general proof. With this message the instructor re-words what in her opinion Raúl had suggested and gives them a hint as to how to write up their solution. However her interpretation of Raúl's solution may not have been correct (see the comments above on Raúl's contribution). This interpretation might have been influenced by what Javier took Raúl as saying.

The instructor's intervention at this time was due to the fact that the homework was due the next day, and the thought that after concentrating enough on this particular problem, the students were ready to read an outline of the proof and have a discussion about it.

> (Day 11, 9:55) *María*: It occurs to me to show that if A and B are similar, then they have the same eigenvalues. If A and B are similar their multiplication also will be similar. I don't think that there is a mistake in the argument, only that we need to add that they are both similar.

María suggests fixing the problem by adding a condition. She might be saying that similarity of the matrices A and B would guarantee them to have the same eigenvalues, hence in her reasoning (although she does not mention this explicitly) to have the same eigenvectors. Therefore according to her, the rest of the argument would remain correct. On the other hand she also seems to be implying that the similarity of A and B would guarantee the similarity of AB and BA, and consequently they would have the same eigenvalues.

In her second message, María continues her own efforts in solving the problem, without mentioning anything about the previous suggestions of the group members. She does not seem to take into account the instructor's message, either.

> (Day 11, 18:50) *John*: So we can conclude that in this problem the mistake is not in the argument itself but in the manner they present the information to be proved? Because if the mistake is that it cannot be assumed that x is the same eigenvector for both matrices, we have an error in giving the information to prove and not necessarily in the proof itself. Right?

Comments: John's reflection comes as a response to the instructor's intervention and reveals that he was considering the part that said "This implies that $Ax = \lambda x$ and $Bx = \mu x$ for some eigenvector x" as something that the problem asks to be proven. This might be due to the form that we are used to seeing in mathematical propositions and statements to be proved. However, if he were taking this sentence as the result to be proven, then he was not paying attention that actually the next step was using this assumption and at the end of the proof another result was being reached.

> (Turned in on Day 12, as it was due) *Group answer*: Final version of the homework:
> That the eigenvector x is considered from the beginning as if it were the same in: $Ax = \lambda x$ and $Bx = \mu x$. In any case what the above outline [referring to the chain of reasoning] shows is that AB and BA for the same eigenvector have the same eigenvalues. That is, the mistake is in the way that the information is presented.
> Do AB and BA have the same eigenvalues? If A and B are similar, then they have the same eigenvalues. If A and B are similar their multiplication will also be similar, therefore AB and BA will have the same eigenvalues.

Comments: This answer shows that the group could not put together enough information to present an answer that solves the problem. Neither did they check up on claims such as María's that if A and B are similar, so are AB and BA. Rather, they chose to mix the different opinions that had not been refuted so far, albeit not in a coherent way, as the time for turning in the homework had come. Apparently they did not read the suggested article. (John later mentioned in a message that he had never received this article in his course package.)

In the following satellite class this problem was commented in detail, underlining the difficulties that the groups displayed in their solution processes. Each group also received corrections to their homework. We believe that after spending so much time on this specific problem and displaying reasonable effort to solve it, students were motivated to find out the answers and understand the solution. In the next section we explain why we think this group of students could not reach a solution and had they had more time and proper guidance, this might have been possible.

4. Conclusions

One interesting thing in all the discussion that took place (as opposed to a traditional interaction between the students and the teacher) is that we have access to what the students say or think, and that is why it is so impressive to follow their "conversations". In this course students were encouraged to express freely what they thought and this contributed to their willingness to share their incomplete attempts to solve the problems. We can see the kind and level of mathematical reasoning that they can employ, where they fail, and how they try to convince each other. These records make it possible for the instructor to consult them whenever necessary for the purpose of identifying those aspects of their understanding that need attention. On the other hand, as is the case with face-to-face discussions, there may be student contributions that are not clear. One pedagogical suggestion that we may offer is to send a follow-up message asking for a clarification. This might motivate more reflection on the part of the student and give us more insight into his/her reasoning.

As a first impression one might think that the reason this group could not reach a solution is because they do not have the conceptual maturity to deal with the given problem and that they are lost. However, the fact that throughout the course this group was quite successful in reaching a consensus about their solutions and that these solutions were generally mathematically correct, leads us into considering other possibilities.

A careful reading of the students' contributions and the local analyses of these discussions offered above suggest that the conception that the students had of what constituted a chain of reasoning differed from what the instructor assumed it to be in giving them this problem. We think that the students saw this argument as having two parts: one part corresponding to the assumptions of a theorem and the result to be proved (that is the statement of the theorem), and another part that consisted in the proof itself. The wording that is used in the argument probably has contributed in interpreting it this way: the use of the expressions "Let—This implies that" pattern correspond to the statement, and the "We have-since" part forms the proof and leads to the conclusion. This assumption was incompatible with what the instructor thought everybody took as shared. María's use of the word "assume" in her first message, John's surprise shown in his last message ("So we can conclude that [...] the mistake is not in the argument itself but in the manner they present the information to be proved? [...] we have an error in giving the information to prove and not necessarily in the proof itself."), Javier's last statement ("I don't see it as a mistake, because if one eigenvector for A is used and another different one for B, we wouldn't have a point of comparison for the proof"), and Raúl's choice of the words in presenting the solution ("That the eigenvector x is considered from the beginning as if it were the same in: $Ax = \lambda x$ and $Bx = \mu x$") all point out to this implicit division of the argument into two parts. It is very possible that it was the first time that these students were dealing with a question of this type. If the only proofs they had seen involving Linear Algebra concepts were presented in their "proper" format, the identification of this argument as something that they were used to dealing with is understandable. As a result of this, the students may not even have questioned whether the statements $Ax = \lambda x$ and $Bx = \mu x$ follow from the previous assumption and might have simply taken it as something to be proved, concentrating on the rest of the argument.

Because of this incompatibility in interpretations, the solution process was not very productive. This phenomenon offers an explanation as to why the group may not have been able to reach a solution. The restrictions of trying to keep up with several group discussions at the same time unfortunately did not allow the instructor to realize the real nature of the difficulty on time to be able to intervene by having a discussion about the students' implicit assumptions and the characteristics of the homework problem.

In a way, not knowing the kind of logical and mathematical reasoning that the students go through when solving problems prevents us as mathematics instructors from discovering that in certain cases the students' reasoning might not coincide with our reasoning at all. We might not realize that the concepts that we teach might take a different form when the students are in the process of constructing them. On the other hand, looking only at the end product, i.e., the answer, might lead us to ignore the richness of their thinking and how much they might have progressed during the process of producing that answer. In summary, we learn a lot from the students' comments as to what ideas can be emphasized so that they can reach a deep understanding of the concepts involved. This can give us opportunities to tap into their understanding of mathematical conventions and help them reorganize their knowledge. However we need experience to get used to the opportunities offered by this medium and new points of reference in order to interpret the situations that we observe.

One aspect of the communication that takes place in this environment is its written and paused nature. This forces the students to elaborate more on their answers which in turn gives rise to interesting discussions. The fact that they have time to read and re-read messages before answering and can do the same with their own messages before posting them might force them to engage in more systemic and analytic thinking, compared to a spontaneous class discussion in which they would communicate verbally. However, we think that there might also be adverse effects of this kind of communication. Let us recall that in our example the instructor in her second message took Raúl's message to be related to the choice of the same x for both A and B. This was probably due to the fact that she had read Raúl's message followed by Javier's interpretation of it, instead of first forming an opinion herself about what Raúl meant to say. The asynchronous nature of the discussions modify the meaning of order and time as we are used to in our regular classes.

As a result, we can say that the on-line media offer possibilities for a real interaction to take place between the students themselves and the instructor if used intelligently. They might provide information which normally would not be available and that can be used to help students develop their mathematical understanding. However we need to be careful in adapting ourselves to this new medium as many of the didactical strategies and interpretations of phenomena that we are used to may change forms in this environment. For this reason it is important to share our experiences openly, discussing the benefits and the limitations of this environment. However we need to be careful in not interpreting the novelty that this medium brings as limitations as a result of our limited experience with it. We need to look into productive and non-productive episodes, teacher decisions, student interactions and their results, our immediate and long-term reactions and interpretations, etc., to be able to understand this new medium in its own terms. This will no doubt help us in using the communication technologies that are available to us for instructional purposes in a more informed manner. More research is needed with respect to this environment's role in the construction of mathematical concepts. We hope that our analysis will contribute to our understanding of the characteristics of this medium and its implications for mathematics education, and of its feasibility as an instructional medium for mathematics.

References

[1] T. Anderson, L. Rourke, D.R. Garrison, W. Archer, Assessing teaching presence in a computer conferencing context, J. Asynchronous Learn. Netw. 5 (2) (2001) 1–17 http://www.aln.org/publications/jaln/v5n2/v5n2_anderson.asp.

[2] H. Anton, Introducción al Álgebra Lineal, Limusa, México, 1997.

[3] W.E. Blanton, G. Moorman, W. Trathen, Telecommunications and teacher education: a social constructivist review, in: P.D. Pearson, A. Iran-Nejad (Eds.), Review of Research in Education, AERA, Washington, DC, 1998, pp. 235–275.

[4] D. Carlson, C.R. Johnson, D. Lay, A.D. Porter, Gems of exposition in elementary linear algebra, College Math. J. 23 (4) (1992) 299–303.

[5] A. Collins, J.S. Brown, S. Newman, Cognitive apprenticeship: teaching the crafts of reading, writing, and mathematics, in: L.B. Resnick (Ed.), Knowing Learning and Instruction: Essays in Honor of Robert Glaser, Lawrence Erlbaum Associates, Inc., Hillsdale, NJ, 1989, pp. 453–494.

[6] D. Crowe, H. Zand, Novices entering mathematics—I. The impact of new technology, Computers Educ. 28 (1) (1997) 43–54.

[7] J.-L. Dorier (Ed.), On the Teaching of Linear Algebra, Kluwer, Dordrecht, 2000.

[8] J.-L. Dorier, Meta level in the teaching of unifying and generalizing concepts in mathematics, Educ. Stud. Math. 29 (2) (1995) 175–197.

[9] L. Figueiras O., Teaching mathematics via internet: written interactions between tutor and student, in: Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, vol. 4, 2000, pp. 57–64.

[10] A. Oktaç, The teaching and learning of linear algebra: is it the same at a distance? in: Proceedings of the ICMI Study Conference on the Teaching and Learning of Algebra, vol. 2, 2001, pp. 501–506.

[11] A. Oktaç, H. Stein, Linear algebra at a virtual university, in: Proceedings of the International Conference on the Teaching of Mathematics, Samos, Greece, 1998, pp. 233–235.

[12] L. Rourke, T. Anderson, Using peer teams to lead online discussions, J. Interactive Media Educ. 1 (2002) 1–21 http://www-jime.open.ac.uk/2002/1.

[13] C. Schahczenski, Experiment substituting in-class discussions with email discussions, in: Electronic Proceedings for Frontiers in Education Conference, 1998. Available from <http://fie.engrng.pitt.edu/fie98/papers/1249.pdf>.

[14] A. Sfard, There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning, Educ. Stud. Math. 46 (1–3) (2001) 13–57.

[15] A. Sierpinska, T. Dreyfus, J. Hillel, Evaluation of a teaching design in linear algebra: the case of linear transformations, Rech. Didactique Math. 19 (1) (1999) 7–40.

[16] F.-K. Wang, C.J. Bonk, A design framework for electronic cognitive apprenticeship, J. Asynchronous Learn. Netw. 5 (2) (2001) 131–151 http://www.aln.org/publications/jaln/v5n2/v5n2_wang.asp.