Designing a Calculus Mobile

Tom Farmer

Tom Farmer (farmerta@muohio.edu) received his Ph.D. in mathematics at the University of Minnesota in 1976 and ever since then has served on the faculty at Miami University in Oxford, Ohio. In addition to gardening, reading, and digital photography, he continues to enjoy learning mathematics and how to teach it. The present paper grew out of discussions of the center of mass in an honors calculus class.

Problem: Design a mobile as in Figure 1. The parts are horizontal slices of a lamina bounded on the left and right by smooth curves $x = f(y)$ and $x = mf(y)$, where $m > 1$ is constant. Each part of the mobile is attached to the one above it by a single connector located along the graph of f, and the parts are intended to hang with their horizontal edges horizontal.

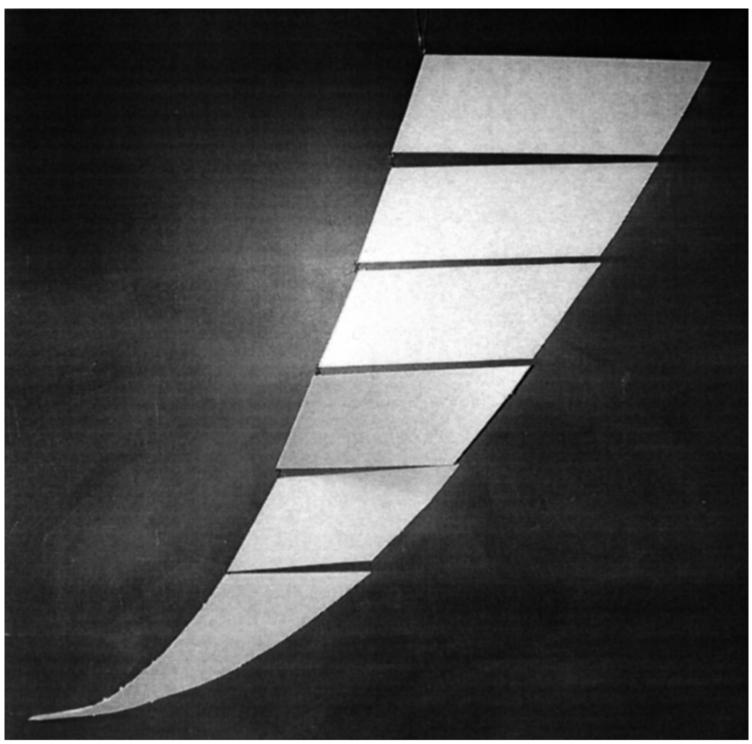

Figure 1.

This problem is connected with several topics from the latter part of first year calculus including the harmonic series, the center of mass of a lamina, and separable differential equations. The purpose of this paper is to show the connections and, in the end, to solve the problem.

Many calculus instructors have used a stack of blocks to illustrate the question of convergence or divergence of a series. Suppose n identical rectangular blocks are stacked as in Figure 2, with the top block extending half its length out from the second block, the second block extending one fourth its length out from the third block, then one sixth, then one eighth, and so on. As n gets large, is there an upper bound on the horizontal distance between the leading edge of the top block and the leading edge of the bottom block? The answer is no, of course. The horizontal distance grows without bound because the series $\sum_{n=1}^{\infty} 1/2n$ diverges.

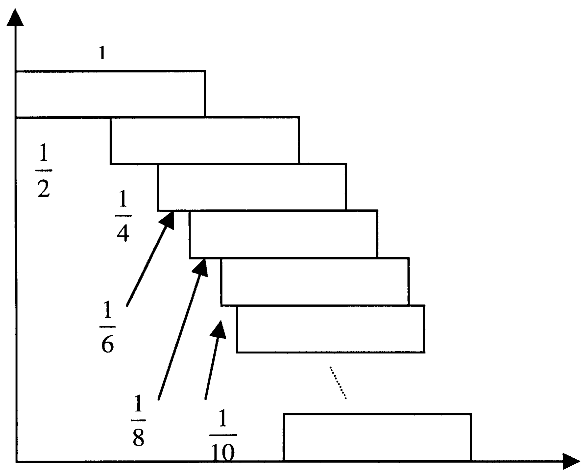

Figure 2.

An interesting, but somewhat hidden, aspect of the block-stacking demonstration is the fact that the blocks are positioned with a certain center of mass property. For example, the center of mass of the top two blocks (the two blocks being treated as one object) is directly above the edge of the third block. In general, the collective center of mass of the top n blocks lies directly above the edge of the next block for any n—the stack is extending out as far horizontally as possible without toppling over. In discussing this center of mass property, notice that, although the stack of blocks is three dimensional, we need only look at the two dimensional side view shown in Figure 2. As a planar lamina with constant density 1, the object represented by the side view of the stack has the property that its center of mass $(\bar{x}, \bar{y})$ has

$$\bar{x} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n},$$

and this is exactly where the edge of the next rectangle is positioned. This formula for $\bar{x}$ is easily proved by induction. In the induction step, we need to calculate the moment M_y of $n+1$ rectangles about the y-axis. But, using the induction hypothesis, the top n rectangles, treated as one lamina, has moment (mass times distance) equal to $n(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n})$. The added rectangle at the bottom, with its center $(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}) + \frac{1}{2}$ units from the y-axis, has moment given by this same numerical value since its mass is 1. Thus, the moment of the $n+1$ rectangles is the sum

$$n\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) + \frac{1}{2}$$

$$= (n+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)+\frac{1}{2}$$
$$= (n+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}+\frac{1}{2(n+1)}\right).$$

It follows that $\bar{x} = \frac{M_y}{M} = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2(n+1)}$, as desired.

A good way to model physically the kinds of problems we wish to consider is to think of a hanging mobile rather than a stack of blocks because then the center of mass property is more apparent. Using a thin but rigid material such as cardboard or foam board, we cut identical rectangles of unit length and hang them in a chain as in Figure 3. The group of n rectangles at the bottom of the chain (for each n) is connected to the lower left corner of the rectangle above by a single connector that is directly above the center of mass of the group. With the connector at this point the group hangs with its edges horizontal and vertical.

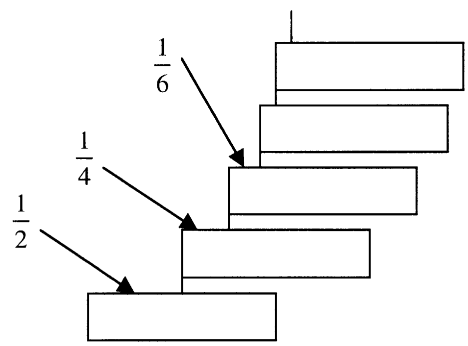

Figure 3.

One of the themes of first year calculus is that regions with curved boundaries can be approximated by unions of rectangles; how about reversing the idea? Imagine that for some large n we construct a mobile consisting of n rectangles of length 1 and width $1/n$. We position the mobile in a coordinate system with the upper left hand corner of the top rectangle at 1 on the y-axis and the lower left hand corner of the bottom rectangle at $-(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2(n-1)})$ on the negative x-axis as in Figure 4. Since

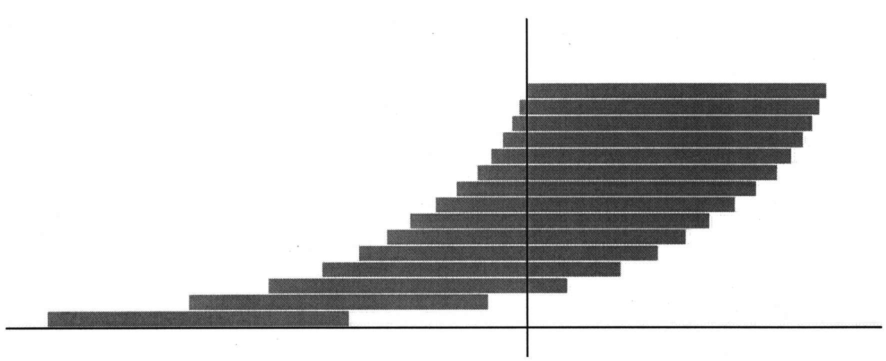

Figure 4.

n is large, this lamina is suggestive of one that is bounded between smooth curves. We could ask what happens as n approaches infinity in this picture but, instead, we just want to motivate the idea of using a lamina with curved boundaries and a center of mass property consistent with what was used for a union of rectangles (see Figure 5). Thus, we seek a lamina bounded on the left and right by $x = f(y)$ and $x = f(y) + 1$, where f is a differentiable function on (0, 1), continuous at 1, and with $f(1) = 0$. The appropriate assumption on the center of mass is that if we make a horizontal cut at $y = t$, for any t in (0, 1], then the part of the lamina that lies below this line should have center of mass $(\bar{x}, \bar{y})$ with $\bar{x} = f(t)$. What is f? We find it as follows:

$$f(t) = \bar{x} = \frac{M_y}{M} = \frac{\int_0^t \frac{f(y)+f(y)+1}{2}(1)\,dy}{\int_0^t (1)\,dy} = \frac{\int_0^t \left(f(y) + \frac{1}{2}\right) dy}{t}.$$

Multiplying both sides by t and differentiating with respect to t yields

$$f(t) + tf'(t) = f(t) + \tfrac{1}{2}.$$

Thus, $f'(t) = \frac{1}{2t}$ and $f(t) = \frac{1}{2}\ln t + C = \frac{1}{2}\ln t$, since $f(1) = 0$. Given the connection of this problem with the series $\sum_{n=1}^{\infty} 1/2n$, our formula for f is not surprising. However, it doesn't seem to have been easily predictable either.

The region bounded between the graphs of $x = f(y) = \frac{1}{2}\ln y$ and $x = g(y) = \frac{1}{2}\ln y + 1$ $(y \in (0, 1])$ causes a problem when it comes to constructing a physical model—it is unbounded. Of course, we could cut the tail off, but the resulting lamina may not be convincing as a representation of the center of mass property that was promised. This drawback motivates the problem, stated at the outset, in which we require a lamina bounded between the graphs of $x = f(y)$ and $x = g(y) = mf(y)$, with $m > 1$ and $f(0) = g(0) = 0$. In order to set up and solve a differential equation satisfied by f, we assume that f is non-negative and twice differentiable on (0, 1) with f' positive. We continue to require the center of mass property just as in Figure 5: for any horizontal cut at $y = t$, with t in (0, 1], the part of the lamina that lies below this line should have center of mass $(\bar{x}, \bar{y})$ with $\bar{x} = f(t)$.

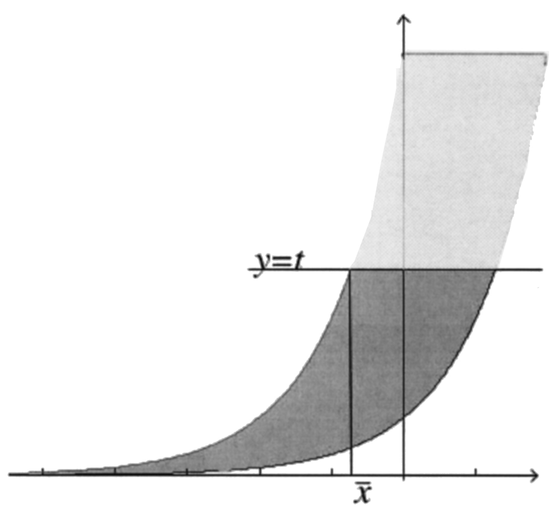

Figure 5.

Since we require $\bar{x} = \frac{M_y}{M} = f(t)$, then

$$f(t) = \frac{\int_0^t \frac{f(y)+mf(y)}{2}\,(mf(y) - f(y))\,dy}{\int_0^t (mf(y) - f(y))\,dy} = \frac{\frac{1+m}{2}\int_0^t (f(y))^2\,dy}{\int_0^t f(y)dy},$$

so

$$f(t)\int_0^t f(y)\,dy = \frac{1+m}{2}\int_0^t (f(y))^2\,dy.$$

Taking the derivative with respect to t on both sides,

$$f'(t)\int_0^t f(y)\,dy + (f(t))^2 = \frac{1+m}{2}(f(t))^2.$$

Finally, if we collect terms, isolate the remaining integral, and differentiate once more, then we obtain the differential equation

$$f(t) = \frac{m-1}{2}\left(\frac{2f(t)\,(f'(t))^2 - (f(t))^2\,f''(t)}{(f'(t))^2}\right)$$

or, equivalently,

$$(m-1)f(t)f''(t) = (2m-4)\left(f'(t)\right)^2. \tag{1}$$

In order to solve this second-order nonlinear differential equation, we need a trick. Note that (1) can be written as

$$(m-1)\frac{f''(t)}{f(t)} = (2m-4)\left(\frac{f'(t)}{f(t)}\right)^2. \tag{2}$$

So let $u = \frac{f'}{f}$ and then $u' = \frac{f''f-(f')^2}{f^2} = \frac{f''}{f} - u^2$. In this way, (2) becomes

$$(m-1)(u' + u^2) = (2m-4)u^2$$

giving us the separable first order equation

$$\frac{u'}{u^2} = \frac{m-3}{m-1} = -Q. \tag{3}$$

The general solution of (3) is $u = \frac{1}{Qt+C} = \frac{f'(t)}{f(t)}$ (for $t > 0$). Integrating again yields $f(t) = D(Qt + C)^{1/Q}$ and the condition $f(0) = 0$ determines $C = 0$. Finally, D could be replaced by $D/Q^{1/Q}$ to provide the simpler form $f(t) = Dt^{1/Q}$. The other boundary of the lamina is $g(t) = mf(t) = mDt^{1/Q}$, where m and Q are linked by (3) or, equivalently, $m = \frac{3+Q}{1+Q}$. As an example, the lamina in Figure 1 was constructed using $Q = 2$ and $m = 5/3$ so the boundary curves are $x = \sqrt{y}$ and $x = (5/3)\sqrt{y}$. Other examples of interest may include the cases

(a) $Q = 1$, so $m = 2$ and the region is bounded by $x = Dy$ and $x = 2\,Dy$; and
(b) $Q = 1/3$, so $m = 5/2$ and the region is bounded by $x = Dy^3$ and $x = (5/2)\,Dy^3$.

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Reference

1. Sydney C. K. Chu and Man-Keung Siu, How far can you stick out your neck?, *College Math Journal* **17** (1986), 122–132.

