Topology Vol. 16. pp. 99–105. Pergamon Press, 1977. Printed in Great Britain

ON LOCAL COMBINATORIAL PONTRJAGIN NUMBERS—I

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(*Received* 15 *November* 1974)

§1. INTRODUCTION

NOVIKOV proved[5] that rational pontrjagin numbers are topological invariants. E. Miller[4] showed that they are the only possible local combinatorial invariants, besides the Euler number, for an oriented triangulated smooth manifold. On the other hand the formula of Baum and Bott[1] suggested that, at least in the complex case, the pontrjagin data could be isolated to arbitrarily small neighborhoods of finite point sets, in a rather natural manner.

The purpose of this note is to prove that the (first) rational pontrjagin number of a triangulated 4-manifold is a local combinatorial invariant. (All manifolds are assumed oriented.) This result was proved independently by the author and by John Morgan (unpublished), using rather similar methods. The method extends readily to show that all pontrjagin numbers are local combinatorial invariants of smooth PL manifolds of any dimension. The proof is in §2. In §3, we apply the result to a few examples.

In the remainder of this section, we shall give the idea of our proof by a mildly amusing derivation of the local combinatorial Euler formula for a surface from the Gauss–Bonnet theorem. In this situation there is no need to deal with cubed manifolds, but we do it for analogy with subsequent argument.

A manifold with polygonal decomposition is *cubed* if each k-cell of the decomposition is a PL k-cube. Any triangulated space is cubable by taking as vertices the barycenters of each simplicial face, as edges the segments joining the barycenters of k-simplices with the barycenters of $(k+1)$-simplices in their stars, etc.

Let M be a compact oriented cubed surface with vertices v_i, with p_i the number of edges (= number of faces) in the star of v_i. Consider the following open cover of $M-\{v_i\}$. Corresponding to each (open) face F we take the open neighborhood U_F of F bounded in M by the segments connecting the barycenters of the 2-faces of M contiguous to F with the common vertices of such a face and F:

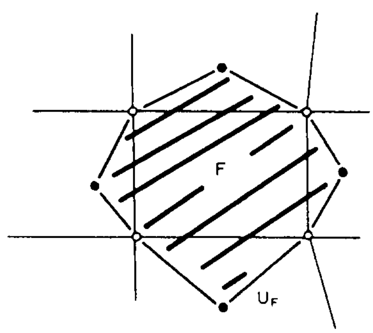

FIG. 1.

This open cover has the properties that $U_F \cap U_{F'} = \emptyset$ unless F and F' are contiguous, and $U_F \cap U_{F'} \cap U_{F''} = \emptyset$ always ($F \neq F' \neq F''$). Make U_F a coordinate patch thus:

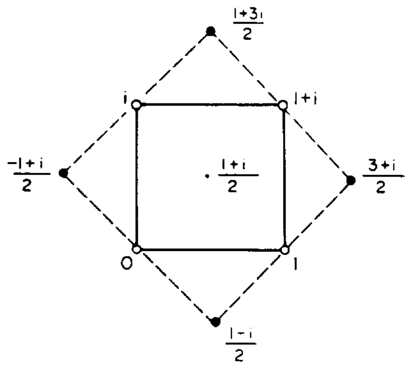

FIG. 2.

Then on $U_F \cap U_{F'}$, $z_F = z_{F'} + C$, and the metric $g' = (\sqrt{-1}/2)\,dz'\,\overline{dz'}$ is globally defined on $M - \{v_i\}$ ($dz' = dz_F$ on U_F), making each face F a unit square. The metric g' defines a continuous distance function d' on M, which fails to be smooth at the v_i.

Let $V_i = \{p \in M : d'(p, v_i) < \epsilon\}$. Coordinatize V_i by $z_i = (z')^{4/p_i}$, where z' is locally a coordinate on the slit disc $V_i - \{\text{some edge in st } v_i\}$ induced by integrating dz'. Then z_i extends continuously across the slit, with $v_i = 0$, and defines a holomorphic differential dz_i. Let $g_i = (\sqrt{-1}/2)dz_i\overline{dz_i}$ on V_i. Let H_i be a *radial* C^∞ *bump function on* V_i *at* v_i; that is:

H_i is a function of r_i alone ($z_i = r_i\, e^{\sqrt{(-1)}\theta_i}$),
$H_i \equiv 1$ near v_i,
$H_i \equiv 0$ outside V_i,
$\partial_r H_i \leq 0$ throughout V_i.

We then have a metric on M: $g = \sum_i H_i g_i + (1 - \sum_i H_i)g'$.

Now the Gauss–Bonnet theorem says that $2\pi\sqrt{-1}c_1(M) = \int_M \bar\partial\partial \log g$ (where by abuse of language $g = (\sqrt{-1}/2)g\,dz\,\overline{dz}$), $\bar\partial\partial \log g$ being the Gaussian curvature of (M, g). It is evident that on the complement of the V_i's, $\bar\partial\partial \log g \equiv 0$, so $2\pi\sqrt{-1}c_1(M) = \sum_i \int_{V_i} \bar\partial\partial \log g$. But in V_i, $g = (\sqrt{-1}/2)[H_i(r_i) + (1 - H_i(r_i))(p_i^2/16)r_i^{p_i/2-2}]\, dz_i\overline{dz_i}$, which is independent of θ_i, so in V_i

$$\bar\partial\partial \log g = \frac{\sqrt{-1}}{2}\left[\frac{\partial^2}{\partial r_i^2} + \frac{1}{r_i}\frac{\partial}{\partial r_i}\right](\log g) r_i dr_i d\theta_i.$$

Hence

$$\begin{aligned}
2\pi\sqrt{-1}c_1(M) &= \sum_i \int_{V_i} \frac{\sqrt{-1}}{2}\left[r_i\frac{\partial^2}{\partial r_i^2} + \frac{\partial}{\partial r_i}\right](\log g) dr_i d\theta_i \\
&= -\sum_i \pi\sqrt{-1}\int_0^R \left[r_i\frac{\partial^2}{\partial r_i^2} + \frac{\partial}{\partial r_i}\right](\log g) dr_i \quad \text{where } R = \epsilon^{4/p_i} \\
&= -\pi\sqrt{-1}\sum_i r_i \frac{\partial}{\partial r_i}\log g\Big|_0^R \\
&= -\pi\sqrt{-1}\sum_i r_i\frac{\partial}{\partial r_i}\left(\log \frac{p_i^2}{16} r_i^{p_i/2-2}\right)\Bigg|_{r_i=R} \\
&= \pi\sqrt{-1}\sum_i 2 - \frac{p_i}{2}.
\end{aligned}$$

Therefore $c_1(M) = \sum_i 1 - (p_i/4)$, which is the local Euler formula for a cubed surface.

§2. LOCALNESS OF THE PONTRJAGIN CLASS

THEOREM. *A cubed oriented* 4-*manifold admits a Riemannian structure determined by the combinatorial structure and whose (first) pontrjagin form vanishes outside arbitrarily small neighborhoods of the vertices.*

The proof mimics the derivation of Euler's formula in the previous section, except that there is no completely canonical choice of metric. The idea is simply that if a sufficiently natural metric is constructed, the pontrjagin form will vanish nearly everywhere. In local terms, we shall simply arrange that enough Christoffel symbols vanish.

At the outset we must make certain choices. We fix once for all a radial C^∞ bump function $h(r)$ at 0 on the unit ball in R, thus inducing a bump function at 0 on the unit ball in R^n. Let $\mathcal{S}_k$ $(k = 2, 3)$ be the (countable) set of combinatorial isomorphism classes of triangulated k-spheres. Let T be a map, also fixed once for all, which assigns to each $\Sigma \in \mathcal{S}_k$ a PL-equivalence with the Euclidean sphere S^k so as to induce a differential structure on Σ with respect to which its triangulation is smooth. T induces differential and Riemannian structures on the $(k+1)$-disc which is the cone on Σ; call the induced metric g_Σ. (It will be observed that apparently T is not a constructable map; it is for this reason that the proof is not constructive. T exists for all $k < 7$, and for $k \geq 7$ if $\mathcal{S}_k$ is restricted to smooth combinatorial spheres. Therefore the proof of the theorem is easily modified to give a result for smooth $4n$-manifolds, $n > 1$. T cannot be extended further without the restriction on $\mathcal{S}_k$, however.)

Construct a metric on M as follows. Mimicking the previous section, cover $M' = M - Sk^2M$ by neighborhoods of the 4-faces no three of which intersect, and map each neighborhood into R^4 in such a way that each 4-face is mapped to the unit cube and that on overlaps coordinates change by rigid motions (unitary transformation plus translation). Let g' be the flat metric on M' induced by these local coordinate maps. As before, g' defines a distance function d' which extends to a continuous distance on M but fails to be smooth on Sk^2M.

Let $\{s_i\}$ be the (open) 2-faces of M. For given s_i, choose local coordinates x', y', u', v' on the (open) 4-cubes in st s_i so that extended from each cube $s_i = \{u' = v' = 0;\ 0 < x', y' < 1\}$. Let $S_i = \{0 < x', y' < 1,\ u'^2 + v'^2 < \epsilon\}$ (fixed small $\epsilon > 0$), and on $S_i - s_i$ let $x_i = x'$, $y_i = y'$, $u_i + \sqrt{-1}v_i = \pm(u' + \sqrt{-1}v')^{4/p_i}$ (up to unitary transformation) where p_i = number of 4-faces in st s_i and the $\pm$ sign depends on the orientation of s_i. Then these coordinates extend smoothly across $s_i = \{u_i = v_i = 0; 0 < x_i, y_i < 1\}$. Let g_i be the induced (flat) metric on S_i, $dx_i^2 + \cdots + dv_i^2$ (note that the above $\pm$ sign becomes insignificant).

Let $\{e_\alpha\}$ be the edges of M. For given e_α, let S_α be the triangulated 2-sphere defined by $]0,1[\times S_\alpha = \mathrm{st}\, e_\alpha \cap \{p \in M: d'(p, e_\alpha) < \epsilon'\}$ (fixed small $\epsilon' > 2\epsilon > 0$). Then we have *a priori* (via the map T) a chosen metric g_{S_α} on the cone on S_α. Let $E_\alpha = \{p \in \mathrm{st}\, e_\alpha : d'(p, e_\alpha) \leq \epsilon'\}$; then E_α has an induced flat product metric $g_\alpha = dx_\alpha^2 + (\epsilon')^2 g_{s_\alpha}$ ($x_\alpha = x'$ coordinate along e_α, g_S independent of x_α).

Finally, for any vertex v let $V = \{p \in M: d'(p, v) \leq \epsilon''\}$, (fixed small $\epsilon'' > 2\epsilon' > 0$). Then V is bounded by a triangulated 3-sphere, whence V is assigned (via T) a flat metric g_v making it an ϵ''-ball.

Define a partition of unity on M thus. On each V we have the bump function $h_v(p) = h(d_v(p, v)/\epsilon'')$, d_v the distance on V induced by g_v. On $E_\alpha - \{V\}$ we have $h_\alpha(p) = h(d_\alpha(p, e_\alpha)/\epsilon')$, and this extends over the cone from v to $E_\alpha \cap \partial V$ by $h_\alpha(q) = (1 - h_v(q))h_\alpha(\epsilon'' q/|q|_v)$ and then over the rest of $E_\alpha \cap V$ by zero. On $S_i - \{E_\alpha\} - \{V\}$ define $h_i(p) = h(d_i(p, s_i)/\epsilon)$. On $(S_i \cap E_\alpha) - \{V\}$ define h_i on the "prism" from e to $\partial E_\alpha \cap S_i$ by $h_i(q) = (1 - h_\alpha(q))h_i(\epsilon' q/|q|_\alpha)$ and extend by zero. Then on $S_i \cap V$ define h_i on the cone from v to $\partial V \cap S_i$ by $h_i(q) = (1 - h_v(q))h_i(\epsilon'' q/|q|_v)$ and extend by zero. Finally, on $C(\{s_i\} \cup \{e_\alpha\} \cup \{v\})$ define $h' = 1 - \sum_i h_i - \sum_\alpha h_\alpha - \sum_v h_v$ on $\{S_i\} \cup \{E_\alpha\} \cup \{V\}$ and $h' \equiv 1$ elsewhere.

Now define the metric g on M by $g = h'g' + \sum_i h_i g_i + \sum_\alpha h_\alpha g_\alpha + \sum_v h_v g_v$. We claim that on the complement of the neighborhoods V of the vertices, the pontrjagin form p_1 vanishes identically. Essentially this is because outside the V the Riemannian structure is flat in some direction.

We compute explicitly. Evidently on $M - \{S_i\} - \{E_\alpha\} - \{V\}$, g is actually flat. Also observe that in local orthogonal coordinates, the only non-vanishing Christoffel symbols are $\Gamma^i_{ii} = (g^{ii}/2)\partial_i g_{ii}$, $\Gamma^i_{ij} = (g^{ii}/2)\partial_j g_{ii} (i \neq j)$, $\Gamma^i_{jj} = -(g^{ii}/2)\partial_i g_{jj} (i \neq j)$. Thus on $S_i - \{E_\alpha\} - \{V\}$,

$$g = dx_i^2 + dy_i^2 + \left(h_i(r_i) + (1 - h_i(r_i))\frac{p_i^2}{16} r_i^{p/2-2}\right)(dr_i^2 + r_i^2 d\theta_i^2)$$

The only possibly non-vanishing Christoffel symbols are Γ^r_{rr}, $\Gamma^r_{\theta\theta}$, $\Gamma^\theta_{\theta r}$. Hence the only non-vanishing curvature form R^i_j is $R_\theta{}^r$, and this is a multiple of $dr_i \wedge d\theta_i$. Therefore

$p_1|_{S_i-\{E_\alpha\}-\{V\}} \equiv 0$. On $E_\alpha - \{V\}$, $g = h'g' + \sum_i h_i g_i + h_\alpha g_\alpha = dx_\alpha^2 + G$, where G is independent of x_α. Hence the non-vanishing Christoffel symbols are at most those not involving x_α. Thus the pontrjagin form can have no term involving dx_α and so vanishes. Hence p_1 vanishes outside the arbitrarily small neighborhoods V of the vertices v of M. This concludes the proof.

§3. EXAMPLES

Example 1. A *local product vertex* v of a cubed 4-manifold is one at which the polyhedral structure has the form of a product $s' \times s''$ where s' and s'' are neighborhoods of vertices of cubed surfaces, or $e \times c$ where e (resp., c) is a neighborhood of a vertex of a cubed curve (resp., 3-manifold). In these cases one can make sufficiently natural choices of metrics to prove directly that the local pontrjagin number $p_1(v)$ vanishes.

First consider the case $e \times c$. Let e_α be any edge of c. Then in V,

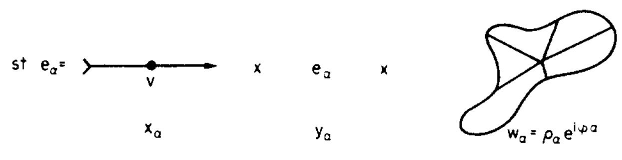

FIG. 3.

coordinatized as indicated, where up to rigid motion $x_\alpha = x'$, $y_\alpha = y'$, $w_\alpha = (w')^{4/q_\alpha}$ with q_α the number of 2-faces in (st $e_\alpha \cap c$). Take $g_\alpha = dx_\alpha^2 + dy_\alpha^2 + (\sqrt{-1}/2)dw_\alpha \overline{dw_\alpha}$. Observe that g_α is a smooth extension of the natural metric on $s_\alpha = e \times e_\alpha$. In the neighborhood E of e (e is actually two edges at v) we have a metric $g_E = dx_E^2 + dy_E^2 + du_E^2 + dv_E^2$, where x_E = coordinate along e and y_E, u_E, v_E are orthonormal coordinates of the metric determined by the triangulated 2-sphere ∂c, $y_E = Y_E(y', u', v')$, $u_E = U_E(y', u', v')$, $v_E = V_E(y', u', v')$. This metric extends smoothly across V, and we take it to be g_v as well. We then observe readily that the metric on V

$$g = h'g' + \sum_i h_i g_i + \sum_{e_\alpha \neq e} h_\alpha g_\alpha + h_E g_E + h_v g_v$$

has the form $g = (dx')^2 + G$ where G is independent of x_α. Hence $p_1(v) = 0$.

Similarly, consider $s' \times s''$. If z' and w' coordinatize $s' - v$ and $s'' - v$ respectively, with $g' = (\sqrt{-1}/2)(dz'\overline{dz'} + dw'\overline{dw'})$, one observes readily that for natural coordinates on a 2-face neighborhood S_i of $s' \times v$ we have $z_i = z'$, $w_i = (w')^{4/p}$; on S_i of $v \times s''$ we have $z_i = (z')^{4/q}$, $w_i = w'$; on S_i of $e' \times e''$ we have $z_i = z'$, $w_i = w'$ (where e' and e'' are edges of s' and s''). In each case $g_i = (\sqrt{-1}/2)(dz_i\overline{dz_i} + dw_i\overline{dw_i})$. Moreover, each edge is contained in s' or s'', and the metrics on the corresponding S_i extend smoothly across the edges. If finally we take $z_v = (z')^{4/q}$, $w_v = (w')^{4/p}$, g_v the by now obvious metric, we conclude that on V, $g = h'g' + \sum_i h_i g_i + \sum_\alpha h_\alpha g_\alpha + h_v g_v$ has the form

$$g = H_1(r_v, \rho_v)dz_v\overline{dz_v} + H_2(r_v, \rho_v)dw_v\overline{dw_v},$$

whence again easily $p_1(v) = 0$.

Example 2. We next compute the pontrjagin number of the 4-*simplex barycenter*. This is the unique vertex type v whose link is the boundary of a singular 4-simplex. To compute $p_1(v)$, construct a cubism of S^4 as follows. Begin by triangulating S^4 as the boundary of a singular 5-simplex. Call the vertices of this triangulation $v_\alpha^0(\alpha = 1, \ldots, 6)$. Introduce as additional vertices the barycenters v_α^k of the k-simplices s_α^k of the triangulation ($k = 1, \ldots, 4$). An edge of the cubism will be a great circle arc $v_\alpha^k v_\beta^{k-1}$ ($k = 1, \ldots, 4$) where $s_\beta^{k-1} \subset \overline{s_\alpha^k}$. A 2-face will be a great S^2-region $v_\alpha^k v_\beta^{k-1} v_\gamma^{k-2} v_\delta^{k-1}$ where

$$\begin{array}{ccccc} & \subsetneq & \overline{s_\beta^{k-1}} & \subsetneq & \\ s_\gamma^{k-2} & & & & \overline{s_\alpha^k} \\ & \subsetneq & \overline{s_\delta^{k-1}} & \subsetneq & \end{array} \qquad (k = 2, 3, 4).$$

3-Faces and 4-faces are determined analogously. (The analogous cubism of S^2 is shown in Fig. 3.)

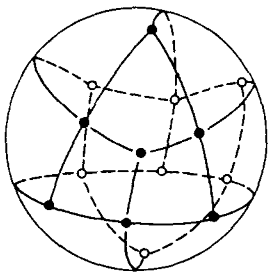

FIG. 4.

Then evidently, each $v_\alpha{}^0$ and $v_\alpha{}^4$ is a 4-simplex barycenter, all with the same orientation, while each $v_\alpha{}^k (k = 1, 2, 3)$ is a local product. We thus have

$$0 = p_1(S^4) = \sum_\alpha v_\alpha{}^0 + \sum_\alpha v_\alpha{}^1 + \sum_\alpha v_\alpha{}^2 + \sum_\alpha v_\alpha{}^3 + \sum_\alpha v_\alpha{}^4$$
$$= 6p_1(v) + 0 + 0 + 0 + 6p_1(v)$$

Hence $p_1(v) = 0$.

Example 3. Consider the orientation-reversing map $S^3 \to S^3: (u, v, x, y) \to (u, -v, -x, -y)$ $(S^3 = \{u^2 + v^2 + x^2 + y^2 = 1\} \subset R^4)$. If T is a triangulation of an (oriented) S^3, denote by T' the image of T under this map. Call a vertex type v *orientation-invariant* if the cubical structure on link v is homotopic to (link v)$'$. The dependence of local pontrjagin numbers on orientation (c.f. [4]) indicates that for an orientation-invariant v, $p_1(v) = 0$. We shall verify this directly. Given an orientation-invariant v, construct a triangulated S^4 by identifying the hemispheres with copies of a small ball around v; from the northern hemisphere the equator will look like link v as ∂ (northern hemisphere) and (link v)$'$ as ∂ (southern hemisphere). Identify these via the homotopy (link v) $\simeq$ (link v)$'$:

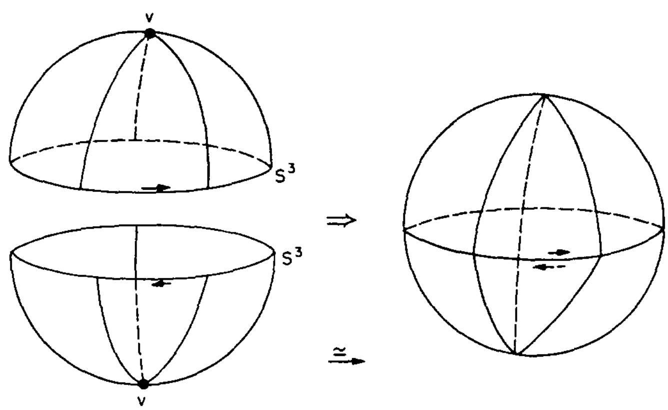

FIG. 5.

Convert this triangulated S^4 into a cubism by the same imposition of barycenters as in the previous example. The vertex types of the cubed S^4 will then be two (oriented) copies of v, several 4-simplex barycenters, several local products. But these latter types of vertex have $p_1 = 0$ and $p_1(S^4) = 0$, so $p_1(v) = 0$ as well.

Example 4. We construct a vertex type v with non-vanishing pontrjagin number. We need the following easy

LEMMA. *If a vertex type v is orientation-invariant, then so is the type of the apex vertex of the barycentric subdivision of the cone on link v.*

Now consider the triangulation of $S^5 \to C^3$:

vertices: $(i^\alpha, 0, 0) = v_\alpha{}^1 \qquad \alpha = 0, 1, 2, 3$
$(0, i^\alpha, 0) = v_\alpha{}^2 \qquad i = \sqrt{-1}$
$(0, 0, i^\alpha) = v_\alpha{}^3$

edges:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j}$, $v_\alpha^{\,j} v_\beta^{\,k}$	$j, k = 1, 2, 3, j \neq k$ $\alpha, \beta = 0, 1, 2, 3, \alpha \neq \beta$
2-faces:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k}$	
	$v_\alpha^{\,1} v_\beta^{\,2} v_\gamma^{\,3}$	
3-faces:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_{\beta+1}^{\,k}$	
	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_\gamma^{\,m}$	
4-faces:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_{\beta+1}^{\,j} v_\gamma^{\,m}$	
	etc.	

This is the analog of the octahedral triangulation of S^2:

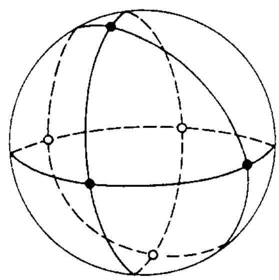

FIG. 6.

Divide out by the Hopf map $S^5 \to P^2$: $(z^1, z^2, z^3) \to [z^1, z^2, z^3]$ to obtain a cellular decomposition of P^2:

vertices:	$v^1 = [1, 0, 0]$,	$v^2 = [0, 1, 0]$, $v^3 = [0, 0, 1]$
edges:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j}$	collapses to v^j
	$v_\alpha^{\,j} v_\beta^{\,k}$	$\to$ edge with vertices v^j, v^k (there are four such edges connecting v^j and v^k)
2-faces:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k}$	$\to$ 2-cell bounded by two edges $v^j v^k$
	$v_\alpha^{\,1} v_\beta^{\,2} v_\gamma^{\,3}$	$\to$ 2-simplex with vertices v^1, v^2, v^3
3-faces:	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_{\beta+1}^{\,k}$	collapses to two 2-faces
	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_\gamma^{\,m}$	$\to$ 3-cell bounded by four 2-cells, one $v^j v^k$, one $v^j v^m$, two $v^j v^k v^m$:

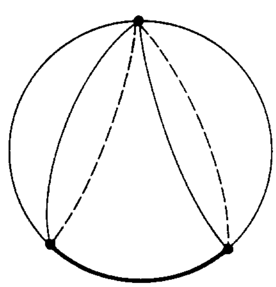

FIG. 7.

4-faces	$v_\alpha^{\,j} v_{\alpha+1}^{\,j} v_\beta^{\,k} v_{\beta+1}^{\,k} v_\gamma^{\,m}$	$\to$ 4-cell bounded by four 3-cells of the above type (one potential bounding 3-cell having collapsed to two 2-cells)

A 4-face is bounded by an S^3 whose stereographic projection to R^3 looks like this:

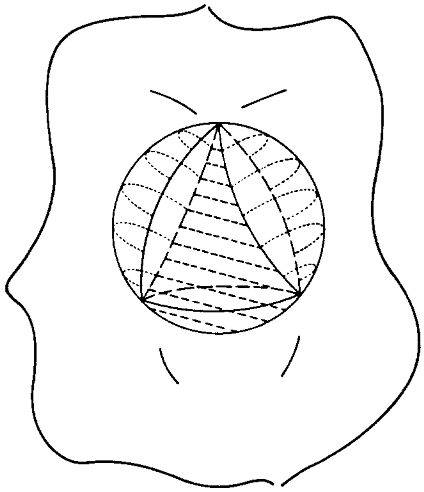

FIG. 8.

It is verified that exactly sixteen such 4-cells cover P^2 (without overlaps). One also verifies that the vertices v^i each have the link type of a product:

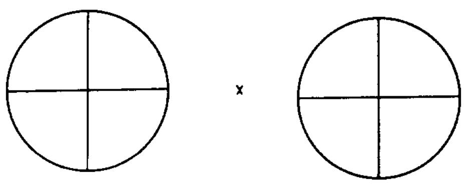

FIG. 9.

which in particular is orientation-invariant.

Next convert this decomposition to a triangulation by barycentric subdivision, and the triangulation to a cubism by the method of example 2. By the lemma, the original three vertices have vanishing pontrjagin numbers. The link type of the vertices created by barycentric subdivision does not change under the conversion to a cubism. On the other hand, each of these barycenters except the barycenters of 4-cells has the link type of a product and hence pontrjagin number zero. Finally, just as before, the vertices created in converting the triangulation to a cubism all have pontrjagin number zero. Therefore the only nonvanishing pontrjagin numbers might be associated with the sixteen isomorphic types of the barycenters of the original 4-cells. But $p_1(P^2) = 3$. Therefore, a vertex whose link is the barycentric subdivision of the S^3 of Fig. 8, has pontrjagin number 3/16.

Remark that as a result of this computation, the link type in question is not orientation-invariant. The reader may verify this directly quite easily. In particular, notice that the antipodal map on the link has the same effect as rotation through π around the horizontal (in the figure) axis.

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