The Map and the Universe
The Work of Maurits Cornelis Escher from a Cultural-Historical Approach

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ABSTRACT
The purpose of this work is to investigate the use of mathematics in the work of Maurits Cornelis Escher (1898-1972). From a cultural-historical approach, a periodization is proposed; it consists of four phases, each of which reveals the different conceptions that the artist assumed in the application of mathematics for the elaboration of his engravings, woodcuts and drawings: in the form of a utility instrument; in as much principle ordaining / syntactic; as a germ of the artistic form; and as the main means of representation. This classification is a contribution, as it offers a look at his creative life through a sense approximation. It incorporates a section with an examination about the meaning of the images and artistic objects from the Transcurssive Logic (LT). Finally, we presented some emerging discussions and final reflections.

Keyword: Escher, mathematics, art, cultural history, Transcurssive Logic

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Tyger, Tyger, burning bright,
In the forests of the night;
What immortal hand or eye,
Could frame thy fearful symmetry?'
William Blake. Songs of Experience (1794)

1.0 MAPS AND TERRITORIES

Walter Benjamin (1938) maintained in his childhood memories that the best way to know a city was to get lost in it. Some people, he adds, even if they do not have an accurate map of the labyrinth, glimpsed clues that allow them to reconstruct the path of the urban forest.

In a similar way, from ancient times, the sages tried not to wander blindly. To locate themselves and understand their position (Egyptians, Chaldeans, Hindus, etc.), they analyzed the regular movement of the stars. Subsequently, in the cultural environment of classical rational thought, the early Greek philosophers deployed arguments about the fundamental principle of all elements. The Pythagoreans, for example, argued that the number was the key to the regular harmony of the universe, where each figure corresponded to a musical note (Gomperz, 2000, Martinez, 1973).

Subsequently, Alfred Korzybski (1994), conceived the idea that the map is not the territory, just as a word is not the object it represents. In his semantic analysis, he identified three additional features: 1) A map cannot represent the whole territory but only a parcel of it; 2) Each map also includes everything that helps the mapping of a territory, including the analysis of psychology and the philosophical and ideological substratum of the map maker (his assumptions and presumptions, his technical skills, his vision of the world, etc.).); 3) Maps have the feature of being self-reflective, since as an "ideal" one should include a map of the map, in a successive and infinite fractal repetition. In addition, he argued that languages, as systems of formulation, are maps and only maps of what they pretend to represent (Korzybski, 1994: xvii). This awareness led to the three premises of general semantics that can be extended to various languages (artistic, mathematical, etc.).

From the perspective of cultural history (Bonnell and Hunt, 1999; Hunt, 1989), Escher's work, in a metaphorical sense, is an attempt to aesthetically represent, through symbols, allegories and artifacts, the map of the universe. In this sense, his engravings, drawings and woodcuts are beautiful reflections, at times disturbing, of the geometric order that is imposed on the perception of a subject immersed in that complex reality - although not irrational - that he tries to represent and understand. The concept of culture includes the study of mentalities, ideologies, symbols and rituals, in relation to both the high culture and the popular. The system of symbolic and linguistic representation of a society is understood through culture. We must add that the theme of "representation" has long been recognized as the central problem of cultural history. One of its questions can be posed in the following way: what is the relation between a work of art (painting, novel, etc.) and the world it pretends to represent? (Bonnell and Hunt, 1999: preface; Hunt, 1989, 16-17).

2.0 ART AND MATHEMATICS: AN APPROACH TO ESCHER

Mathematics is a formal science, with an ambiguous relation to the world. It begins and develops in its early stages with exclusively practical purposes: counting, measuring, arithmetic operations. However, it is later discovered that they also facilitate the systematic study of natural phenomena (Berlinghoff and Gouvea, 2004) and provide models of representation of emerging patterns in nature (Novak, 2006).

While sciences would be impossible to imagine without the support of mathematics in its formalization, there is no general agreement on the epistemological condition of mathematics. It is so that many schools of thought have reached different conclusions, mainly on the correspondence between mathematical entities and objects of the universe. Are these mathematical entities real or ideal, empirical or the result of our mental operations, proper to intuitions and potential human cognition or simple fictions? For example, for Kant, it is our very perception that structures reality in this way. In other words, only what is reflected as reality in our mind obeys the mathematical rules; but nothing can be known from the outside world, only some phenomenal aspects, always insufficient, partial and scarce (Eilenberger, 1986).

However, since Pythagoras, as an emblematic figure of geometric knowledge, mathematics has become one of the highest expressions of human imagination in the effort to understand the ultimate structure of reality (Cucker, 2013). It is clear from the reference made by Iamblichus (1818) that at its deepest level, reality is of mathematical nature. For Pythagoras the first essence was the nature of numbers and proportions which extend through all things, according to which everything is harmoniously disposed and properly ordered.

Other authors have insisted on the aesthetic dimension of mathematics, perhaps due to the inherent harmony emanating from the construction of axiomatic systems, theorems, and demonstrations. It has even been claimed that the precision of formal definitions and deductions carry some beauty. In the words of Gian-Carlo Rota (1997), the conquest of beauty distinguishes mathematics from the rest of the sciences.

Robert Kaplan and Ellen Kaplan (2003) are a key reference on intellectual and aesthetic pleasure in mathematics, since they published on the discovery of zero and the finding of irrational (algebraic / transcendent) numbers. Besides, H.S. Coxeter (1999) should be remembered for the diffusion of his essays on the perfection of geometry; to Heinz-Otto Peitgen and Peter Ritcher (1986) for his analysis of the quasi-perfection of fractals in complex dynamical systems and to Ulianov Montano (2014), who developed an aesthetic theory of mathematics. We must also mention the influence of scientists Henri Poincaré and Geoffrey Hardy on the artistic world, especially the first, since Escher used the ‘Poincaré disk model’ of hyperbolic geometry to establish the patterns of his series ‘Circle Limit’ (Dunham, 2007; Coxeter, 1979).

The arts have frequented and used, since ancient times, tools related to mathematics (Cucker, 2013, Capecchi et al., 2010). As Lynn Gamwell (2016) points out, mathematicians and artists have long sought an understanding of the physical world presented to them and abstract objects they know only with thought. A broader work would require a reference to the philosophical ideas that drive the discipline (Platonic idealism, intuitionism, constructivism, structuralism, etc.) and of a cultural and intellectual history, in order to recover the semantic itinerary of mathematical concepts and socio-historical contexts in which they have been expressed by artists (measure, proportions, perspective, infinity, fractal dimensions, symmetry, geometric configurations).

The relationship between mathematics and the arts can be seen from the 'inspiration of two-way'. For instance, the link between geometrician H.S. Coxeter and artist Escher shows a reciprocal influence and fertile combinations of complementary perspectives (Coxeter, 1980; 1979). Undoubtedly, mathematics have encouraged the development of the arts in various ways. The visual arts, on the other hand, have collaborated in the generation of mathematical formulations through the aesthetic representation of theoretical and instrumental paths, and even, they have advanced with their intuitions to the proposal of new scientific hypotheses. For example, relations between relativistic physics, deep psychology, and art (cubism, surrealism, etc.) are notorious.

An example of the synergy between science and art is revealed by Escher, who was one of the plastic creators, designer and illustrator, who has better graphically reflected modern mathematical and topological thinking. The mathematical interest of Escher, leads the artist to discover in the mathematics the ultimate structure of reality.

The Dutch artist intuitively agreed to some mathematical ideas that aesthetically seduced him and allowed him to understand the logic of patterns, regularities, symmetries, and repetition of motifs, as he observed them in the intricate Euclidean plane of mosaics and tessellations of Granada (Schattschneider and Emmet, 2003; Critchlow, 1999). The impact of Islamic culture on Escher cannot be overemphasized. Although he had already traveled the peninsula during the 1920s, it was during his trip to Granada in 1936, stopping several days to contemplate the wonders of Andalusí art, that Escher was excited by majolica of the Alhambra and the set of the Nazari palaces, all of which generated a permanent effect on his work, particularly on his tessellations. (Abas, 2010). This aesthetic discovery will allow you to move from the elaboration of natural panoramas and mountainous villages to mental landscape.

3.0 PHASES IN THE WORK OF ESCHER

There are a lot of studies on Escher, who stop with fascination in the analysis of their skill in drawing or of their skills in engraving. Schattschneider (2010) asks: "How (Escher) did it?", and with this question tries to reproduce the attitude that arises with all plainness and disbelief before the astonishment that produces his work.

However, this work has already been developed by many authors especially by geometers who have sought to elucidate the scientific foundations of their work (Abas, 2003, Coxeter 1979, Emmer 2006, Hart, 2017 and Schattschneider 1990; 2003; Schattschneider and Emmer, 2003). For this reason, the contribution of the present article is to inquire on the orientation and reasons of such a trajectory. The different perceptions on the artistic production of Escher raise the problem of the "reception" of the works of art. We argue that excessive emphasis has been placed on its use of geometric techniques and on mathematics as an instrument.

According to Bruno Ernst (1976), Escher employs figures of mathematical inspiration in three primordial areas: a) The structure of space (landscapes, world comprehension, mathematical bodies and abstract forms); b) The structure of the surface (metamorphosis, interlaced figures of birds and fish, cycles and approaches to infinity); c) The projection of three-dimensional space in the plane (traditional pictorial representation, perspective and impossible figures).

Another classification of his work, indicates that the evolution in the tracing of Escher's creative production comprises the following stages:

I - *Realistic Phase or Mathematics as an Instrument (1922-1937)*
It is characterized by the representation of landscapes, still lifes and self-portraits. It starts with panoramas of small towns in Italy and Mediterranean coasts, and then revives them at various times. The most famous works are: "*Castrovalva*" (1930), lithograph representing a small city of Abruzzo; "*Still Life with Mirror*" (1934); "*Hand with reflecting sphere*" (1935), "*Three Spheres II*" (1946), and "*Three worlds*" (1955).

In these works, M. C. Escher uses mathematics as an instrument because the mathematical mechanisms allow him to reproduce, alter and falsify perspectives, depths and proportions. In this, the uses of geometry applied since Giotto (1266-1337) are carried to an end. Here, the instrument refers to the set of techniques that allow you to do the works you imagine. He becomes a good craftsman. He has learned from Piero della Francesca, Giotto, and the architectural master Alberti, etc.

Escher enlists the help of mathematics and strength until he finds the forms he seeks. But the conception of the work appears as prior to the application of the mathematical device. Mathematics is definitely outside the work of art, like the other utensils used in its design and elaboration.

II - *Phase of metamorphosis or mathematics as pragmatic (1937-1945)*
The essence of representations is the transformation of the two-dimensional into three-dimensional, of figures that change in other figures, of optical and geometrical illusions. The more representative works of this period are: "*Metamorphosis I*" (1937), "*Day and Night*" (1938), "*Doric columns*" (1945), "*Magic mirror*" (1946), "*Drawing Hands*" (1948).

Escher begins to use mathematics to dictate the syntax of works and generates the contents with formulas: appear repetitions, iterations, conversions, mutations. The artist has become a mathematical researcher (Schattschneider, 2010). It seems to interpret, in an embryonic way, Galileo's statement that nature is written in mathematical language.

This conception leads him as an artist to precision, being carried away by the dictates of mathematics. This means, in addition, that the initial concept is executed according to an imposed logic, in which the author relies blindly. In the construction of the artistic object, Escher accepts the mathematical mandate, so that the forms, dyes and relations between the represented entities respond to an external and rigorous pragmatics.

III - *Phase of dichotomies or mathematics as metaphysics* (1946-1956)
Escher explores unconventional points of view and experiments with new methods and techniques to represent perspectives, depths, vanishing points and convergence, mainly in prints in fractal dimensions and with perspectives. The parallel existence of physical and abstract objects evidences the dichotomy that the artist discovers between manifest reality and mental constructs.

In Escher's paintings and drawings many abstract entities, purely mathematical and dissociated from the representations of reality emerge. Escher is again one, but now the conflict is between modes of representation. It does not reach a synthesis, and it is forced to look for new (non-conventional) points of view. Without systematic philosophy, though he thinks through visual images, the artist discovers that there are entities that can be and are fully described by mathematics.

The works representative of this phase are "*Other World*" (1946), "*Smaller and smaller*" (1946), "*Gallery*" (1946), "*Crystal*" (1947), "*Stars*" (1950),"*Convex and Concave*" (1955), "*Print Gallery*" (1956), "*Ascending and Descending*" (1960), "*Waterfall*" (1961); and a series of Moebius tapes, etc.

IV - *Phase of approximations to infinity or mathematics as universe-world* (1956-1970)
These works use a concept of hyperbolic geometry and fractal principles. It can be seen that the more Escher immerses in technical problems, each work tends to reveal the more from the artifices of its construction. Thus, the work flows from formal norms and rules, from the mathematical recipe that sustains it, and reality (what the artist once considered to be real) vanishes.

Escher asks himself: "... this work belongs to the realm of mathematics, or to the realm of art?" The answer is in his works. He is no longer interested in using mathematics to do his work, and he does not want to express himself in mathematics; the universe (and not just its world, the restricted world of its art) is mathematical, mathematics is the ultimate reality.

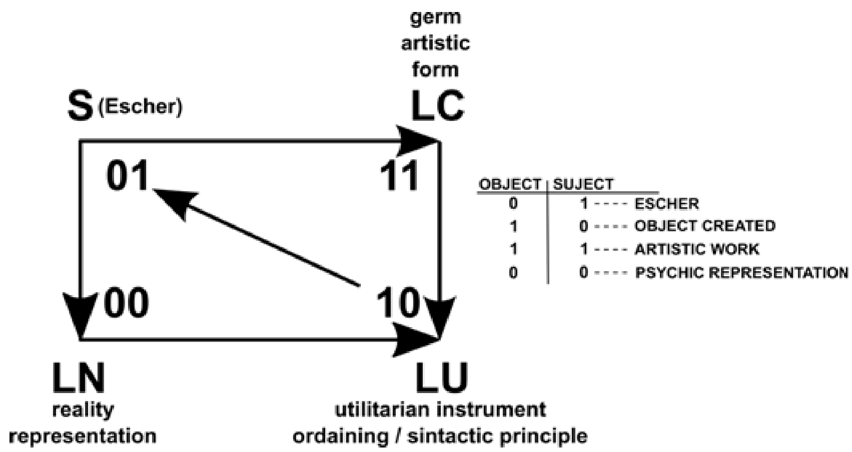
Significant works of the period are "*Three Spheres I*" (1945), the series of *Circle Limit*, in particular "*Circle Limit III*" (1959), "*Path of life III*" (1966), "*Metamorphosis III*" (1969).

The creativity in the work of Escher according to the Transcurssive Logic
According to Transcurssive Logic, human creativity emerges from the common substrate that constitutes the fundamental aspects that underpin subjective reality. This 'universal language' indicates: what are the aspects of behavior that once integrated by the cognitive, appear on the surface as a determined behavior (Salatino, 2017, p. 278).

In turn, creativity can be equated with a true language, where the 'universal language' of frank biological rootedness would behave as its 'syntactic aspect'. A natural language (Salatino, 2012) that characterizes the affections that strengthen both the volitional and the cognitive in our psyche, would act as a 'semantic aspect'. Finally, a conventional language enabled for communication in the sociocultural environment, would support its 'pragmatic aspect'.

The 'backbone' of generic language representing creativity lies in the close relationship that exists between the deep elements of its three components (Salatino, 2017, pp. 279-280). That is, experiencing or the visceral organization of knowledge through the 'universal language.' The intuition or affective integration of the three basic elements through the 'natural language', and 'creating' or social projection of feelings and motivations are combined with the help of conventional language. Figure 1 records this subjective approach and its application to the stages of Escher's artistic production

Fig. 1- PAU of creativity in Escher.



References:
S: subject – LC: conventional language – LN: natural language – LU: universal language

4.0 DISCUSSIONS ON THE ESCHER'S LAST PHASE

In the last stage of Escher's work, Mathematics no longer only facilitates the representation of reality but became the substantial structure of the universe as a reflection of an "implicate order" (in the semantics of David Bôhm, 1980).

Escher's works anticipate theories such as Max Tegmark's 'Mathematical Universe Hypothesis' (2014), which plainly states: "*Our external physical reality is a mathematical structure*" and present some unknowing resonance with the perspective on the 'universe geometric 'of Penrose (see Huggett, 1998).

Whatever the descendants or the coincidences of their legacy, Escher's vision is a return to neo-Pythagorean conceptions (perhaps with Neoplatonic components). Here infinity is a plausible idea, although it is never reached in phenomenal reality. With these concepts closes the cycle initiated by the school of Crotona 25 centuries ago, where it is affirmed that there is nothing that is not mathematics. Or perhaps, better expressed: everything that exists can be thought with mathematical methods. Thus, it can be said that the ultimate purpose of his works was to expand the 'doors of perception' (Aldous Huxley), to model aesthetic expressions that represent the mathematical mapping of the universe.

But, Escher goes even further. The artist has seen the higher order of the universe, he has glimpsed the key of that knowledge, the figure of the structure. The pictorial works of this phase are an attempt to capture the "truth" of this vision. In part, it is the symbolic dichotomy presented by Danto (1973), between artworks and real things. Escher expresses this contradiction through aesthetic language, despite understanding (or precisely because of) that the senses perceive always partial and laboriously. As Blake wrote: "*In the infinite void, but tied to the earth by its narrow perceptions.*" (William Blake. *The Book of Urizen*: Chapter IX, v. Four). Escher intuited - in the lexicon of Alfred Korzybski - that the map, that is, the language (artistic or mathematical) is not the territory, it's just a representation of it.

For these reasons, the work of art, although it reaches the canons of beauty, is ultimately a "failure." At this point, a deeper philosophy is accessed, as the author becomes aware of the limits imposed by external physical reality to language and cognitive ability (Penrose, 1999).

5.0 CONCLUSION

I. An approach to Escher's work from cultural history facilitates the reconstruction of the process by which mathematical ideas intertwine with the visual arts. This approach also reflects the possibilities of a "reasoned" history by identifying the intersections between geometric objects, mathematical concepts, works and philosophical genealogies.

II. Escher's artistic production combines a set consisting of impossible constructions, geometrical paradoxes, space structure, imaginary worlds, metamorphoses, symmetries and tessellations, etc. These areas are reflected in a series of disturbing beauty, which allow the author not only to play with optical illusions but to challenge the usual modes of representation.

III. Escher 'transfigured the commonplace' (Danto, 2003), by illustrating with (from) his artistic images that the structure of reality is deeper, complex and diverse than that which is imprinted by the 'veracity' of sensory processes and the "normal" discernment of the outside world.

IV. Escher's relationship with mathematics changed in the different phases of his work, from a period where they play an instrumental role to another where they are the key to make the essential order of the cosmos intelligible. Escher intuited the mathematical structure of the universe and tried to represent this idea through his artistic production. In its last stage, mathematics (the map) constitutes the ultimate structure of reality (the universe).

V. From the perspective of the TL, Escher's work interpellates: what is actually perceived? Moreover, how does the subject perceive and represent the reality (geometrical) in which he is immersed (and at the same time trying to understand)? From this point of view, the creativity that in Escher is an eminent manifestation, gives answer to these questions from the intuition and the experience of life.

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