Loxodromes: A Rhumb Way to Go

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A rhumb is a course on the Earth of constant bearing. For example, to travel from New York to London a voyager could head at a constant bearing 73° east of north. *Loxodrome* is a Latin synonym for *rhumb*, and has come to be used more as a geometric term—the course is a rhumb, the curve is a loxodrome. On a surface of revolution, *meridians* are copies of the revolved curve; on the earth, they are north-south lines of constant longitude. A loxodrome intersects all the meridians at the same angle. A circle of constant latitude is a loxodrome (perpendicular to meridians). Any other loxodrome can be continued forwards and backwards, winding infinitely often around the poles in limiting logarithmic spirals, as in FIGURE 1.

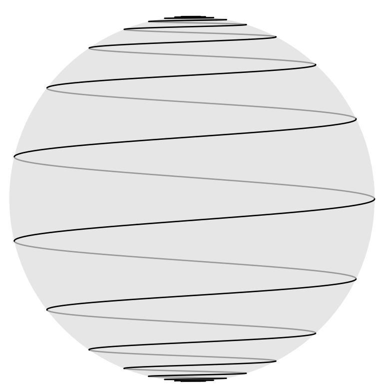

Figure 1 Loxodrome on a semi-transparent sphere. Compare also M. C. Escher's "Sphere Surface with Fish" and "Sphere Spirals" [**3**]. All loxodromes, except circles of latitude, look like this, differing only in the slope. Of course, for navigation, only a portion of the full loxodrome is relevant.

A rhumb course is not generally a great circle, and thus not the shortest route from one point on the Earth to another. However, a great-circle course requires continually resetting the bearing, an impossible task in the early days of ocean navigation. Usually a navigator would approximate a great circle by a series of rhumbs. Even today, with the availability of global positioning system (GPS), sailors and airplane pilots must know about rhumbs.

The history of loxodromes goes back to the days when voyagers first realized that the Earth is not flat and they had to take the curvature into account. They had to develop the mathematics of loxodromes. A major development was the introduction of the Mercator map projection—a rhumb is a straight line on a Mercator map. Here we consider three related mathematical problems:

I. Constructing the Mercator projection,

II. determining the rhumb heading from one location to another, and

III. computing the rhumb distance between two locations on the earth.

Some history

Economically, spice was the oil of the early Renaissance in Europe [**1**]. Economies depended on the spice trade. The Dutch and British East India Companies, incorporated for the spice trade, were the Shell and British Petroleum of their times. Spices were used for food preservation and preparation (a major consideration before refrigeration), for medicine, for personal grooming, and for conspicuous consumption. Originally, spices were transported from various points in Asia overland or along the coast of the Arabian Sea, and through the Middle East and Egypt. There were lots of middlemen (and women—in 595, Muhammad married a spice trader) along the route; each one took his (or her) cut, and prices were whatever the market would bear. A small satchel of smuggled pepper could financially set a man up for life. Small wonder that the Iberian countries, at the very end of this long route, took the lead in exploring alternative routes to the Far East.

A water route was a natural for these countries. But navigating the open Atlantic was a different ballgame than the Mediterranean. Perhaps the only certainty about the Atlantic was that it was home to sea monsters. But the economics were compelling, and navigation became a science. In the 1500s, the technology of navigation was cutting-edge for the countries of Europe. Charts, of course, could be developed only for known areas. Less than 100 years earlier, the Canaries and Azores had been *terra incognita*. There was a mystic island of Brazil somewhere out there. The Portuguese had, in a succession of voyages, sailed down the west coast of Africa, eventually losing their view of the North Star, requiring new techniques for determining latitude. Christopher Columbus had traversed the Atlantic in 1492 (but not to the spices of India), and in 1501, Amerigo Vespucci sailed down the east coast of South America. Sometime after, Portuguese sailors made south Atlantic crossings, and realized that the traverse was shorter than charts with equally-spaced meridians indicated.

Much like today, governments acted to protect and enhance technologies important to their economies. In the 1500s, Portugal was a major seafaring country. The Portuguese king prohibited the use of (newly high-tech) globes for navigation, presumably as an export control, to prevent them from falling into foreign hands. In 1534, a Chair of Mathematics was inaugurated at the University of Coimbra, underwritten with the intent of bringing mathematical methods to navigation. The polymath Pedro Nunes (1502–1578) was appointed to the new position. Nunes (pronounced noo′ nush) originally took a degree in medicine, but then held chairs in moral philosophy, logic, and metaphysics at the University of Lisbon before moving to Coimbra. He wrote treatises on mathematics, cosmography, physics, and navigation. He was appointed Royal Cosmographer in 1529 and Chief Royal Cosmographer in 1547.

Navigators came to realize in the early 1500s that a course of constant bearing is not the same as a great circle. A (probably apocryphal) tale is that Nunes was introduced to the issue by a sea captain complaining that when he set out, say, eastward, and tied the rudder to hold a straight course, he would find his ship turning towards the equator [**12**]. In 1537, Nunes published two treatises analyzing the geometry of such courses, followed by a synthesis and expansion in 1566 in Latin [**11**], where he first used the word *rumbo*. Nunes's work was controversial. Indeed, his analysis met opposition. At one point, claiming calumny on the part of his critics, he wrote, "I have decided, for this reason, to polish up some of the things I have written and set about studying philosophy and abandon mathematics, in the study of which, I have irretrievably lost my health" [**12**, translated]. The word *loxodrome* (Greek *loxos* = *oblique*, *dromos* = *bearing*) is a 1624 Latinization by Willebrord Snell (Snel) (1580–1626) [**15**] (of Snell's law in optics) of the Dutch word *kromstrijk* (curved direction), used by Simon Stevin in his description of Nunes's work.

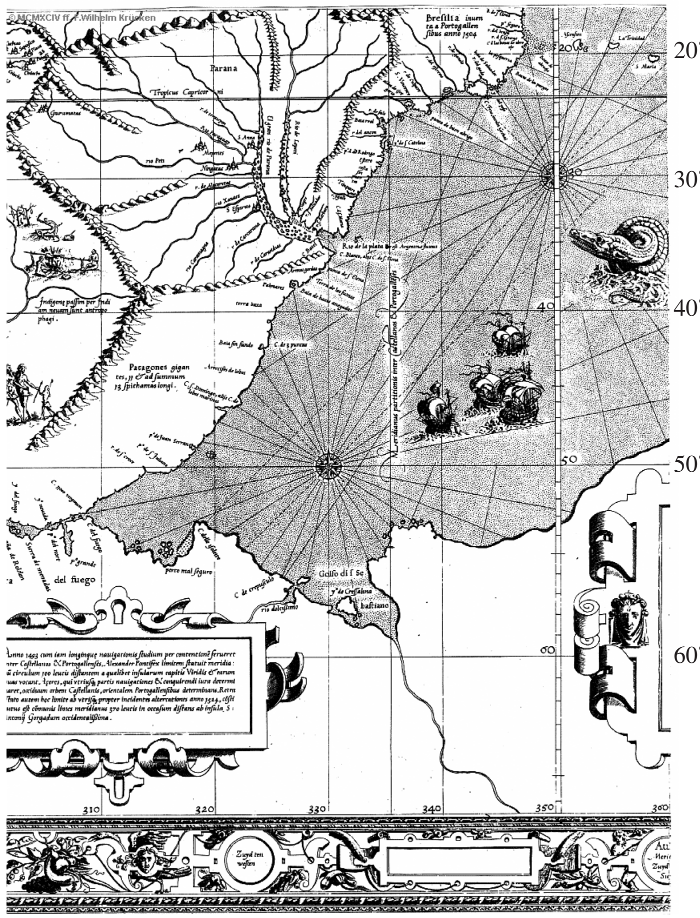

Figure 2 A detail of Mercator's 1569 map, showing a portion of South America, including latitude lines for 20 through 60 degrees South. Although the concept of "increasing latitudes," that is, the uneven spacing of the parallels of latitude, was developed early in the 1500s, Mercator figured how to space the latitudes so that a loxodrome appears as a straight line. For a complete map and other details, see [**9**]. ©F. Wilhelm Krücken, used with permission.

Another major development was the construction in 1569 by Gerhardus Mercator (Latinized from Gerhard de Cremer) of his world map [**9**]. His map, a portion of which is shown in FIGURE 2, had the great virtue that a straight line is a rhumb and (not coincidentally) angles on the map equal angles on the earth. To set a course from one location to another, a navigator drew a straight line on the map, determined the bearing, and set off. The Mercator projection became the standard for navigation for centuries, and, perhaps unfortunately given its distortions [**5**], also for atlases, wall maps, and geography books. The mathematical problem, the "true division of the nautical meridian"—the exact spacing of the meridians—remained into the next century [**8, 10**]. (Side note: The Inquisitions were in full bloom in the 1500s, and affected both Nunes and Mercator. Many Jews in Spain were converted to Roman Catholicism; such people were called *conversos*. Nunes was converted as a child. The primary targets of the later Spanish inquisition were descendants of conversos, who were persecuted under the vague charge of *judaizing*. According to one authority, this happened to Nunes's grandsons in the early 1600s [**14**, p. 96]. Mercator was (evidently falsely) accused of *lutherye* in an inquisition originated by Queen Maria of Hungary and spent seven months in 1544–45 imprisoned in the castle at Rupelmonde, narrowly escaping execution [**2**, chap. 15]. He moved to Duisburg in 1552 to escape further persecution.)

The importance of navigation, and hence of these problems, in the 16th and 17th century is hard to overestimate. Geometry was premier applied mathematics. At the time, calculus was not available. Today, these problems are straightforward applications of calculus to geometry and navigation.

Calculations

Mercator mathematics On a Mercator map, meridians are vertical lines. They do not converge at the poles. Although, as is well known, the Mercator map distorts distances, it does not distort angles. At any point, it distorts east-west directions exactly the same as north-south directions—the distortion factor depends only on position, not on direction. Such a map is called *conformal* [**4**]. Moreover, the distortion depends only on the latitude, not on longitude. A line of longitude, or meridian, is of course half a great circle, running between the north and south poles. On the other hand, latitudes are parallel circles, but shrinking in radius away from the equator. On a spherical Earth, the parallel of latitude L (in degrees north or south of the equator) is shrunk by the factor $\cos L$ compared to the equator. The equator has length $2\pi R$, where $R \approx 6371$ km ≈ 3959 mi is the mean radius of the earth. Inversely, at latitude L, distances on the Earth are stretched by the reciprocal $\sec L$ on the Mercator map.

More generally, on any surface of revolution, let $\sigma(L)$ denote the *local stretching factor* at latitude L, the reciprocal of the amount that parallels of latitude L are shrunk, compared to the equator. On a spherical Earth, $\sigma(L) = \sec L$. Some geodesists' computations are more exact, and take into account the fact that the earth is more an ellipsoid than a sphere. In this case, R is set equal to the equatorial radius 6378 km ≈ 3963 mi, and

$$\sigma(L) = \frac{(1 - e^2)\sec L}{1 - e^2 \sin^2 L},$$

where the eccentricity $e \approx .081$ specifies how far off from spherical the earth is. Here L is the *geodetic latitude*, the complement of the angle of the perpendicular to the surface with the axis of the earth.

Let $\Sigma(L) = \int_0^L \sigma(\ell)\, d\ell$ denote the *total stretching* from the equator to L, as in FIGURE 3. Here and below, all angles in the analysis are represented in radians. The total stretching solves problem I. The east-west scale on the equator sets the scale for the map. The parallel at latitude L is placed at north-south distance $\Sigma(L)$ from the equator. If a standard schoolhouse ruler is laid N-S on a Mercator map, the distance markings on the ruler are proportional to Σ. Thus Σ serves as a linear coordinate on a Mercator map.

Rhumb directions The Mercator mathematics of the previous section provides the basis for determining rhumb courses. Thus, if λ denotes longitude, the east-west coordinate, a straight line on a Mercator map has the form

$$\Sigma = m(\lambda - \lambda_1) + b. \qquad (1)$$

Since the Mercator projection is conformal and meridians are parallel, such a line corresponds to a curve of constant bearing on the Earth, that is, a loxodrome. Thus, the rhumb line between two points of longitudes λ_1 and λ_2 and latitudes L_1 and L_2 has

$$m = \frac{\Sigma_2 - \Sigma_1}{\lambda_2 - \lambda_1}, \qquad b = \Sigma_1.$$

In particular, the bearing from location 1 to location 2 is (one of the two values of)

$$\theta = \operatorname{arccot} \frac{\Sigma_2 - \Sigma_1}{\lambda_2 - \lambda_1} \text{ (degrees)} \tag{2}$$

from north, solving problem II.

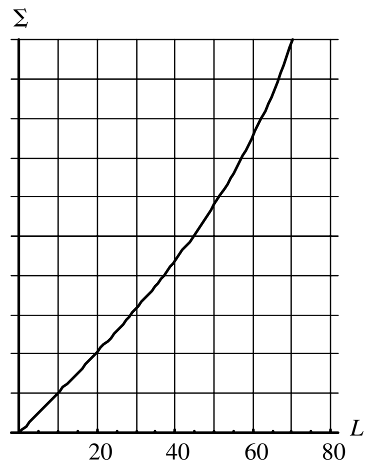

Figure 3 The graph of $\Sigma(L)$, solving problem I. The horizontal axis is measured in degrees and the vertical axis is scaled accordingly. As $L \to 90°$, $\Sigma \to \infty$ and the parallels of latitude spread apart more and more. Edward Wright [**16**] realized Σ was obtained by "perpetuall addition of the Secantes," and created a table using increments of one minute of arc, essentially amounting to numerical integration [**8, 13**]. The identification of an explicit expression for Σ (equation (6)) has a story of its own [**13**].

Rhumb distances In a Riemannian geometry, the differential of arc length, ds, is integrated over a rectifiable curve to obtain its length. By considering small (infinitesimal) right triangles, with sides aligned with latitudes and longitudes, one finds that, in terms of latitude and longitude, the square differential of arc length is

$$\left(\frac{ds}{R}\right)^2 = dL^2 + \left(\frac{d\lambda}{\sigma(L)}\right)^2.$$

Thus

$$\frac{ds}{dL} = R\sqrt{1 + \frac{1}{\sigma^2(L)}\left(\frac{d\lambda}{dL}\right)^2} = R\sqrt{1 + \frac{1}{\sigma^2(L)}\left(\frac{d\lambda}{d\Sigma}\right)^2\left(\frac{d\Sigma}{dL}\right)^2} = R\sqrt{1 + \frac{1}{m^2}},$$

so that the rhumb distance (up to sign) between the two points is

$$D_{\text{rh}} = \int_{\text{start}}^{\text{end}} ds = \int_{L_1}^{L_2} \frac{ds}{dL} dL = R\sqrt{1 + \frac{1}{m^2}} \int_{L_1}^{L_2} dL = R\sqrt{1 + \frac{1}{m^2}}\,(L_2 - L_1). \tag{3}$$

Since, from equation (1), m is the cotangent of the bearing θ, equation (3) can be written

$$D_{\mathrm{rh}} = R\,|L_2 - L_1|\,|\sec\theta|. \tag{4}$$

An alternative form for equation (3) is

$$D_{\mathrm{rh}} = R\frac{\sqrt{\Delta\Sigma^2 + \Delta\lambda^2}}{|\Delta\Sigma/\Delta L|}. \tag{5}$$

Any of (3), (4), or (5) solves problem III. Table 1 gives some examples. Equation (5) can be read as the "Mercator Euclidean" distance divided by the relative total stretching between the two latitudes. From formula (5), in the limiting case $L_2 = L_1 = L$, the distance $D_{\mathrm{rh}} = R|\Delta\lambda|/\sigma(L)$, as we already knew. Formula (5) is easy to program into a calculator or computer. Note also that a loxodrome extending all the way to the poles has finite length, although it spirals infinitely.

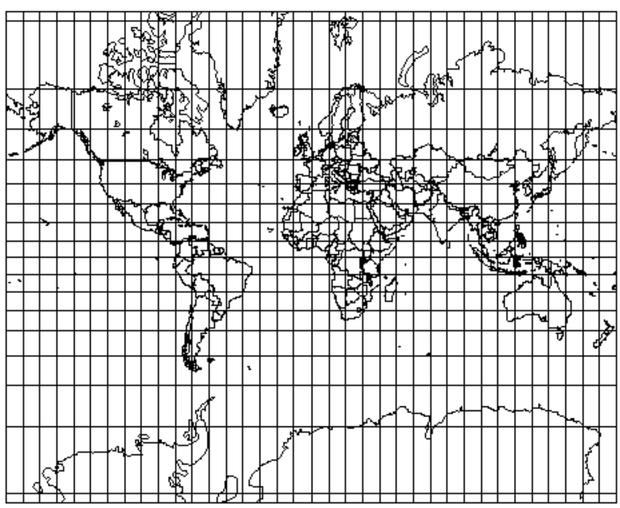

Figure 4 A modern Mercator map, vertically centered at the equator, with 10° latitude and longitude grid. The vertical spacing of the parallels of latitude is given by Σ, as graphed in FIGURE 3. The area distortion at high latitudes is evident. A rhumb course from New York to Beijing goes almost directly west across the 40th parallel of north latitude, while the great circle route comes close to the North Pole.

For a spherical earth, $\sigma(L) = \sec L$, so

$$\Sigma(L) = \Sigma_{\mathrm{sphere}}(L) = \ln(\sec L + \tan L) = \ln\left(\tan\frac{1}{2}\left(\frac{\pi}{2} + L\right)\right), \tag{6}$$

and the rhumb distance (5) can be computed by algebra (code can be downloaded from the web for calculators or PDAs). For an ellipsoidal earth, the stretching factor can also be integrated explicitly:

$$\Sigma_{\mathrm{ellipsoid}}(L) = \ln[\sec L + \tan L] - \frac{e}{2}\ln\left(\frac{1 + e\sin L}{1 - e\sin L}\right),$$

and more precise loxodromic calculations can be made. In fact one could push further; the earth is slightly pear-shaped (technically, the spherical-harmonic coefficient J_3 of the earth is nonzero), and one could make even more precise loxodromic calculations. However, to this level of precision, there are variations that depend on longitude and the precision becomes meaningless.

A rhumb between two points of the same longitude is an arc of a great circle (for instance, New York and Bogotá); a rhumb deviates most from a great circle when the two points have the same latitude. For comparison, the great-circle distance between two locations on a spherical earth is

$$D_{gc} = R \arccos\big(\sin L_1 \sin L_2 + \cos L_1 \cos L_2 \cos(\lambda_2 - \lambda_1)\big)$$

(as can be obtained from a dot product of position vectors), which can also be programmed into a calculator. Some examples of distances from New York (latitude/longitude $= 40°45'N/73°58'W$) are given in Table 1.

TABLE 1: Great circle and rhumb distances between New York and selected cities (distances in kilometers)

City	Latitude/Longitude	Comments	D_{gc}	D_{rh}
London	$51°32'N/0°10'W$		5,564	5,802
Bogotá	$4°32'N/74°5'W$	colongitudinal	4,024	4,030
Beijing	$39°55'N/116°23'E$	colatitudinal	11,019	14,380
Canberra	$35°31'S/149°10'E$		16,230	16,408

If the rhumb is to cross the international date line, the calculation must be modified slightly. In equation (2), 2π must be added to or subtracted from the denominator. In equation (3) or (5), m or $\Delta\lambda$ must be calculated appropriately. Otherwise, one determines the rhumb that goes around the earth in the opposite direction. In fact, there are an infinite number of rhumbs between any two non-polar, non-colatitudinal, points, differing in the number of times they encircle the earth, say indexed by the number of times they cross the international date line (positively or negatively). These can all be visualized by putting an infinite number of Mercator maps side by side. The lines from one point to the infinite number of copies of the other point correspond to the infinite number of rhumbs. The headings and distances are calculated by modifying the longitudinal differences in (2) and (5) by the index times 2π. If the index is very large, the rhumb will come close to the target point, but then swing away for another cycle around the earth.

Loxodrome equations

A loxodrome can be represented parametrically. Inverting equation (6) and using equation (1) with $\lambda_1 = b = 0$ yields $L = \pi/2 - 2\arctan(\exp(-m\lambda))$. Substituting into spherical coordinates and simplifying with various trigonometric formulae yields $x = R\cos\lambda \operatorname{sech} m\lambda$, $y = R\sin\lambda \operatorname{sech} m\lambda$, $z = R\tanh m\lambda$. Thus can loxodromes be graphed, as in FIGURE 1 (where $m = .075$). The stereographic projection to the tangent plane at the pole is the logarithmic spiral $r = Re^{m\theta}$, a result found by Edmond Halley by synthetic construction [**6, 7**], but an easy exercise with analytic geometry.

A question

These results provoke the question: for what positions for locations 1 and 2 are loxodromic distances the largest compared to great circles? Briefly we consider a restricted case, leaving a larger answer to the interested reader. Suppose 1 and 2 are at the same latitude L (in radians), but 180° apart in longitude. The great circle route goes over the pole, with length $D_{gc} = R(\pi - 2L)$. The loxodromic distance is $D_{rh} = \pi R \cos L$. The ratio D_{rh}/D_{gc} increases from 1 at the equator to $\pi/2$ at the pole. On the other hand, the difference $D_{rh} - D_{gc}$ is greatest when $\sin L = 2/\pi$, so $L \approx .69 \approx 40°$ (see New York-Beijing). This seems to be a maximum for locations that are 180° apart in longitude (with no constraints on the latitudes).

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50 Years Ago in the MAGAZINE

from "On the Stability of Differential Expressions," by S. W. Ulam and D. H. Hyers, Vol. **28**, No. 2, (Nov.–Dec., 1954), 59:

> Every student of calculus knows [!] that two functions may differ uniformly by a small amount and yet their derivatives may differ widely. On the other hand, it is more or less obvious intuitively, and quite easily proved, that if a continuous function f on a finite closed interval has a proper maximum at a point $x = a$, then any continuous function g sufficiently close to f also has a maximum arbitrarily close to $x = a$.

An interview with Ulam appeared in the June 1981 issue of the *College Mathematics Journal*, known at the time as *The Two-Year College Mathematics Journal*, **12**: 3 (1981), 182–189.