Gaussian Guesswork ....

.... or Why 1.19814023473559220744 . . . is Such a Beautiful Number

Article Author: Adrian Rice
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...and for that to happen, all kinds of experimentation,

observation, invention and, indeed, imagination

must come into play."

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- Kept a "mathematical diary" for nearly 20 years (just before his 19th birthday in 1796 until July 1814)

Gauss' May 30, 1799 Discovery:

Amazing relationship between three particular numbers:

- a sophisticated form of average
- a particular value of an elliptic integral
- the ratio of the circumference of a circle to its diameter (π)

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  $\ldots$ and $\sin(u + 2\pi) = \sin u$.

Hmmmmmm . . .

. . . what happens with similar integrals?????

$$\text{If } u = \int_0^x \frac{dt}{\sqrt{1-t^n}}, \text{ then } x = \ldots$$

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Follow the $\sin u$ pattern to define a *leminiscate sine*!!!

- If $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$, then $x = \sin u$.

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$$\omega = 2.62205755429211981046\, etc.$$

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Gauss' May 30, 1799 Discovery:

Amazing relationship between three particular numbers:

- the ratio of the circumference of a circle to its diameter: π
- a particular value of an elliptic integral: $\omega = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}$
- a sophisticated form of average

The Arithmetic-Geometric Mean of a, b

Define two sequences:

$$n = 0: \quad a_0 = a \qquad b_0 = b$$

$$n = 1: \quad a_1 = \tfrac{1}{2}(a_0 + b_0) \qquad b_1 = \sqrt{a_0 b_0}$$

$$n \geq 1: \quad a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}) \qquad b_n = \sqrt{a_{n-1} b_{n-1}}$$

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  Call this limit the *arithmetic-geometric mean of* a, b:

$$M(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Why $1.1981402347355922074 4\,etc$ is a Beautiful Number

We have established that

the arithmetic-geometric mean

between 1 and $\sqrt{2}$ is π/ω to 11 places;

the proof of this fact will certainly

open up a new field of analysis.

Gauss's Diary, May 30, 1799

May 1800 - Gauss completes the proof

Lemma 1. Let $a, b > 0$, $a = a_0$, $b = b_0$ and set

$$I(a,b) = \int_0^{\pi/2} \frac{dq}{\sqrt{a^2 \cos^2 q + b^2 \sin^2 q}}.$$

Then $I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right)$

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Theorem. $M(1\,,\,\sqrt{2}) = \frac{\pi}{\omega}$.

Further Reading

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