Integration Policy:
Cultural Transmission with Endogenous Fertility.*

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"But the more they were oppressed, the more they multiplied and spread"
(Exodus 1, 12).

Abstract: We live in heterogeneous societies with many cultural and ethnic minorities. The cultural composition of our societies changes over time as a result of immigration fertility choices and cultural assimilation. Studying such population dynamics, we examine the effect of integration policies, which increase the cost of direct cultural transmission, on the size of the cultural minority. We show that integration policies, while often aimed at reducing the minority's size, may have the opposite effect of increasing minority fertility and its growth rate.

Key words: Minorities, Fertility, Cultural Transmission, Integration Policies.

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1. Introduction

The biblical excerpt *"But the more they were oppressed, the more they multiplied and spread"* (Exodus 1, 12) has become central in Jewish culture. It is studied in Jewish schools, referred to in Jewish history classes and quoted in contemporary political debates. The excerpt describes the Israelites in Ancient Egypt (16^{th} century B.C.) who were oppressed as slave laborers on the one hand, but on the other hand grew in numbers. The phenomenon described in Exodus 1, 12 is clearly counterintuitive. Is it possible that oppressing a minority leads to an increase in its size? This question is not merely an historical one. We live in heterogeneous societies with different types of minority groups. The cultural composition of our societies keeps changing over time, and many governments implement different forms of integration and assimilation policies aiming to change the cultural composition of their societies. The interpretation of the biblical excerpt in a modern day context, is whether it is possible that such integration policies may result in even larger minorities?

Integration policies are commonly used to affect the fabric of society. These policies are usually designed to speed up or even force the assimilation process of minority groups.[1] In the ancient world (and in some countries even these days) physical oppression was commonly used in order to force social integration. In present times the emphasis is on legislation that affects minorities' cultural identity. Such policies target minorities' cultural institutions, and make the process of identity preservation and transmission more costly. Other policies may target minority segregation forcing minorities to interact with the majority group. Examples for integration policies include restricting the teaching or usage of minority languages (e.g., the Russian language in Estonia and the Basque language in the Basque province in Spain)[2]. In other cases governments may impose restrictions or eliminate subsidies for minority schools, theaters or cultural clubs, and may even prohibit

[1] Although exceptions exist in the form of "multicultural policies" through which the government encourages a pluralistic society, rather than attempting to assimilate its minorities.
[2] For example a governmental integration program in Estonia mandated that all Russian-speaking schools teach at least 60% of their curriculum in the Estonian language, starting the 2007/8 school year. See Krimpe (2001).

certain cultural rituals or dress codes (the Burqa Ban in France or Turkey is a prominent example).[3] Evidences that integration policies are not always effective are described in Bisin, Patacchini, Verdier and Zenou (2007) who study the ethnic identity of Muslim immigrants in the UK and show that Muslims integrate less and more slowly than non-Muslims.

The paper studies the possible effects of integration policies in a society characterized by a minority and a majority group[4]. We do so in a framework of population dynamics, where individuals choose both their fertility level and their investment in their children's cultural identity. Our main results demonstrate the need to exercise caution when introducing integration policies, as these may "backfire". Namely, when minority fertility is endogenous, a policy aimed at minority assimilation may result in a larger fertility rate and a larger minority group.

The population dynamics of minority groups are governed by three (possibly related) processes: (i) migration which is beyond the scope of this paper, (ii) the relative fertility of each cultural group and (iii) the preservation of group identity of future generations through cultural transmission.[5]

The role of cultural identity and ethnicity in determining minorities' fertility rates and the difference between minority and majority fertility rates have been extensively discussed in the Sociological and Demographic literature. Three major explanations for minorities' relative fertility patterns dominate the literature.[6] (i) The assimilation (or characteristics) hypothesis suggests that differences in fertility rates between majority and minority groups are due to differences in socioeconomic and demographic factors. Therefore, once these differences in characteristics vanish, the fertility rates of the two groups would converge. (ii) The cultural hypothesis

[3] Yet, even policies that aim to support multiculturalism and strengthen the minority's identity may lead to social unrest. This is usually the case when the minority group's lifestyle is in odds with that of the majority group. See Buruma (2006) for a discussion on such policies in the Netherlands and Vaisse (2006) for a discussion on such policies in France.
[4] By cultural integration we refer to a decline in the size of minority groups. Under this definition there is no change in the set of cultural identities but only in the relative size of each identity group. An alternative view is presented in Kuran and Sandholm (2008).
[5] Intermarriage between cultural groups can be viewed as part of this process (see Bisin, Topa and Verdier (2004).
[6] For an extensive discussion see Goldscheider and Uhlenberg (1969), Ritchey (1975), Mosher and Goldschider (1984) Kollehlon (1989) and Forste and Tienda (1996).

postulates that difference in fertility rates between cultural groups are due to different norms and attitudes towards fertility which is typically referred to as “taste for children”.[7] (iii) The “minority group status hypothesis” asserts that minority group status exerts an independent influence on fertility behavior (see Goldscheider and Uhlenberg (1969)).[8] The economic approach to fertility emphasizes individuals’ rational fertility choice (see Becker (1970), Becker and Lewis (1973), Becker and Tomes (1976) and for surveys see Easterlin (1978) and Nerlove, Razin and Sadka (1988)).[9] Fertility, in this approach, depends on the economic variables that determine the alternative cost of bearing and rearing children (e.g. women’s wages).

The second process that determines the dynamics of minority groups is cultural transmission or assimilation process by which individuals may choose different cultural identity and this cultural identity is transmitted from one generation to the next. The paper adopts the dynamic cultural transmission framework introduced by Bisin and Verdier (2001).[10] In this setting parents would like their children to inherit their own cultural type. Cultural identity is transmitted from one generation to the next via a two-stage socialization process. The first stage is direct socialization, where parents directly invest in each child, determining the probability that this child would adopt their own cultural identity. Whenever direct socialization fails, children are subject to "oblique socialization", where they adopt the cultural identity of a role

[7] For a discussion on the effect of social norms and social interaction on fertility choice see Manski and Mayshar (2003).

[8] An interesting example was provided by Johnson and Nishida (1980) who considered the following natural experiment: In 1970 in the state of Hawaii the population distribution was such that the Japanese were 28.3%, the Chinese were 6.8% and the whites were 37.9%. At the same time in California the Japanese were only 1.1%, the Chinese were 0.9% while whites were 81.9%.Comparing the fertility rates of all these groups they showed that minority status matters and it may have depressed the fertility of married Japanese and Chinese women in California relative to these groups in Hawaii. Another interesting example was described by Day (1968) who noted that a strong Catholic-Protestant fertility differential was substantially reduced in West Germany after World War II when the partition of Germany gave Catholic and Protestants equal shares of the West German society.

[9] For a discussion on the different sociological and economic approach to fertility see Andorka (1978) and Hammel (1990). While most of the literature asserts that minority fertility rate will be lower this may not be the general case and minority group status may lead in some cases to higher fertility rate.

[10] This population dynamics is based on earlier work in Anthropology by Cavalli-Sforza and Feldman (1973, 1981) and Boyd and Richardson (1985). An alternative setup for population and preferences dynamics would be based on evolutionary sociobiology see Becker (1970), Dawkins (1976) and Frank (1987). According to this approach individuals may change their cultural traits to adopt those of “successful individuals”.

model randomly selected from the entire population.[11] As an example consider the problem of religious intermarriage as discussed by Bisin, Topa and Verdier (2004). In this paper parents are paternalistically altruists such they prefer their children to have the same religion as they do. Marriage is determined by social interaction and parents may exercise some segregation effort in order to reduce the probability of religious intermarriage. This segregation effort is equivalent to the direct socialization in the general model and if it fails children are choosing a partner randomly from the entire population (this is the oblique socialization part).

The reason individuals are engaged in direct socialization is that parents would like their children to inherit their own cultural trait. This has been the standard assumption in the Sociological literature (e.g. Cavalli-Sforza, 1981), and has been further established in the context of ethnic-racial socialization (e.g. Speight and Thomas, 1999, Verkuyten, 2002, Baley and Schecter, 2012 and the survey by Hughes et al. 2006). Following Bisin and Verdier (2001), we refer to this assumption as *paternalistic altruism*. This assumption implies that parents would like their children to carry their main cultural traits (e.g. religion and cultural heritage).[12]

Yet, when individuals have several children, paternalistic altruism may take on different forms. Individuals from different cultures may have different intolerance to having only some of their children adopting different cultural identities. For example, what is the relative disutility of a Catholic, Protestant, Jewish or Muslim individual when only one of their two children converts to another religion relative to the case where both her children convert? In some cultural groups the emphasis is on having at least one child that remains loyal to the parents' cultural group – we denote this type as a "survivalist type". In other cultural groups the emphasis is on having all the children maintaining the group's cultural identity and there may even be a collective penalty on families, where even a single child deviates and adopts a different identity – we denote this type as a "zealot type". Most groups are in between these two

[11] Bisin and Verdier (2000, 2001) identify the conditions for having a stable population with a non-degenerate cultural structure.

[12] Religion, for example, represents a cultural trait that most people are keen to transmit to their children (see for example Glazer 1997).

extreme characterizations.[13] Note that the meaning of a zealot type in our setting is only with respect to the attitude towards conversion of one child and not with respect to fertility norms.

The main result of the paper is that individuals' fertility decision and their direct socialization effort crucially depend on their degree of zealousness. While there is not much empirical literature on the degree of zealousness of different cultural groups we believe that this is an important and interesting cultural variable that explain fertility, socialization and segregation choices of individuals from different cultures.

As an example one can compare the following two cultural groups: the Amish and the Orthodox Jewish groups. Both have high fertility rates and both are considered to be very conservative with strict behavior and dressing codes. However these two groups have different types of paternalistic altruism. In the Orthodox Jewish tradition there are very strong social norms against any conversion. These norms are best illustrated by the following biblical quotations:

> "Do not intermarry with them. Do not give your daughters to their sons or take their daughters for your sons, for they will turn your children away from following me to serve other gods, and the LORD's anger will burn against you and will quickly destroy you." Deuteronomy 7:3-4.

> "If thy brother, the son of thy mother, or thy son…. Saying let us go and serve other gods,…, Thou shalt not consent unto him; …, But thou shalt surely kill him; thine hand shall be first upon him to put him to death." Deuteronomy 13:6-10.

Consequently, in the Orthodox Jewish community there is a collective harsh penalty for any conversion. If one of the children converts, marry with a non-Jewish person or becomes a secular the marriage prospects for all of his/her siblings are affected, the family may be boycotted by people from the community, their children may be expelled from their schools etc. This reality is illustrated in a BBC documentary on the high cost of leaving ultra-orthodox Judaism:[14] "The fear of secular society is so strong that if a son or daughter chooses to leave, for parents it can be "the end of the

[13] When there are several cultural groups an individual may be zealot with respect to conversion of her child to group A and relatively survivalist with respect to group B. We believe that measuring the different degrees of zealousness between different cultural groups is an important part of studying the dynamics and the coexistence of these groups.

[14] See http://news.bbc.co.uk/2/hi/8435275.stm

world", she says. If one child leaves, it can harm the marriage prospects of their brothers and sisters, or influence siblings to make the break too, she explains. In one case she knows of, a father told his daughter he would rather kill her than see her become secular. She eventually committed suicide."[15] Given these preferences against any conversion the Orthodox Jewish group would be considered as zealous in terms of our model.

In contrast to the Orthodox Jewish community in the Amish community there is no such penalty and young Amish are encouraged to sample other ways of life before voluntarily committing to the Amish church. This aspect of Amish culture is best described in the book on Amish society by Hostetler (1993):

> "During this period the young person must come to terms with two great decisions: whether to join the Amish church, and whom to marry. To make these decisions, the individual must establish a certain degree of interdependence from family and community. The family relaxes some of its control. The church has no direct control over the young person who has no voluntarily become a member… The young people are thereby allowed some freedom to taste the outside world …" (p. 177).

As a result some young people leave the Amish community but there is no collective punishment imposed on their family. Clearly this freedom is not given in the Orthodox Jewish community that tries to segregate young individuals from the world outside their community. This different attitude does not mean that the Amish and the Orthodox Jewish communities have different levels of Paternalistic Altruistic preferences, it just emphasis that these preferences have a different structure (i.e. different zealousness).

Combining the fertility and the cultural transmission processes, parents in our model have two important decisions that may affect the evolution of their cultural group; fertility rate and direct socialization effort. We focus on integration policy that affects the direct socialization cost and makes it more costly or less effective. An alternative approach would be to focus on a policy that targets the oblique socialization process by reducing segregation and facilitating more social interaction between members of different cultural groups. The focus of this paper is on the

[15] The first story in this documentary was about an Orthodox Jewish man that became secular and as a result was unable to see his two children for five years.

conditions under which integration policies are ineffective and result in even larger minority groups.[16] For example, whenever individuals are relatively survivalists an integration policy may induce higher fertility rates as individuals would switch from having one child with a high (and costly) direct socialization level to having two children with much lower direct socialization level. This change may result in larger minority size whenever the effect of higher fertility is stronger than the effect of lower socialization level.

Our results depend on having endogenous fertility and assuming that whenever there are several children the paternalistic preferences are not separable. Bar-El et al. (2013) study secularization dynamics in a cultural transmission framework with endogenous fertility assuming that the minority and the majority groups have different preferences for fertility. Extending their analysis would indicate that integration policies would have the expected of effect of resulting in smaller minority size. Bisin and Verdier (2001) also consider a model with endogenous fertility but they assumed that the utility from children is additive and independent. That is, the conversion of one child does not affect the utility function from the second child. In their model endogenous fertility is relevant because there is a cost of raising children which is a function of the number children. Considering integration model in such a setup would always result in having smaller minorities.

2. A Model of Cultural Transmission with Endogenous Fertility Choice

Consider a society which consists of two cultural groups; the minority and the majority groups denoted by *r* and *m* respectively. The fraction of the minority in the population is $q_r \in (0, 0.5)$ (and respectively $q_m = 1 - q_r$). Our cultural transmission model is based on Bisin and Verdier (2000, 2001) into which we introduce endogenous fertility and integration policy. Individuals are asexual and live for two

[16]An interesting example is given by Clark (1981) that documented that the Basque Nationalist Party was growing only slightly in areas where Basque is widely spoken, but is making significant gains in area of low usage. There are clearly other forms of integration policies that are aimed to affect the relationship between cultural groups, the acceptance of cultural minorities or the attitude of minorities towards the majority group. As it was documented by Joppke (2009) there are also limits to such integration policies.

periods. In the first period, the childhood period, their cultural identity is determined. In the second period, the adulthood period, individuals choose the number of their children (fertility choice) and engage in socialization activities that may affect their children's cultural type. We assume for simplicity that individuals may have either one or two children.

We assume that individuals have paternalistic altruistic preferences such that each individual would like her children to be of her own type (or belong to her own cultural group).[17] When an individual of type $i \in \{r, m\}$ has only one child we let $V_i(j)$ be her utility from having a child of type $j \in \{r, m\}$. This utility includes all the costs of child bearing, as well as the joy of having a child, and can therefore be either positive or negative. We let $\beta = V_r(r) - V_r(m)$ be the gain of a minority group member from having one child who maintains his parent's type. Paternalistic altruism implies that $\beta > 0$.

When people have two children we let $V_i(j_1 j_2)$ be the value for type i individual of having children of types j_1 and j_2, where $i, j_1, j_2 \in \{r, m\}$. We focus on the minority group choices and frame our presentation in terms of group r – the minority group. Clearly the choices of the majority group are analogous. Paternalistic altruistic preferences imply that $V_r(rr) \geq V_r(rm) \geq V_r(mm)$. We let $B \equiv V_r(rr) - V_r(mm)$ captures the gains of a minority type individual from having two children of the minority type relative to a situation in which both her children convert to the majority type. We define $\delta \in (1, 2]$ such that $B = \delta\beta$. We assume that $V_r(rr) > V_r(r)$ which captures a taste for children, i.e., individuals prefer two children of their own type to one child of their own type. We further use the normalization $V_r(m) = V_r(mm)$.[18] Similar inequalities hold for the majority group.

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[17] While this is a standard assumption in the Sociological and Economic literature there are clearly counter examples, such as new immigrants that would like their children to be assimilated into the larger society. But even in such cases parents may prefer that their children will carry on with their main cultural traits, like religion.

[18] We need to normalize the utility from having one child and two children as the fertility decision will be based on such a comparison. We therefore assume that $V_r(m) = V_r(mm)$. We can change this normalization letting $V_r(m) = \xi V_r(mm)$ without any qualitative change in our analysis.

An important aspect of paternalistic altruism is the parents' attitude towards having children of mixed types. That is, how would parents react, or what would be their disutility, when only one of their children converts to the other group. We distinguish between two extreme cases. In the first, the emphasis is on having at least one child who maintains the parent's cultural type. In such societies the emphasis is on continuation and survival of the cultural trait. On the other hand, there are societies in which there is a strong emphasis on having all children maintaining their parent's type. In such societies there might be a collective penalty on families whenever one of their children converts to the majority type. We denote these two extreme types as "Survivor" and "Zealot". Specifically,

(i) **Survivor type**: The most important consideration for this type is to have at least one child of her own type. We capture such preferences by assuming that for this type $V_r(rm) = V_r(rr)$.

(ii) **Zealot type**: For an individual in this type of societies, the main consideration is that all her children will be of her own type i.e., $V_r(rm) = V_r(mm)$.[19]

The general paternalistic preferences are in between the above two extreme types. In order to capture this aspect of paternalistic preferences we introduce the measurement $\mu \in [0,1]$ that captures the cultural group zealousness. Formally we let μ be the proportion of B which is "lost" by individual of type r when one of her two children converts to the opposite cultural type.[20] That is $V_r(rr) - V_r(rm) = \mu B$ (and respectively $V_r(rm) - V_r(mm) = (1-\mu)B$). Therefore a survivor type is characterized by $\mu = 0$ while a zealot type is characterized by $\mu = 1$. We refer to μ as the level of zealousness of members of the minority group r. We assume that μ is given and not a subject to evolutionary forces.

[19] Note that under our assumptions for the zealot type having two children one of his own type and one of the other type is worse than having one child of his own type. While this may be viewed for some readers as an extreme assumption it would be interesting to note that in some zealot communities it is customary to treat the converted child as dead to mourn him/her and never to meet with him/her.
[20] Clearly the minority and the majority type may have different zealousness levels but for convenience we use hereinafter the term μ for the minority type.

Remark: Since there are only two cultural groups in our setting for individuals of group *r* there is only one type of conversion; from group *r* to group *m*. But when there are several cultural groups, paternalistic preferences may depend on the new identity of the converted child. For example, a Jewish parent might have a different degree of zealousness if her child would convert to Islam or Catholicism (see for example Bisin, Topa and Verdier (2004) for a discussion on the attitude of different cultural groups to intermarriage of their children). Therefore whenever there are multiple cultural groups, paternalistic altruistic preferences need to define the parents' attitude towards different combinations of mixed type children.[21]

Clearly different fertility rates may be the outcome of different tastes for children, that vary across societies (see Andorka (1978) and Hammel (1990) for a summary of the debate on the effect of social norms on fertility decisions and Manski and Mayshar (2003) for the effect of social interaction and norms for bearing children on the fertility rate). It is possible that such fertility norms will be the major factor that determines the fertility of individuals our focus is on the decision to deviate from these norms. The fertility norm is typically vague and does not necessarily distinguish between six or seven children and therefore there is still a possibility for individuals to change their fertility choice as a respond to changes in the environment in which they live.

Cultural Transmission: There are two types of cultural transmission. The first is **direct socialization** which occurs either inside the family or at special schools. This socialization is an outcome of parental costly effort. We denote the degree of direct socialization of group *r* by τ_r, which represents the probability that a child of a type *r* parent becomes type *r* through the process of direct socialization. We assume that the cost of such direct socialization is $\alpha\tau_r^2/2$ and assume that $\alpha > \beta$.[22] When a household has two children we assume that it cannot discriminate between the children and thus

[21] This would clearly complicate our analysis. We thus chose to simplify by assuming that individuals have either one or two children, and there are only two cultural groups.
[22] Assuming $\alpha > \beta$ guarantee an internal solution for the direct socialization problem.

it must provide for both of them the same level of direct socialization.[23] This is clearly a simplifying assumption but it can be modified in different ways and still capture the main intuition of our results.[24] We also assume that there are no economies of scale in direct socialization and the cost of direct socialization of two children is $\alpha\tau_r^{\,2}$.[25]

Children whose cultural type has not been determined by the direct socialization process are subject to **oblique socialization** - they randomly choose a role model from the entire population, and adopt the cultural identity of this role model. Letting $p_r(j)$ be the probability that a child of type *r* parent becomes type *j* individual (where $j \in \{m, r\},$) we have:

$$(1_a) \quad p_r(r) = \tau_r + (1-\tau_r)q_r$$

$$(1_b) \quad p_r(m) = (1-\tau_r)(1-q_r)$$

When there are two children we assume that there is no correlation between the children's conversion probabilities beyond the one generated from having the same τ_r. That is, our assumption that households cannot discriminate between children with respect to direct socialization clearly creates a conversion correlation between siblings. Our assumption is that for a given τ_r there is conversion correlation.[26] Consequently,

$$(1_c) \qquad p_r(j_1 j_2) \equiv p_r(j_1)\cdot p_r(j_2), \quad \text{where } j_1, j_2 \in \{m, r\}.$$

We begin by examining the direct socialization effort for each fertility choice, and then proceed to find the optimal fertility choice.

[23] While in principal individuals are free to discriminate between their children the question is if empirically this is a common behavior and if parents are comfortable with such discrimination.
[24] The critical part of this assumption is that parents are not able to condition the direct socialization effort of their second child on the cultural trait choice of their first child. This assumption can be easily justified as in most cases the relative small age difference between children will make such a conditioning impossible.
[25] The intuition of our results will hold even when there are economies of scale as long as direct socialization of two children is more costly than direct socialization of one child.
[26] The intuition of our main results would hold even when there is such a correlation as long as it is not a perfect correlation.

2.1 The Direct Socialization Choice

The optimal direct socialization choice depends on the number of children, the size of the minority group and its zealousness level μ.

One child: The utility from having one child and choosing the direct socialization level τ_r is denoted by $u_r^1(\tau_r \mid q_r)$ and is given by:

$$(2) \quad u_r^1(\tau_r \mid q_r) \equiv [\tau_r + (1-\tau_r)q_r]V_r(r) + (1-\tau_r)(1-q_r)V_r(m) - \frac{1}{2}\alpha\tau_r^{\ 2}.$$

Maximizing (2) with respect to τ_r yields the optimal direct socialization effort when an individual has one child, denoted $\tau_r^1(q_r)$:

$$(3) \quad \tau_r^1(q_r) = Min\left\{\frac{(1-q_r)(V_r(r)-V_r(m))}{\alpha};1\right\} = Min\left\{\frac{1-q_r}{\alpha}\beta;1\right\}$$

Our assumption that $\alpha > \beta$ guarantees an interior solution of (3).

Two children: The utility of an individual of type μ who has two children is denoted by $u_r^2(\tau_r \mid q_r, \mu)$ and is given by[27]:

$$(4) \quad \begin{aligned} u_r^2(\tau_r \mid q_r, \mu) = {} & [\tau_r + (1-\tau_r)q_r]^2 V_r(rr) + (1-\tau_r)^2(1-q_r)^2 V_r(mm) + \\ & 2(1-\tau_r)(1-q_r)[\tau_r + (1-\tau_r)q_r]V_r(rm) - \alpha\tau_r^{\ 2}. \end{aligned}$$

Maximizing (4) yields the optimal direct socialization level with two children, $\tau_r^2(q_r, \mu)$:

$$(5) \quad \tau_r^2(q_r, \mu) = \frac{B(1-q_r)\left[q_r\mu + (1-q_r)(1-\mu)\right]}{\alpha - (1-q_r)^2(2\mu - 1)B}.$$

Our assumption that $\alpha > B$ implies that $0 \le \tau_r^2(q_r, \mu) < 1$.

Since cultural substitution plays an important role in describing the cultural transmission process our first proposition identifies the conditions under which this property holds (or does not hold) in our population dynamics setup.

[27] Note that μ enters (4) via $V_i(ij)$ (as $V_i(ij) = V_i(jj) + (1-\mu)B$).

Proposition 1 (cultural substitution):

(i) When individuals have only one child the direct socialization effort $\tau_r^1(q_r)$ is decreasing with q_r (cultural substitution).

(ii) When individuals have two children the cultural substitution property does not necessarily hold. Specifically, when $\mu > 0.5$ there is a critical $q^e(\mu)$ such that whenever $q_r < q^e(\mu)$ the optimal direct socialization effort $\tau_r^2(q_r, \mu)$ increases with q_r.

Proof: See the Appendix.

The intuition of part (i) is simple. When q_r increases there are lower incentives to invest in direct socialization since a higher q_r implies that if the direct socialization fails there is a higher probability that the child will still maintain the minority culture trait through the process of oblique socialization. This is the cultural substitution effect which holds whenever individuals have one child (see Bisin and Verdier, 2001). But when there are two children the probability of having one child of type *r* and one child of type *m*, denoted by $p_r(rm) \equiv p_r(r) \cdot p_r(m)$, is non-monotonic in q_r. While $p_r(r)$ increases in q_r and $p_r(m)$ decreases in q_r, $p_r(rm)$ increases for low values of q_r and decreases for high values of q_r. This implies that for low levels of q_r, an increase in q_r results in a relatively higher level of $p_r(rm)$, which increases the incentives for a relatively zealous type to invest in direct socialization, as the biggest gain for such a type is to transform his children from (*rm*) types to (*rr*) types. Thus for a small and zealous minority the cultural substitution effect does not hold whenever there are two children.

Do parents invest more (per child) in direct socialization when they have one child than when they have two children? This question relates to the discussion on quantity/quality tradeoffs in child education (see for example Becker and Lewis (1973) and Becker and Tomes (1976)). The typical tradeoff suggests that when there are more children the education "investment" per child is lower. However, when accounting for the effect of the individuals' level of zealousness, this tradeoff may not hold.

Proposition 2: Zealot type individuals (high μ) will have a higher direct socialization effort when they have two children than when they have only one child. Specifically, there is a $\mu(q_r)$, such that for every $\mu > \mu(q_r)$ the optimal direct socialization effort with one child is lower than the optimal direct socialization effort with two children. On the other hand, when $\mu < \mu(q_r)$, the optimal direct socialization effort is higher whenever there is only one child.

Proof: See appendix.

The intuition of the above result is based on the complementarity between the two children's types. When a parent is relatively zealous (high μ) and one child maintains the parent's type, the marginal utility from the type of the second child is high and increases with the parent's zealousness. Since the optimal effort is a function of the minority's relative size, then the threshold zealousness, from which the direct socialization effort will be higher for two children, is a function of the minority's size as well.

Throughout the remainder of the paper we will continue to focus on the minority group, omitting the subscript *r* whenever it does not cause ambiguity.

2.2 The Fertility Choice.

We now turn to individuals' fertility decision. Let $U^1(q)$ be the expected utility from having one child and choosing the optimal direct socialization level $\tau^1(q) = (1-q)\beta / \alpha$.

$$(6) \quad U^1(q) = \frac{\beta^2}{2\alpha}(1-q)^2 + \beta q + V_r(m).$$

Similarly, let $U^2(q,\mu)$ be the expected utility of a minority group individual of type μ that has two children and chooses the optimal direct socialization level $\tau^2(q,\mu)$.

$$(7) \quad U^2(q,\mu) = \frac{B^2(1-q)^2[q\mu + (1-q)(1-\mu)]^2}{\alpha - (1-q)^2(2\mu-1)B} + qB[q\mu + (2-q)(1-\mu)] + V_r(mm)$$

Individuals' fertility decision is based on the comparing the utilities (6) and (7). In figure 1 we describe the utility from having one child and two children as a function of q. Since we focus on the minority group, we restrict our attention to

$q < 0.5$. We graph the utility for three levels of zealousness: for the zealot type $(\mu = 1)$, for the survivor type ($\mu = 0$) and for a moderate type μ.[28]

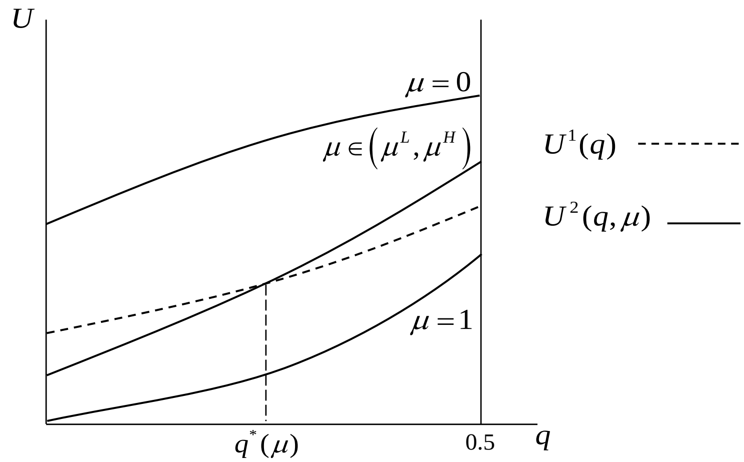


Figure 1: the utility from one vs. two children

Comparing (6) and (7) yields that when individuals are of the survivor type (low μ) their utility from having two children is always higher than their utility when they have one child. The reason is that the cost of direct socialization is convex and they suffer little or no utility loss when one of two children switches to the other type. Thus the survivor type always prefers having two children on having one child. On the other hand, an individual with a high μ (zealot type individuals) suffers a large utility loss whenever one of their two children converts to the other type and thus wishes to minimize the risk of such a conversion. Thus, as long as he is part of the minority group, the zealot type will always prefer having one child (eq. (6) is above eq. (7)). However when a zealot individual is part of the majority type, and q_m is sufficiently large, she would be better off having two children.[29]

Note that the utility from having two children or one child are always increasing with q as a larger minority leads to a higher probability of a successful oblique socialization. Also note that the utility from having two children (eq. (7)) is

[28] By an intermediate type we mean $\mu \in (\mu^L, \mu^H)$, where μ^L and μ^H are defined in proposition 3.

[29] Note that when q=1 all players would be better off having two children as in this case it is guarantee that these children would be of their own type and there is no need for direct socialization. In this case the utility does not depend on the level of zealousness.

decreasing in the level of zealousness μ. In terms of Figure 1, the line that represents the utility from having two children as a function of q shifts downward when μ increases.

As seen in Figure 1, for a moderate level of μ there is an intersection of the two utility lines. When q is relatively large, the moderate types will choose to have two children, and when q is small, they will choose to have only one child. Furthermore, an increase in the level of zealousness μ would increases the range of q for which individuals would choose to have only one child, and decreases the range of q for which individuals choose two children. This intuition is summarized in the Proposition 3.

Proposition 3: There are μ^L, μ^H ; $\mu^L < \mu^H$ such that:

(i) For $\mu < \mu^L$, $U_r^2(q,\mu) > U_r^1(q)$ for all q. That is, survivors (low μ) always choose to have two children regardless of their size q.

(ii) For $\mu > \mu^H$, $U_r^2(q,\mu) < U_r^1(q)$ for all q. That is, zealots (high μ) always choose to have one child.

(iii) Whenever $\mu^L \le \mu \le \mu^H$ there exists $q^*(\mu,\alpha) < 0.5$, such that $U_r^2(q,\mu) > U_r^1(q)$ whenever $q > q^*(\mu,\alpha)$ and $U_r^2(q,\mu) < U_r^1(q)$ for $q < q^*(\mu,\alpha)$. That is, for such a μ, a low q induces individuals to have only one child, while for a large q they will have two children. Furthermore, $q^*(\mu,\alpha)$ is increasing in μ.

Proof: See appendix. ■

The interpretation of Proposition 3 is not necessarily that zealot types would have less children but that given the group's fertility norm and other potential factors that affect fertility (like economic success), zealousness may provide incentives to reduce the number of children. Therefore having zealot types with many children (like the Orthodox Jewish community) does not contradict Proposition 3.

3. Integration Policy and Endogenous Fertility Choice

Integration policy is a policy that aims to reduce the size of the minority group. There are several types of integration policies. A government may attempt to make direct socialization more difficult or more costly. In terms of our model this integration policy would increases α. Such a policy may be implemented by reducing or eliminating government subsidies for special minority clubs, books, newspapers, theater and other cultural activities that are instrumental in maintaining the minority's cultural identity. An alternative integration policy would be to try to affect the oblique socialization process. In our model we assume that oblique socialization process is random without any segregation biases. In reality many minority groups try to affect this socialization process by adopting different types of segregation practices. Minorities sometimes prefer to live in segregated neighborhoods or send their children to special schools. These segregation practices affect the oblique socialization process as they increase the probability that it will be biased in favor of one's own cultural group. Integration policies such as promoting neighborhood integration and enforcing uniform public schooling may interfere with these segregation practices. A third type of integration policy may target the minority incentives to maintain its own cultural trait and therefore it may reduce their incentives to invest in direct socialization.

We focus in this paper only on the first type of integration policies targeting direct socialization. When fertility is exogenous any increase in the cost of direct socialization will lower the optimal socialization effort, thereby decreasing the size of the minority group in the following period. When fertility is endogenous the effect of integration policy on fertility depends on the assumed paternalistic preferences. When the utility from children is additive and independent such that the utility from one child does not depend on the type of the other child, as in Bisin and Verdier (2001), then increasing the cost of direct socialization leads to lower levels of direct socialization and possibly to a reduction in fertility rates. The question however, is whether raising the cost of direct socialization could result in a higher fertility rate.

Proposition 4 (Integration Policy and Fertility Rate): There exists $\mu^{\alpha}(q)$, such that for minority type μ; $\mu < \mu^{\alpha}(q)$ increasing α decreases $q^{*}(\mu,\alpha)$. That is, increasing the cost of direct socialization, α, may encourage members of the minority group to increase their fertility rate.[30]

Proof: See appendix. ■

The intuition of the above result is as follows: Consider a relative survivalist individual (i.e., $\mu < \mu^{\alpha}(q)$). Assume that for a given α, this individual would prefer to have one child, and in order to ensure a high probability that this child would retain the minority identity this individual chooses a high level of direct socialization. But when direct socialization becomes more expensive this individual might prefer to change her fertility choice and to have two children with a low level of direct socialization, hoping that at least one of the these two children would maintain the minority identity.

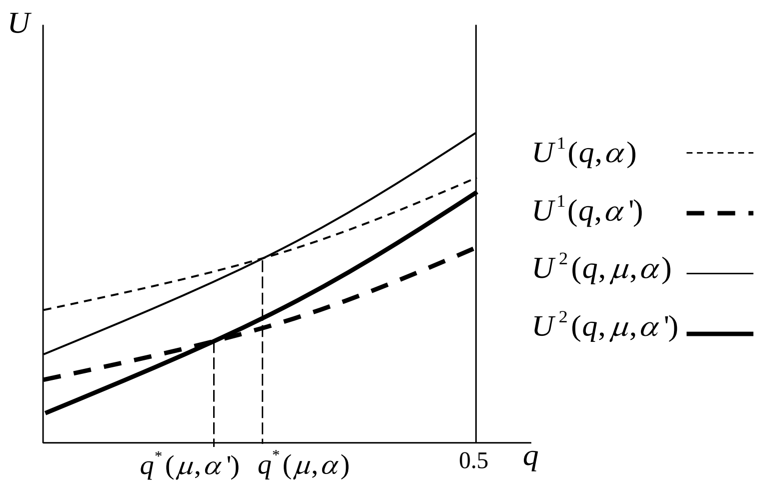


Figure 2: The utility from one vs. two children, before and after an increase in the cost of direct socialization ($\alpha' > \alpha$), for an α-policy that encourages minority fertility.

[30] Note that $q^{*}(\mu,\alpha)$ is defined by Proposition 3 as the critical fraction of population such that whenever q is below $q^{*}(\mu,\alpha)$ minority members choose to have only one child and if q is above this level they choose to have two children. A lower $q^{*}(\mu,\alpha)$ implies that there is a larger range of population fractions for which individuals choose to have two children. Note that this result will hold only for μ such that $\mu \in (\mu^{L}, \mu^{H})$, that is a μ for which optimal fertility depends on the size of the minority population q.

Figure 2 depicts the utility for each fertility choice for two levels of α. An increase of α to α' (where $\alpha' > \alpha$) implies a decline of these utility functions as the cost of direct socialization goes up. When two children are chosen, the level of zealousness mitigates the effect of an increase in cost. Thus, for a low level of zealousness both the decline in the optimal effort and the resultant utility when having two children is relatively lower compared to the decline in effort and utility with one child. Consequently, when the minority is not too zealous, an increase in the cost of direct socialization may cause individuals to switch from having one to two children whenever $q \in \left(q^*(\mu,\alpha'), q^*(\mu,\alpha)\right)$.

Numerical Example: Consider the fertility choice problem when the parameter values are $\alpha = 2$, $\beta = 1$ and δ=1.4. For this case $\mu^\alpha(q) \approx 0.7$, and thus whenever $\mu^L \le \mu < 0.7$ there exists a range of minority sizes for which integration policy will induce an increase in the fertility rate. For example, whenever $\mu = 0.65$ and the minority group is 26.5% of the population, members of the minority group choose to have one child when $\alpha = 2$. Increasing the cost of direct socialization by 10% to $\alpha' = 2.2$ will induce them to switch to two children.

4. Integration Policy and the Minority Growth Rate

The growth rate of the minority group is determined by the fertility rates of the majority and the minority groups as well as the cultural transmission process of both groups which determine the rate of conversion from one group to the other. Our analysis will be carried out in two steps. We first examine the effect of integration policy on the minority's one period growth rate. We will then consider the stable steady state structure.

4.1 The minority's one period growth rate

Our focus is on the effect of integration policy on the one period growth rate defined by q_r^{t+1} / q_r^t. One period dynamics is of particular interest as in our setting a period is actually a generation. One period dynamics are thus the main interest of a policy maker, who wishes to assess the effect of different policies on the minority's expected growth rate from one generation to the next. Maintaining our focus on the minority's decisions we assume that members of the majority group have only one child and apply no direct socialization effort, relying solely on oblique socialization.

In the previous section we consider the effect of integration policy on fertility. But a high fertility minority does not necessarily grow faster as high fertility may be accompanied by a low level of direct socialization which may result in a high conversion rate that may, in turn, reduce the size of the minority group. We therefore start by stating the conditions under which a low fertility minority would enjoy a faster growth rate than a high fertility minority.

Lemma 1: There exist $q^g\left(\alpha,\beta,\delta\right)$ and $\mu^g(q,\alpha,\beta,\delta)$ such that a minority in which individuals have only one child would grow faster than a minority in which individuals have two children whenever:

(i) $q < q^g\left(\alpha,\beta,\delta\right)$ and $\mu > \mu^g(q,\alpha,\beta,\delta)$, or –

(ii) $q > q^g\left(\alpha,\beta,\delta\right)$ and $\mu < \mu^g(q,\alpha,\beta,\delta)$

Proof: See Appendix. ■

A one-child minority will grow faster than a two-child minority only when individuals choose a significantly higher level of direct socialization effort when having one child. In these cases the effect of a high direct socialization effort overrides the effect of lower fertility. As Lemma 1 states, this is characteristic of a small and zealous minority who will invest heavily in the direct socialization of their only child. It is also characteristic of a large and survivalist minority, which will put in only a little effort whenever choosing to bear two children, relying on oblique socialization. Alternatively, note that a small and survivalist minority will choose a relatively low level of direct socialization when having either one or two children, and will grow at a faster rate when choosing to have two children. Finally, note that for a large and zealous minority, the two effects are aligned. This minority will choose a high fertility rate along with high levels of direct socialization effort, which will result in a higher growth rate when two children are chosen.

When integration policy does not change the minority fertility choice it will always result is a lower growth rate as it induces a lower direct socialization effort. But, as discussed above, integration policy may change minority’s fertility rate. We identify two situations in which integration policy results in a higher growth rate. As in Lemma 1, the first case occurs when an integration policy induces higher fertility rates, which result in higher growth rates. In the second case, the integration policy induces lower fertility rates, but the conditions of Lemma 1 hold and such lower fertility rates are accompanied by high levels of the direct socialization effort, leading to higher growth rates.

Case 1 (Higher growth rate with higher fertility rate): Assume a minority characterized by $\mu < \mu^{\alpha}(q)$ (where $\mu^{\alpha}(q)$ is defined in Proposition 4). Figure 3a graphs the minority's growth rate as a function of its relative size, and the optimal fertility choice, for two levels of the direct socialization cost α and α', such that $\alpha' > \alpha$.

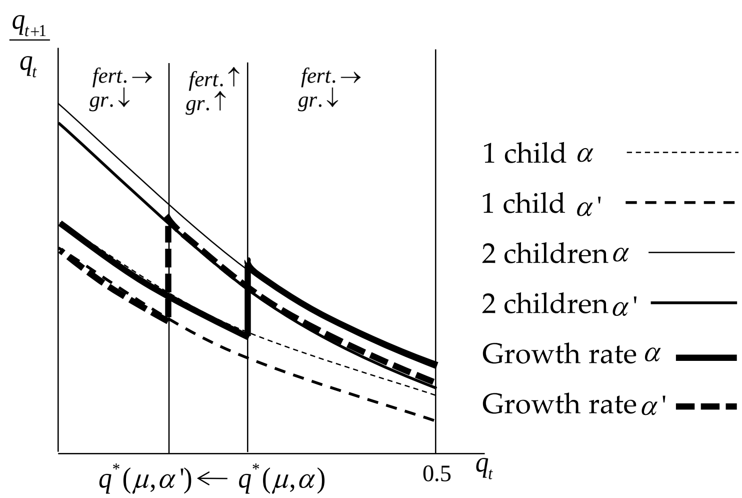


Figure 3a: Minority growth rate as a function of its size and fertility choice, pre and post integration policy- the case of increased growth rate due to higher fertility.

When $\mu < \mu^{\alpha}(q)$ Proposition 4 implies that $q^*(\mu,\alpha') < q^*(\mu,\alpha)$. Thus there are three regions of q (see Figure 4a):

(i) When $q \le q^*(\mu,\alpha')$ integration policy does not affect fertility and members of the minority group choose to have one child. As a result of higher cost of direct socialization these individuals reduce their direct socialization effort, resulting in a lower growth rate.

(ii) When $q \in q^*\left((\mu,\alpha'),q^*(\mu,\alpha)\right]$ the higher cost of direct socialization induces minority group members to increase their fertility and have two children (see Proposition 4). Thus the higher cost of direct socialization results in a higher minority growth rate.

(iii) When $q > q^*(\mu,\alpha)$ individuals choose to have two children for both α and α'. A higher cost of socialization α' leads to lower socialization effort and to a lower growth rate.

Case 2 (Higher growth rate with lower fertility): Assume a minority characterized by $\mu > \mu^{\alpha}(q)$. Figure 4b graphs the minority's growth rate as a function of its relative size, and the optimal fertility choice, for two levels of the direct socialization cost α and α', such that $\alpha' > \alpha$. When $\mu > \mu^{\alpha}(q)$ Proposition 4 implies that $q^*(\mu,\alpha') > q^*(\mu,\alpha)$.

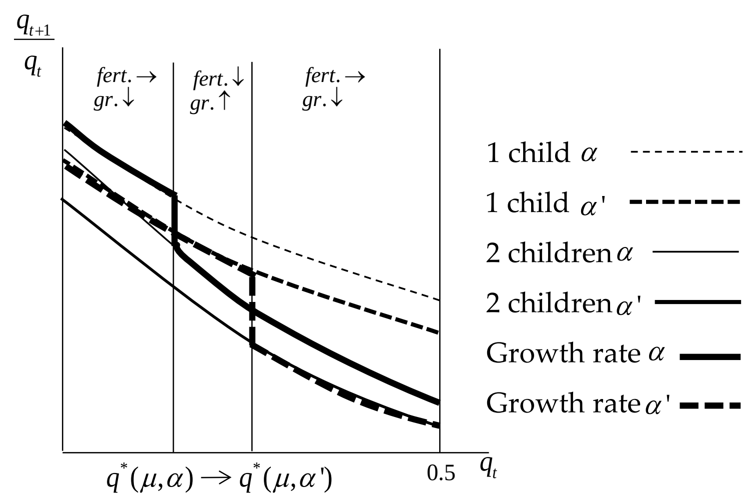


Figure 3b: Minority growth rate as a function of its size and fertility choice, for two levels of socialization cost $\alpha < \alpha'$ - increased growth rate with lower fertility.

Whenever $q > q^*(\mu,\alpha')$ or $q \le q^*(\mu,\alpha)$ the integration policy does not affect the fertility choice and therefore it results in a lower growth rate due to a lower direct socialization effort. However whenever $q \in \left(q^*(\mu,\alpha),q^*(\mu,\alpha')\right]$ integration policy induces a reduction in the minority's fertility rate from two children to one child (see

Proposition 4). Whenever the conditions of Lemma 1 hold, this lower fertility leads to a higher growth rate as depicted in Figure 3b.

We summarize the effect of integration policy on the minority growth rate in the following Proposition:

Proposition 5 (Integration policy and minority growth rate): Increasing the cost of direct socialization, α , will increase the minority's growth rate in the following cases:

(i) When it induces a switch from one to two children and the fertility effect dominates the lower direct socialization effect, such that a two-child minority grows faster (Figure 3a).

(ii) When it induces a switch from two children to one but with a higher direct socialization effort, and the direct socialization effect dominates the fertility effect, such that a one-child minority grows faster (Figure 3b).

Numerical Example: To illustrate the possibility that an integration policy may increase the minority's growth rate, we consider the baseline case (in which $\alpha = 2$, $\beta = 1$, $\delta = 1.4$ and $\mu = 0.6$) and examine the effect of a 10% increase in the cost of socialization to $\alpha' = 2.2$ (see Table 2). When the minority is only 10% of the population the integration policy does not affect fertility and minority members choose to have one child. The higher costs of socialization induce a reduction in the direct socialization effort which results in a lower growth rate. A similar effect occurs when the minority is 30% of the population, and members of the minority choose to have two children. The interesting effect occurs for a mid-range minority. For example when the minority is 17.5% of the population the integration policy induces individuals to switch from one child to two children but to lower their direct socialization effort. The overall effect however is a higher growth rate.

q	α, α'	Fertility choice	τ	q_{t+1} / q_t
$q = 10\%$	$\alpha = 2$	1	0.4500	1.405
	$\alpha' = 2.2$	1	0.4091	1.368
$q = 17.5\%$	$\alpha = 2$	1	0.4125	1.340
	$\alpha' = 2.2$	2	0.2500	1.351
$q = 30\%$	$\alpha = 2$	2	0.2420	1.261
	$\alpha' = 2.2$	2	0.2185	1.235

Table 2: Integration policy and growth rate ($\beta = 1$, $\delta = 1.4$ and $\mu = 0.6$)

4.2 Steady state population structures

We turn to discuss the effect of integration policy on the long-run steady state population structure. We assume that the majority group is characterized by $\mu_m = 0.5$ and that both the minority and majority group members choose their direct socialization effort and fertility level optimally. The steady state minority size, denoted by q^s, should therefore satisfy the following steady state condition:

$$(8)\ q^s = \frac{q^s N_r p_r(r) + \left(1-q^s\right) N_m p_m(r)}{q^s N_r + \left(1-q^s\right) N_m}$$

Where $p_r(r)$ and $p_m(r)$ are respectively the probability that a minority child would maintains his type and the probability that a majority child would switch to the minority's type. The fertility choice of the two groups are given by N_r and N_m.

Members of the majority and minority groups may differ in their child-bearing costs and benefits. Throughout this section group subscripts will be added to the cost and benefit parameters, now denoted by α_i and δ_i, for $i \in \{r,m\}$.[31] The only symmetry between the groups is assumed with respect to β, the utility loss when having one child that switches types, i.e., $\beta = V_r(r) - V_r(m) = V_m(m) - V_m(r)$.

We now turn to consider the fertility decision of the majority group. Since $q_m > 0.5$ and $\mu_m = 0.5$ the majority's fertility choice depends on δ_m.

Lemma 2: There exists δ_m^* such that for $\delta_m \geq \delta_m^*$ members of the majority group choose $N_m = 2$ regardless of its size. When $\delta_m < \delta_m^*$, there exists a threshold $q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$ such that when $\delta_m < \delta_m^*$ and $q_m \geq q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$, members of the majority group choose $N_m = 2$, and only when $\delta_m < \delta_m^*$ and $q_m < q_m^*\left(\mu_m, \alpha_m, \delta_m\right)$ they choose $N_m = 1$.

Proof: See Appendix. ■

[31] Recall that $B_r = \delta_r \beta$.

Note that while, in principle, the majority group may choose to have one child, this would occur only for very low values of δ_m. For most parameter values that we examined δ_m^* is close to 1. A choice of high fertility seems intuitive for the majority group, as its type of $\mu_m = 0.5$ implies a relatively low penalty for a deviation of one child, which, combined with the group's high probability of successful oblique socialization, results in an overall low child-bearing risk. We will therefore restrict our analysis hereinafter to the case of $N_m = 2$.

We now proceed to study the steady state population structures in two steps. At the first step we take the fertility rate as given and show that for each fertility choice pair $(N_r, 2)$ there exists a unique stable steady state population structure denoted by $q_{N_r,2}^s \in (0,1)$ (see Lemma A1 in the Appendix). We then check for each $q_{N_r,2}^s$ whether for such a minority size the individuals' optimal fertility choice would indeed be N_r. In order to guarantee that at the steady state $q_r < 0.5$ such that the minority maintains its minority group status, we need to further assume that $\alpha_m/\delta_m < \alpha_r$ and $\mu < 0.5 + 2(\alpha_r/\delta_r\beta - \alpha_m/\delta_m\beta)$.

Proposition 6: The population will converge to one of the stable structures $q_{1,2}^s$ or $q_{2,2}^s$ depending on the initial population structure q^0:

(i) When $\mu_r \le \mu^L$ the population will converge to $q_{2,2}^s$.

(ii) When $\mu_r \ge \mu^H$ the population will converge to $q_{1,2}^s$.

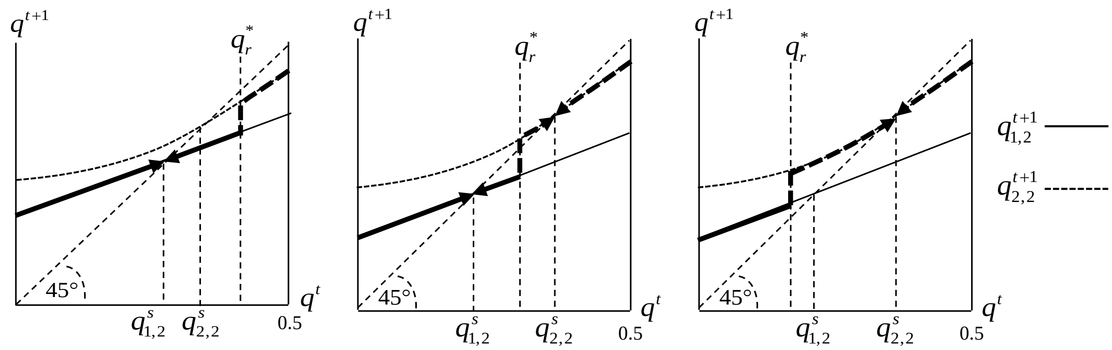
(iii) When $\mu^L < \mu_r < \mu^H$:

a. If $q_{1,2}^s < q_{2,2}^s \le q_r^*(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q_{1,2}^s$.

b. If $q_{1,2}^s \le q_r^*(\mu_r, \alpha_r, \delta_r) < q_{2,2}^s$ the steady state will depend on the initial population structure q^0. If $q^0 \le q_r^*(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q_{1,2}^s$ and if $q^0 > q_r^*(\mu_r, \alpha_r, \delta_r)$ the population will converge to $q_{2,2}^s$.

c. If $q_r^*(\mu_r, \alpha_r, \delta_r) < q_{1,2}^s < q_{2,2}^s$ the population will converge to $q_{2,2}^s$.

Part (i) and (ii) of Proposition 6 are immediately derived from Proposition 3 according to which whenever $\mu_r \le \mu^L$ individuals always choose to have two children and whenever $\mu_r \ge \mu^H$ they always prefer to have one child. Part (iii) is illustrated in Figure 4. Figures 4a and 4c correspond to cases (iii)a and (iii)c (respectively), and Figure 4b corresponds to case (iii)b where the steady state depends on the initial population structure. For this latter case, when the initial minority size is $q^0 \leq q_r^*\left(\mu_r, \alpha_r, \delta_r\right)$ the population will converge to the $q_{1,2}^*$ steady state and when the initial minority size is $q^0 > q_r^*\left(\mu_r, \alpha_r, \delta_r\right)$ the population will converge to the steady state of $q_{2,2}^*$, which is characterized by a larger minority.



(a) High μ_r (b) Intermediate μ_r (c) Low μ_r

Figure 4: Convergence to the steady state, from different initial minority sizes.

An integration policy which imposes a higher α_r, may change all the threshold levels in Proposition 6. We therefore do not describe all the possible cases, but provide one example where the onset of an integration policy results in a new steady state, with a larger long-run minority.

Proposition 7: Integration Policy may result in a larger minority in the long run steady state. Specifically, whenever $q_{1,2}^s \leq q_r^*\left(\mu_r, \alpha_r, \delta_r\right) < q_{2,2}^s$, and $q^0 \leq q_r^*\left(\mu_r, \alpha_r, \delta_r\right)$ the stable steady state is $q_{1,2}^s$. In this case, introducing an integration policy could lower $q_r^*\left(\mu_r, \alpha_r, \delta_r\right)$, and may thus increase the minority's fertility rate, thereby leading to convergence to the new steady state $q_{2,2}^s$, with a larger minority.

Numerical example: Assume that $\alpha_r = 4,\ \alpha_m = 2,\ \beta = 1,\ \delta_r = \delta_m = 1.5$. This case is illustrated in Figure 4b. For these parameter values, the threshold minority size above which the minority chooses high fertility is 27.5% of the population. Therefore a minority which is just below the threshold and consists of 27.4% of the population will converge to the steady state of $q_{1,2}^{s}$, with a long run population size of 25%. Following the onset of an integration policy which increases the minority's cost of direct socialization by 10%, the threshold $q_r^*(\mu_r, \alpha_r, \delta_r)$ decreases to 27.3%. For the above minority, this implies a change from low to high fertility, and will thus result in a convergence to the higher steady state of $q_{2,2}^{s}$ with a minority size of 29% of the population.

5. The effect of minority's zealousness level on its growth rate

Combining the result on fertility choice with the result on growth rate we can now discuss the effect of minority zealousness on its growth rate. Figures 3a and 3b graph the minority's growth rate as a function of its relative size for two minority types, μ_1 and μ_2; $\mu_1 < \mu_2$. Figures 5a and 5b differ in the maximal benefit from two children, δ. When δ is sufficiently large, individuals apply a higher effort, and the growth rate with two children is higher than the growth rate with one child, for all minority sizes. However, when δ is small, individuals apply a smaller effort with two children, and there exists a range of minority sizes for which a two-child minority will exhibit a lower growth rate than a one-child minority.

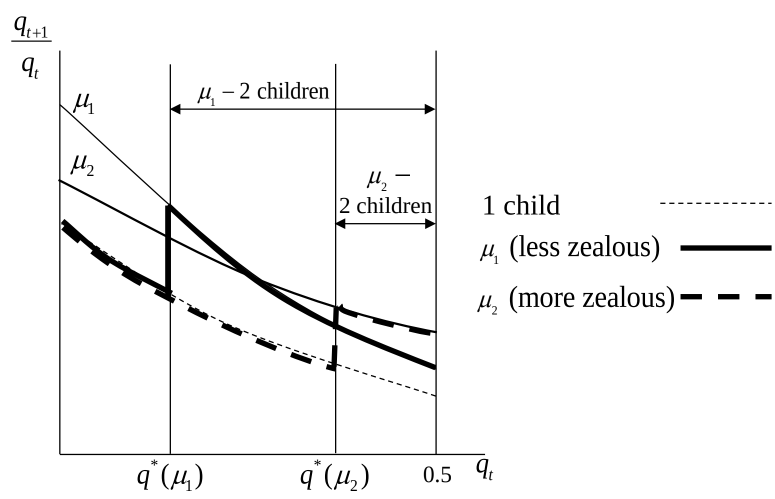


Figure 5a: Minority growth rate depending on fertility choice, $\mu_1 < \mu_2$.

Figure 5a presents the case in which the fertility effect dominates the direct socialization effect and consequently a two-child minority grows faster than a one-child minority (Lemma 1 does not hold). For this case, we compare the growth rate of the two minority types, for three ranges of q:

(i) $q \le q^*(\mu_1)$ - both types choose to have one child and consequently have the same growth rate.

(ii) $q \in \left(q^*(\mu_1), q^*(\mu_2)\right]$ - only the less zealous minority (i.e., μ_1) chooses to have two children while the more zealous minority (μ_2) chooses to have only one child. In this case the less zealous minority grows faster.

(iii) $q > q^*(\mu_2)$ - Both minority types will choose to have two children, and the more zealous minority grows faster due to a higher level of direct socialization effort.

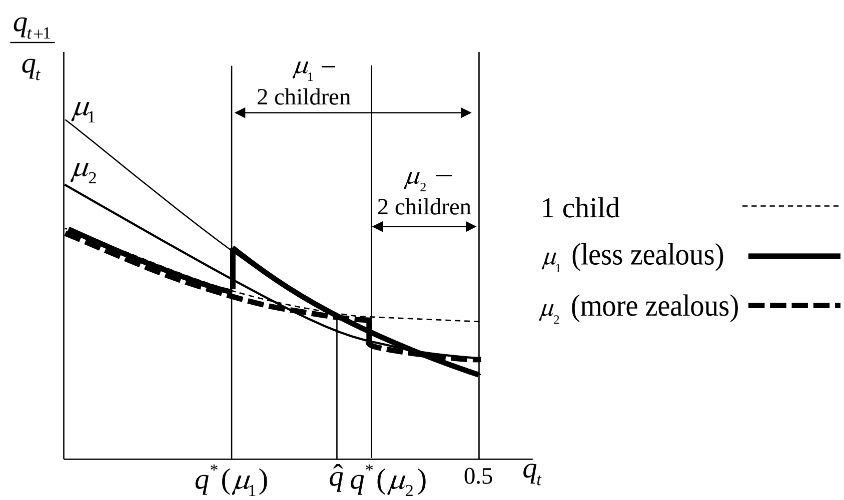


Figure 5b: Minority growth rate depending on fertility choice, $\mu_1 < \mu_2$.

Figure 5b demonstrates the case in which a two-child minority will not necessarily enjoy a faster growth rate than a one-child minority, due to the lower direct socialization effort applied with two children as described in Lemma 1. Again, we compare the growth rate of the two minority types, for three ranges of q:

(i) $q \le q^*(\mu_1)$ - Both types choose to have one child and have the same growth rate.

(ii) $q \in \left(q^*(\mu_1), q^*(\mu_2)\right]$ - Only the less zealous minority (characterized by μ_1) chooses to have two children. There is a $\hat{q}$, such that:

a) For $q \in \left(q^*(\mu_1), \hat{q}\right)$- the less zealous minority has a higher growth rate.

b) For $q \in \left(\hat{q}, q^*(\mu_2)\right]$- the more zealous minority enjoys a higher growth rate despite its lower fertility, as the direct socialization effect dominates the fertility effect.

(iii) $q \geq q^*(\mu_2)$ - Both minority types choose to have two children. The minority type which applies a higher effort will grow at a faster pace. Following Lemma 1, the more zealous minority applies a higher effort only when the minority's size is large enough.

Concluding Remarks

The focus of our paper has been on the effects of an "Integration policy" on the fertility, assimilation process and growth of minority groups. The integration policy is modeled as an exogenous increase in the cost of the direct socialization effort, which could be the outcome of a decrease in government subsidy to the minority's schools, newspapers, theater or other cultural establishments. A naive policy maker would expect such policies to decrease the socialization effort chosen by the minority, and a subsequent decrease in the minority's size. However, taking into account both fertility and effort choices, we show that the integration policy could have the opposite effect. Our results therefore suggest that integration policies may have counter intuitive results and may lead to a larger minority.

The cultural transmission model assumes only two forms of socialization; direct socialization and oblique socialization, with individuals choosing only the direct socialization effort. The standard cultural transmission framework may be generalized by allowing individuals to affect the oblique socialization level as well. This may be done by introducing different levels of voluntary segregation. Such a segregation implies that when the direct socialization process fails, the probability that a child adopts the majority's identity would also be a decreasing function of the segregation level imposed by his parents. Such a framework may give rise to yet another detrimental effect of integration policies, as these policies could result in a more segregated minority.

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Appendix:

Proof of Proposition 1:
(Note that subscripts are omitted from this and following proofs whenever it causes no ambiguity.)

The optimal effort with one child is given by $\tau^1(q) = \frac{\beta}{\alpha}(1-q)$, and the derivative w.r.t. q is $\frac{d\tau^1(q)}{dq} = -\frac{\beta}{\alpha} < 0$.

The optimal effort with two children is given by

$\tau^2(q,\mu) = \frac{B(1-q)\left[q\mu + (1-q)(1-\mu)\right]}{\alpha - (1-q)^2(2\mu-1)B}$, and the derivative w.r.t. q is given by:

$$\frac{\partial \tau^2(q,\mu)}{\partial q} = B\frac{\alpha\left[-q\mu + (1-q)(3\mu-2)\right] - B(1-q)^2(2\mu-1)\mu}{\left[\alpha - B(1-q)^2(2\mu-1)\right]^2}$$

This derivative is negative if and only if the following inequality holds:

$$\alpha\left[-q\mu + (1-q)(3\mu-2)\right] - B(1-q)^2(2\mu-1)\mu < 0$$

When $\mu \le 0.5$, the above inequality holds for all values of *q*.

When $\mu > 0.5$, the above inequality holds for

$q > q^e(\mu),$ where $q^e(\mu) = 1 - \frac{\alpha}{\mu B} + \frac{1}{\mu B}\sqrt{\alpha^2 - \alpha\mu B\frac{\mu}{2\mu-1}} < 1.$

Summarizing,

$$\text{For } \mu > 0.5: \ \frac{\partial \tau^2(q,\mu)}{\partial q} \le 0 \text{ iff } q \ge q^e(\mu)$$

$$\frac{\partial \tau^2(q,\mu)}{\partial q} > 0 \text{ iff } q < q^e(\mu)$$

$$\text{For } \mu \le 0.5: \ \frac{\partial \tau^2(q,\mu)}{\partial q} < 0 \ \text{ for all } q \in [0,1].$$

■

Proof of Proposition 2:
We compare the optimal direct socialization effort with one child to the optimal effort with two children. The optimal effort with one child is larger whenever:

$$\frac{\beta}{\alpha} > \frac{\delta\beta\left[q\mu + (1-q)(1-\mu)\right]}{\alpha - \left(1-q\right)^2(2\mu-1)\delta\beta}$$

Solving yields the threshold μ^e, given by:

$$\mu^e \equiv \frac{\alpha + \delta\beta(1-q)^2 - \alpha\delta(1-q)}{\alpha\delta q + 2\delta\beta(1-q)^2 - \alpha\delta(1-q)}$$

Such that for $\mu > \mu^e$ the optimal effort with two children is higher, and for $\mu < \mu^e$ it is

lower than the effort with one child.
Note that

$$\frac{\partial \mu^e}{\partial q} = \frac{\alpha\delta\left(2-\delta\right)\left(2\beta(1-q)-\alpha\right)}{\left[\alpha\delta q + 2\delta\beta(1-q)^2 - \alpha\delta(1-q)\right]^2}$$

Therefore, μ^e (weakly) increases in q whenever $\alpha \le 2\beta(1-q)$, and decreases in q otherwise. ■

Proof of Proposition 3:
We write the utility for minority group members for extreme values of q, for each fertility choice:

$$U^1(0) = \frac{\beta^2}{2\alpha} + V_r(m)$$

$$U^1(0.5) = \frac{\beta^2}{8\alpha} + \frac{\beta}{2} + V_r(m)$$

$$U^2(0,\mu) = \frac{B^2(1-\mu)^2}{\alpha-(2\mu-1)B} + V_r(mm)$$

$$U^2(0.5,\mu) = \frac{B^2}{16\alpha-4(2\mu-1)B} + 0.5B\left(1.5-\mu\right) + V_r(mm)$$

Since $U^1(q)$ and $U^2(q,\mu)$ are monotonically increasing in q, we show that there is a single crossing of $U^1(q)$ and $U^2(q,\mu)$ for $q \in \left(0,0.5\right)$, by showing that $U^1(0) > U^2(0,\mu)$ and $U^1(0.5) < U^2(0.5,\mu)$.
The first condition, $U^1(0) > U^2(0,\mu)$, yields the following inequality:

$$\frac{\beta^2}{2\alpha} - \frac{B^2(1-\mu)^2}{\alpha-(2\mu-1)B} > 0.$$

This inequality is quadratic in μ and yields μ_1, μ_2 such that for $\mu_1 \le \mu \le \mu_2$ the inequality holds. Since $\mu_2 > 1$ and $0 < \mu_1 < 1$, we denote $\mu^L \equiv \mu_1$:

$$\mu^L = \frac{2\alpha B - \beta^2 - \beta\sqrt{\beta^2 + 2\alpha^2 - 2\alpha B}}{2\alpha B}.$$

And $U^1(0) > U^2(0,\mu)$ for all $\mu > \mu^L$.
Substituting $B = \delta\beta$, $\delta \in (1,2]$, we can show that for $\delta > \sqrt{2}$, $\mu^L > 0.5$, and for $\delta < \sqrt{2}$, $\mu^L < 0.5$.
The second condition, $U^1(0.5) < U^2(0.5,\mu)$, yields the following inequality:

$$\frac{B^2}{16\alpha - 4B(2\mu-1)} + 0.5B(1.5-\mu) > \frac{\beta^2}{8\alpha} + 0.5\beta \quad .$$

This inequality is quadratic in μ as well, and yields μ_1, μ_2 such that for $\mu < \mu_1$ and $\mu > \mu_2$ the inequality holds. Since $\mu_2 > 1$ and $0 < \mu_1 < 1$, we denote $\mu^H \equiv \mu_1$:

$$\mu^H = \frac{8\alpha(\alpha+B) - 2\alpha\beta - \beta^2 - \sqrt{\left[8\alpha(\alpha-B) + 2\alpha\beta + \beta^2\right]\left[8\alpha^2 + 2\alpha\beta + \beta^2\right]}}{8\alpha B}.$$

And $U^1(0.5) < U^2(0.5,\mu)$ for all $\mu < \mu^H$.

We have therefore identified the lower and upper bounds for μ: For $\mu^L < \mu < \mu^H$ there is single crossing of $U^1(q)$ and $U^2(q,\mu)$ for $q \in (0, 0.5)$. Further note that $\mu^L < \mu^H$ is non-empty – a sufficient condition for $\mu^L < \mu^H$ is $B < \frac{(2\alpha+\beta)^2 + 2\beta^2}{8\alpha}$ or $\delta < \frac{1}{2} + \frac{\alpha}{2\beta} + \frac{3\beta}{8\alpha}$.

Denote the single crossing of $U^1(q)$ and $U^2(q,\mu)$ by $q^*(\mu,\alpha)$. It remains to show that $q^*(\mu,\alpha)$ is increasing in μ. Clearly, $q^*(\mu,\alpha)$ increases in μ if and only if $U^2(q,\mu)$ decreases in μ. We write the derivative of $U^2(q,\mu)$ w.r.t μ:

$$\frac{\partial U^2(q,\mu)}{\partial \mu} = \frac{-2B(1-q)\left[q\alpha + B(1-q)^2(1-\mu)\right]\left[\alpha - B\mu(1-q)\right]}{\left[\alpha - B(1-q)^2(2\mu-1)\right]^2} < 0$$

which completes our proof. ■

Proof of Proposition 4:

Increasing α decreases both $U^1(q)$ and $U^2(q,\mu)$. When the effect of α on $U^1(q)$ is stronger $q^*(\mu,\alpha)$ decreases, and there exist minority sizes for which this implies a change in the fertility choice from one to two children. We proceed by identifying the condition for $\frac{\partial U^2(q,\mu)}{\partial \alpha} - \frac{\partial U^1(q)}{\partial \alpha} > 0$:

$$\frac{\partial U^2(q,\mu)}{\partial \alpha} - \frac{\partial U^1(q)}{\partial \alpha} = (1-q)^2\left[\frac{\beta^2}{2\alpha^2} - \frac{B^2[q\mu + (1-q)(1-\mu)]^2}{[\alpha - B(1-q)^2(2\mu-1)]^2}\right] =$$

$$= (1-q)^2 \underbrace{\left[\frac{\beta}{\sqrt{2}\alpha} + \frac{B[q\mu + (1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)}\right]}_{>0} \underbrace{\left[\frac{\beta}{\sqrt{2}\alpha} - \frac{B[q\mu + (1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)}\right]}_{\equiv A}$$

We denote $A \equiv \frac{\beta}{\sqrt{2}\alpha} - \frac{B[q\mu + (1-q)(1-\mu)]}{\alpha - B(1-q)^2(2\mu-1)}$.

$\frac{\partial U^2(q,\mu)}{\partial \alpha} - \frac{\partial U^1(q)}{\partial \alpha} > 0$ if and only if $A > 0$. Further arranging yields that $A > 0$ if and only if $\mu < \mu^\alpha(q)$, where:

$$\mu^{\alpha}(q) = \frac{\alpha + \delta\beta(1-q)^2 - \sqrt{2}\alpha\delta(1-q)}{2\delta\beta(1-q)^2 - \sqrt{2}\alpha\delta(1-2q)}$$

This completes our proof. ■

Proof of Lemma 1:

We solve $q^{t+1}\big|_{N_r=1} > q^{t+1}\big|_{N_r=2}$:

$$\frac{(1-q)q + q\left[\tau^1 + (1-\tau^1)q\right]}{(1-q)+q} > \frac{(1-q)q + 2q\left[\tau^2 + (1-\tau^2)q\right]}{(1-q)+2q}$$

This reduces to $\tau^1 > \frac{2}{1+q}\tau^2$:

$$\frac{\beta}{\alpha}(1-q) > \frac{2}{1+q}\frac{\delta\beta(1-q)\left[q\mu + (1-q)(1-\mu)\right]}{\alpha - \delta\beta(1-q)^2(2\mu-1)}$$

$$\alpha(1+q) - 2\alpha\delta(1-q) + \delta\beta(1-q)^2(1+q) > \mu\underbrace{\left[2\alpha\delta(2q-1) + 2\delta\beta(1-q)^2(1-q)\right]}_{(*)}$$

The expression marked (*) is increasing in q, and there exists $q^g(\alpha,\beta,\delta)$ such that (*) is negative for all $q < q^g(\alpha,\beta,\delta)$ and positive for all $q > q^g(\alpha,\beta,\delta)$.

The threshold value for μ is derived by dividing the LHS of the inequality by (*).

Denote this threshold by μ^g:

$$\mu^g \equiv \frac{1}{2} + \frac{\alpha(1-\delta+q)}{2\alpha\delta(2q-1) + 2\delta\beta(1-q)^2(1+q)} \quad .$$

We conclude that $q^{t+1}\big|_{N_r=1} > q^{t+1}\big|_{N_r=2}$ whenever:

(i) $q < q^g(\alpha,\beta,\delta)$ and $\mu > \mu^g(q,\alpha,\beta,\delta)$, or –

(ii) $q > q^g(\alpha,\beta)$ and $\mu < \mu^g(q,\alpha,\beta,\delta)$. ■

Proof of Lemma 2:

We begin by writing a majority group member's utility from having one child and from having two children (under the assumption that $\mu_m = 0.5$):

$$U_m^1(q) = \frac{\beta^2}{2\alpha_m}(1-q_m)^2 + \beta q_m + V_m(r)$$

$$U_m^2(q) = \frac{\delta_m{}^2\beta^2}{4\alpha_m}(1-q_m)^2 + \delta_m\beta q_m + V_m(rr)$$

Two children are chosen whenever:

$$\frac{\delta_m{}^2\beta^2(1-q_m)^2}{4\alpha_m} + q_m\delta_m\beta + V_m(rr) > \frac{\beta^2}{2\alpha_m}(1-q_m)^2 + \beta q_m + V_m(r)$$

Arranging yields:

$$\left(\frac{\delta_m{}^2}{2}-1\right)\frac{\beta^2\left(1-q_m\right)^2}{2\alpha_m}+q_m\left(\delta_m-1\right)\beta>0$$

Note that when $\delta_m \geq \sqrt{2}$ the above inequality holds for all q_m, and more generally, there exists a threshold δ_m, denoted $\delta_m^* \in \left(1,\sqrt{2}\right)$ such that when $\delta_m \geq \delta_m^*$ the inequality holds for all q_m and the majority will always choose $N_m = 2$.

Solving the inequality for q_m, we find that one root is positive and smaller than 1 and the other is larger than 1. Denote these roots by $q_1 < q_2$. When $q < q_1$ or $q > q_2$ the inequality holds. We denote the smaller root by $q_m^*\left(\alpha_m,\delta_m\right)$ and conclude that for $\delta_m < \delta_m^*$ and $q_m < q_m^*\left(\alpha_m,\delta_m\right)$ the majority will choose $N_m = 1$, and otherwise $N_m = 2$.

■

Lemma A1: For each fertility choice pair $\left(N_r,2\right)$ there exists a unique stable steady state population structure, with $q_{N_r,2}^s \in \left(0,1\right)$:

(i) For $\left(1,2\right)$: $q_{1,2}^s = \alpha_m / \left(\alpha_m + \delta_m\alpha_r\right)$, thus $q_{1,2}^s < 0.5$ iff $\alpha_m/\delta_m < \alpha_r$.

(ii) For $\left(2,2\right)$: $q_{2,2}^s \in \left(0,1\right)$. $q_{2,2}^s < 0.5$ iff $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r\beta} - \frac{\alpha_m}{\delta_m\beta}\right)$.

Proof of Lemma A1:

The condition for a stable population structure, given a fertility choice pair $\left(N_r,N_m\right)$ is $N_r\tau_r^{N_r} = N_m\tau_m^{N_m}$. We solve this condition for fertility choice pairs $\left(1,2\right)$ and $\left(2,2\right)$:

(i) For $\left(1,2\right)$: $\tau_r^1 = 2\tau_m^2$ implies –

$$\frac{1-q}{\alpha_r}\beta = 2\frac{\delta_m\beta q}{2\alpha_m}$$

Arranging yields $q_{1,2}^s = \frac{\alpha_m}{\alpha_m + \delta_m\alpha_r}$. Note that $q_{1,2}^s < \frac{1}{2}$ iff $\alpha_m/\delta_m < \alpha_r$.

Furthermore, this steady state is stable, since $q < q_{1,2}^s$ implies $\tau_r^1 > \tau_m^1$, and thus minorities that are smaller than $q_{1,2}^s$ will grow towards this size, while larger minorities will decrease towards $q_{1,2}^s$.

(ii) For $\left(2,2\right)$: $2\tau_r^2 = 2\tau_m^2$ implies –

$$\frac{\delta_r\beta\left(1-q\right)\left[q\mu+\left(1-q\right)\left(1-\mu\right)\right]}{\alpha_r-\delta_r\beta\left(1-q\right)^2\left(2\mu-1\right)} = \frac{\delta_m\beta q}{2\alpha_m}$$

Solving for q is equivalent to finding the zeros of the following function:

$$f_{2,2}(q,\mu) = q^3\delta_r\delta_m\beta(2\mu-1) + q^2 2\delta_r(\alpha_m+\delta_m\beta)(1-2\mu) + q\left[6\alpha_m\delta_r\mu - 4\alpha_m\delta_r - \alpha_r\delta_m + \delta_r\delta_m\beta(2\mu-1)\right] + $$
$$+ 2\alpha_m\delta_r(1-\mu)$$

We find values of $f_{2,2}(q,\mu)$ for $q \in \{0, 0.5, 1\}$:

$f_{2,2}(0,\mu) = 2\alpha_m\delta_r(1-\mu) \geq 0$

$f_{2,2}(1,\mu) = -\alpha_r\delta_m < 0$

$f_{2,2}(0.5,\mu) = \frac{1}{8}\delta_r\delta_m\beta(2\mu-1) + 0.5\alpha_m\delta_r - 0.5\alpha_r\delta_m < 0$ iff $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r\beta} - \frac{\alpha_m}{\delta_m\beta}\right)$

By the intermediate value theorem, we can state that there exists a zero in $(0,1)$. Denote this zero by $q^s_{2,2}$. Furthermore, $q^s_{2,2} \in (0, 0.5)$ when $f_{2,2}(0.5,\mu) < 0$, i.e. if and only if $\mu < \frac{1}{2} + 2\left(\frac{\alpha_r}{\delta_r\beta} - \frac{\alpha_m}{\delta_m\beta}\right)$. Also note that $q^s_{2,2}$ is the unique zero of $f_{2,2}(q,\mu)$ in $(0,1)$ (this is straight forward for $\mu \geq 0.5$ and can be shown numerically for $\mu < 0.5$).

Stability of $q^s_{2,2}$ follows from similar arguments as in (i). ■